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Sliding Modes in Sampled-data Systems

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The sliding mode application in discrete-time systems can result in unwanted oscillations of the controlled variable (so called chattering). To avoid above-mentioned oscillations a new approach in the design of sliding mode control is proposed in this paper. In the proposed approach the calculation of the equivalent control is not necessary while the influence of the system uncertainty and chattering are reduced. The proposed method is applicable to linear as well as nonlinear systems. It allows the design of the control without transformation of the system description to the discrete-time form (z-domain). Upper bound of the sapling time is determined from the switching function changes during the sampling period. The systems with state observers are analyzed. Experimental and simulation results are presented to clarify the design procedure and the features of the proposed algorithm.

Key words: discrete-time systems, linear systems, Liapunov design, nonlinear systems, sliding mode, variable structure systems

1 INTRODUCTION

Ideal sliding mode, characterized by the motion in sliding mode manifold can occur in real systems rarely. In continuous time systems real sliding mode motion is characterized by high frequency oscillations within a boundary layer of the sliding mode manifold, but averaged motion is kept in the sliding mode manifold. For continuous time, design of the sliding mode with discontinuous control requires the information about the upper bound of the equivalent control and the position of the system state with respect to the sliding mode manifold (the signs of all components of the switching function vector) [1].

Due to the »hold« processes in the control loop and unpredictable changes of the external disturbances in discrete-time systems the ideal sliding mode can occur rarely. Therefore, like in any discrete-time system, the state can be kept within a boundary layer of the sliding mode manifold. The motion in this boundary layer is accepted as a sliding mode motion [2, 3, 4], and sometimes it is described as so-called quasi-sliding mode [5, 6].

This paper deals with the implementation of sliding mode control for continuous system with discrete-time implementation of the control algorithm by maintaining sliding mode. A considerable amount of work has been done analyzing discrete-time sliding modes [2–9]. In [5], Milosavljevic studied quasisliding in the vicinity of the sliding manifold due to the discretization of continuous time signals. In [2], Utkin and Drakunov proposed, for control law design, the discrete-time equivalent control that directs the state onto sliding mode manifold after a finite number of sampling intervals, due to the boundness of the control input. The resulting control appears to be non-switching. Other related works can be found in [3, 4, 6, 7, 8, 9] all addressing discrete-time sliding modes from different prospective. Most of the proposed control strategies use, in one or another way, the calculation or estimation of the discrete-time equivalent control explicitly, which requires the transformation of the plant model into a discrete-time form. That leads to calculation intensive requirements, and it is sensitive to the plant parameters change.

This paper is an extension of our previous work [12–15] in the motion control systems and is mainly concentrated on two problems. The first is to formulate a design procedure for control input calculation that will maintain the state in the ε-vicinity of the sliding mode manifold without calculating equivalent control and the other, to find the upper bound for the sampling interval while using the proposed control algorithm and for given intersampling deviation from the sliding mode manifold. It will be demonstrated that such a control input can be designed using only information about the distance from the sliding mode manifold. The upper bound for the sampling interval can be determined such that the prescribed intersampling deviation is maintained.

The paper is organized as follows: in section 2, the problem statement along with some background results is considered. In section 3, the design procedure is presented in details and the stability analysis for systems with and without computational delay is shown. The approximated algorithm is introduced and stability conditions for it are presented as well. The conditions for the sampling interval selection are determined. In section 4, the equations of motion of the system with proposed control are derived and the influence of the design parameters is analyzed. In section 5, the issue of robustness of the system motion under proposed control is analyzed. In section 6, behavior of the systems with state observers is presented. The simulation and experimental results are discussed shortly in order to show the implementation issue for the proposed algorithm.

2 PROBLEM STATEMENT

Given a continuous nonlinear dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x},t) + \mathbf{B}(\mathbf{x},t)\mathbf{u}(\mathbf{x},t), \ \ \mathbf{x} \in \mathbf{R}^n, \ \ \mathbf{u} \in \mathbf{R}^m \quad (1)$$

where all elements of vector $\mathbf{f}(\mathbf{x},t)$ and matrix $\mathbf{B}(\mathbf{x},t)$ are continuous and bounded having continuous and bounded first order time derivatives; $rank(\mathbf{B}(\mathbf{x},t) = m$, $\forall \mathbf{x}, t > 0$; all components of the control input $\mathbf{u}(\mathbf{x},t)$ are bounded by known functions $u_i(\mathbf{x},t)_{\min}$ and $u_i(\mathbf{x},t)_{\max}$ $i = 1,2,...,$m.

Assume that the desired specification of closed loop system is achieved if the system state satisfies algebraic constrain $\sigma(\mathbf{x},t) = 0$, $\sigma \in \mathbf{R}^m$ where all components of function $\sigma(\mathbf{x},t)$ are continuous. Then, the design goal is to stabilize the system motion on the smooth manifold

$$\mathbf{S} = \left\{ \mathbf{x} : \sigma(\mathbf{x},t) = \mathbf{0}; \mathbf{G}(\mathbf{x},t) = \frac{\partial \sigma}{\partial \mathbf{x}}, \ \sigma \in \mathbf{R}^m, \right.$$
$$\left. rank\mathbf{G}(\mathbf{x},t) = m, \forall \mathbf{x}, t > 0 \right\} \quad (2)$$

where all elements of vector $\frac{\partial \sigma}{\partial t}$ and matrix $\mathbf{G}(\mathbf{x},t)$ are, by assumption, continuous and bounded.

In the sliding mode approach with discontinuous control action, the time derivative of function $\sigma(\mathbf{x},t)$ is forced to have discontinuities so that, in general, this function and its first time derivative can have opposite signs. In this situation the system state will oscillate in the ε-vicinity of manifold (2). This motion is known as real sliding mode motion. In the ideal case the oscillations have an infinitely small amplitude and the system state remains in manifold (2) after reaching it (so called ideal sliding mode). If the initial state $\mathbf{x}(t_0)$ for $t = t_0$ belongs to the manifold (2), then to satisfy condition $\mathbf{x}(t) \in S$ for $\forall t > t_0$, it is enough to find control input which provides that time derivative $\frac{d\sigma(\mathbf{x},t)}{dt} = 0$ for $\mathbf{x}(t) \in S$ for $\forall t > t_0$, and then, further motion will remain in manifold (2) [1]. According to this idea, if **GB** is a regular matrix, the control input, so called equivalent control, is determined from

$$\frac{d\sigma(\mathbf{x}, \mathbf{u}_{eq}, t)}{dt} = 0$$

and the system dynamics (1) as

$$\mathbf{u}_{eq} = -(\mathbf{GB})^{-1} \left(\frac{\partial \sigma}{\partial t} + \mathbf{Gf} \right).$$

To determine the equations of motion this control input should be substituted into (1) to obtain

$$\frac{d\mathbf{x}}{dt} = (\mathbf{E} - \mathbf{B}(\mathbf{GB})^{-1}\mathbf{G})\mathbf{f} - \mathbf{B}(\mathbf{GB})^{-1} \frac{\partial \sigma}{\partial t},$$

which along with $\sigma(\mathbf{x},t)$ describe the motion of system (1) in manifold (2). This equation is the so-called ideal sliding mode equation. It has been proven in [10] that the motion of the closed loop system is independent on all disturbances presentable as $\mathbf{f} = \mathbf{B}\boldsymbol{\lambda}$. This so called matching condition is a very powerful tool in the design of the sliding mode systems. Sliding modes possess very attractive properties like disturbance rejection, very low sensitivity on the changes of the system parameters etc. This procedure shows the sliding mode equations of motion and the existence conditions, but it does not have any clue for the control input selection. The control input selection is related to the stability of the projection of the system motion on the subspace whose coordinates are the components of the sliding mode function.

For a continuous time system with discontinuous control the design procedure can be briefly stated as follows: the stability of the solution $\sigma(\mathbf{x},t)$ will be guarantied if control is selected so that the candidate Lyapunov function $v_d = \frac{\sigma^T \sigma}{2}$ has time derivative

$$\frac{dv_d}{dt} = \sigma^T \frac{d\sigma}{dt} = -\sigma^T \Gamma \mathbf{sign}(\sigma), \ \Gamma \rangle 0.$$

Then, if **GB** is a regular matrix, the control input can be determined as $\mathbf{u} = \mathbf{u}_{eq} - (\mathbf{GB})^{-1}\Gamma\mathbf{sign}(\sigma)$ or for $\max((\mathbf{GB})^{-1}\Gamma) = F(\mathbf{x},t,\Gamma)$ then

$$\mathbf{u} = \mathbf{u}_{eq} - F(\mathbf{x},t,\Gamma)\mathbf{sign}(\sigma).$$

For motion in the neighborhood of manifold (2), $F(\mathbf{x},t,\Gamma)$ can be very small if the equivalent control is calculated on line. Otherwise, if only bounds of the equivalent control are known $F(\mathbf{x},t,\Gamma)$ can be determined from inequality

$$|F(\mathbf{x},t,\Gamma)| > \max_i (u_{ieq}) \ i = 1,2,...,m.$$

Then control input has the form

$$\mathbf{u} = -F(x,t,\Gamma)\mathbf{sign}(\sigma).$$

In real systems the control input will have finite frequency switching which result in the oscillatory motion (the chattering) of the closed loop system in an ε-vicinity (so called boundary layer) of the sliding mode manifold.

The procedure for discrete-time sliding mode design [2] begins with a transformation of the plant description to the discrete-time form

$$\mathbf{x}(kt+T)=\mathbf{x}(kT)+\mathbf{f}^*+\mathbf{B}^*\mathbf{u}(kT)$$

where T is sampling interval,

$$\mathbf{f}^*=\int_{kT}^{kT+T}\mathbf{f}(x,\tau)\mathrm{d}\tau;\ \mathbf{B}^*=\int_{kT}^{kT+T}\mathbf{B}(x,\tau)\mathrm{d}\tau.$$

Then, so-called discrete-time equivalent control can be calculated from

$$\sigma(kT+T)=\mathbf{Gx}(k+1)=0$$

as

$$\mathbf{u}_{eq}(kT)=-(\mathbf{GB}^*)^{-1}(\mathbf{Gx}(kT)+\mathbf{Gf}^*).$$

It should be noted that in this case no computational delay is taken into account. The intersampling change for equivalent control is of $O(T)$ order. The equivalent control tends to infinity if $T\to 0$ for $\sigma\neq 0$, since $(\mathbf{GB}^*)^{-1}\to\infty$ while $(\mathbf{Gx}(kT)+\mathbf{Gf}^*)$ takes a finite value. This requires the introduction of limits for the control action. Taking into account that, control action is bounded by assumption, the control algorithm can be expressed as $\mathbf{u}(kT)=sat(\mathbf{u}(kT))$ where $\min(sat(\bullet))=u_{\min}$ and $\max(sat(\bullet))=u_{\min}$. It has been proven that the selected control will force system state to stay in ε-vicinity of sliding mode manifold (2) with a thickness of the boundary layer of $O(T^2)$ order. To avoid cumbersome explanations in discrete-time control systems, from now on, term »continuous control« is used in the sense that intersampling change of the control input is of $O(T)$ order.

In some applications, like power electronics, switching is the »way of life« and motion in a boundary layer is unavoidable regardless of the technique one can use in the control system design. In these systems chattering (often called ripple) is the structural property of the system. In some other systems (like mechanical, or process systems) discontinuity of the control action is not so desirable (or easy to achieve) from many points of view (actuator wearing, excitement of unmodelled dynamics, etc.). In these systems properties that can be achieved by introducing sliding mode motion are very attractive, so a design procedure that will allow to attain these properties while discontinuity of the control action is avoided is most desirable. Further the design procedure will be demonstrated in details along with stability proofs for systems with and without computational delays. The design procedure begins with a selection of the candidate Lyapunov function and the form, which the time derivative of the candidate Lyapunov function should satisfy. From these two selections the control input is determined. In sampled data systems the satisfaction of the stability conditions is determined at the moment renewed control is applied (usually the beginning of the sampling interval) and at the end of the sampling interval in order to determine the sampling interval and allowed computational delay.

3 DESIGN

For system (1) asymptotic stability of the solution $\sigma(\mathbf{x},t)=\mathbf{0}$ can be assured if one can find a control input such that the Lyapunov stability criteria are satisfied. Natural selection of the candidate Lyapunov function is a quadratic form $v=\frac{\sigma^T\sigma}{2}$. The design procedure can be started from the requirement that the time derivative of the Lyapunov function should have the following form

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-\sigma^T\mathbf{D}\sigma,\ \mathbf{D}>0.$$

Then

$$v(\mathbf{x},t)>0,\ \frac{\mathrm{d}v(\mathbf{x}^*,t)}{\mathrm{d}t}<0\ \forall\mathbf{x}\notin S,$$

$$v(\mathbf{x}^*,t)=0,\ \frac{\mathrm{d}v(\mathbf{x}^*,t)}{\mathrm{d}t}=0\ \forall\mathbf{x}^*\in S,\forall t>0$$

and solution $\sigma(\mathbf{x},t)=\mathbf{0}$ is asymptotically stable on the trajectories of the system (1). A control input that satisfies given requirements can be calculated from

$$\sigma^T\frac{\mathrm{d}\sigma}{\mathrm{d}t}=-\sigma^T D\sigma\Rightarrow\sigma^T\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}+D\sigma\right)=0$$

gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}+D\sigma=0,\quad\forall\sigma\neq 0.$$

By substituting

$$\frac{\mathrm{d}\sigma(\mathbf{x},\mathbf{u}_{eq},t)}{\mathrm{d}t}=\mathbf{Gf}+\mathbf{GBu}+\frac{\partial\sigma}{\partial t}$$

into

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}+D\sigma=0$$

and by solving this equation for an unknown control u, it is easy to obtain

$$\mathbf{u}(\mathbf{x},t)=-(\mathbf{GB})^{-1}\left(\frac{\partial\sigma(x,t)}{\partial t}+\mathbf{Gf}\right)-(\mathbf{GB})^{-1}\mathbf{D}\sigma.\quad(3)$$

To have a unique solution for the control input, matrix GB must be regular. That is the same requirement needed to find equivalent control. Selected control input guaranties that the motion of system

(1) satisfies the dynamical constrain $\frac{\mathrm{d}\sigma}{\mathrm{d}t}+D\sigma=0$. That means all distances from the manifold (2) exponentially tend to zero and the system motion will remain in ε-vicinity of the manifold (2) after reaching it. Strictly speaking, control (3), being continuous, does not provide the sliding mode motion on manifold (2) because that manifold could be reached only for $t\to\infty$. Taking into account that the equivalent control, required to keep the motion in the manifold S if initial state belongs to this manifold, can be expressed as $\mathbf{u}_{eq}=-(\mathbf{GB})^{-1}\left(\frac{\partial\sigma}{\partial t}+\mathbf{Gf}\right)$, equation (3) can be rewritten as

$$\mathbf{u}(\mathbf{x},t)=\mathbf{u}_{eq}(\mathbf{x},t)-(\mathbf{GB})^{-1}\mathbf{D}\sigma. \tag{4}$$

In (4) the resulting control action is continuous (the equivalent control is continuous and function $\sigma(\mathbf{x},t)=\mathbf{0}$ is continuous by assumption) and $\mathbf{u}(\mathbf{x},t)=$ $=\mathbf{u}_{eq}(\mathbf{x},t)$ for $\sigma(\mathbf{x},t)=\mathbf{0}$. In the implementation of algorithms (3) or (4) full information about system dynamics and external disturbances is required (for equivalent control calculation). Because of this these algorithms are not practical for application. They are used here as intermediate results to show the procedure in the development of simpler and more useful control strategies. From $\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\mathbf{GB}(\mathbf{u}-\mathbf{u}_{eq})$ equivalent control can be substituted into (4) to obtain

$$\mathbf{u}(\mathbf{x},t)\equiv\mathbf{u}(\mathbf{x},t)-(\mathbf{GB})^{-1}\left(\mathbf{D}\sigma(\mathbf{x},t)+\frac{\mathrm{d}\sigma(x,t)}{\mathrm{d}t}\right). \tag{5}$$

This form of expressing the control input is very instructive. It shows that in order to force the system to reach ε-vicinity of sliding mode manifold (2) and to stay within ε boundary layer the control input should be modified by the term

$$(\mathbf{GB})^{-1}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}+\mathbf{D}\sigma\right).$$

at every instant of time. The control (5) takes the value of the equivalent control for $\sigma(\mathbf{x},t)=0$.

3.1 Discrete-time realization

Algorithm (5) can be easily modified for the application in the discrete time systems with no computational delay. In such a system relations between measured and computed variables are depicted in Figure 1. where measurement taken before the calculation of new value of the control input are denoted as $\bullet(kT^{-})$ and all variables immediately after new value of the control input is applied are denoted by $\bullet(kT^{+})$, (from now on all variables will be written shorter so $\sigma(kT)=\mathbf{0}$ means $\sigma(\mathbf{x}(kT),kT)=\mathbf{0}$. Note that all continuous functions and variables satisfy $\bullet(kT^{-})=\bullet(kT^{+})$.

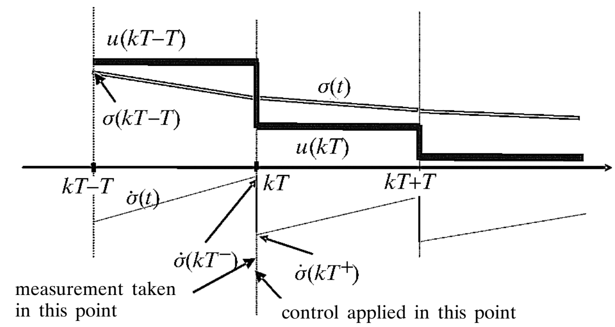

Fig. 1 The relations between measured and calculated variables for discrete time systems without computational delay

By taking into account the relationship depicted in Figure 1, algorithm (5) can be rewritten in the following form

$$\mathbf{u}(kT^{+})=\mathbf{u}(kT^{-})-(\mathbf{GB})^{-1}\left(\mathbf{D}\sigma(kT^{-})+\frac{\mathrm{d}\sigma(kT^{-})}{\mathrm{d}t}\right). \tag{6}$$

The satiability conditions should be analyzed at the moment the control input is applied to the system and at the end of the corresponding sampling interval. The value of the switching function and its derivative at k-th sampling interval e.g. at $t=kT$ and $t=kT+T$ should be determined and value of the selected Lyapunov function and its derivative should be calculated at both instants. By calculating the derivative of function $\sigma(\mathbf{x},t)$ at $t(kT^{+})$ one can find

$$\frac{\mathrm{d}\sigma(kT^{+})}{\mathrm{d}t}=\frac{\partial\sigma(kT^{+})}{\partial t}+\mathbf{G}(\mathbf{f}(kT^{+})+\mathbf{Bu}(kT^{+})). \tag{7}$$

From (6) and (7) it follows

$$\frac{\mathrm{d}\sigma(kT^{+})}{\mathrm{d}t}=\left[\frac{\partial\sigma(kT^{-})}{\partial t}+\mathbf{G}(\mathbf{f}(kT^{-})+\mathbf{Bu}(kT^{-}))\right]-$$
$$-\mathbf{D}\sigma(kT^{-})-\frac{\mathrm{d}\sigma(kT^{-})}{\mathrm{d}t}$$

and finally

$$\frac{\mathrm{d}\sigma(kT^{+})}{\mathrm{d}t}=\frac{\mathrm{d}\sigma(kT^{-})}{\mathrm{d}t}-\mathbf{D}\sigma(kT^{-})-\frac{\mathrm{d}\sigma(kT^{-})}{\mathrm{d}t}=$$
$$=-\mathbf{D}\sigma(kT^{-})=-\mathbf{D}\sigma(kT^{+}). \tag{8}$$

The time derivative of the Lyapunov function, at $t=kT^{+}$, can be expressed as

$$\frac{\mathrm{d}v(kT^{+})}{\mathrm{d}t}=-\sigma(kT^{+})^{T}\mathbf{D}\sigma(kT^{+}).$$

This shows that at the moment immediately after new control is applied (the beginning of the sampling interval) the stability conditions are satisfied.

For $t \in [kT, kT+T]$ the control input is constant. The change of the system state during an intersampling interval can be expressed as

$$\mathbf{x}(kT+t) = \mathbf{x}(kT) + \int_{kT}^{kT+t} \mathbf{f}(\mathbf{x},\tau)\mathrm{d}\tau + \left[\int_{kT}^{kT+t} \mathbf{B}(x,\tau)\mathrm{d}\tau\right]\mathbf{u}(kT)$$
$$\mathbf{x}(kT+t) = \mathbf{x}(kT) + \xi(t); \tag{9}$$

and the change of the distance from the sliding mode manifold (2) is

$$\sigma(kT+t) = \sigma(kT) + \int_{kT}^{kT+t} \mathbf{Gf}(x,\tau)\mathrm{d}\tau + \left[\int_{kT}^{kT+t} \mathbf{GB}(x,\tau)\mathrm{d}\tau\right]\mathbf{u}(kT)$$
$$\sigma(kT+t) = \sigma(kT) + \varsigma(t). \tag{10}$$

By the assumption $\mathbf{f}, \mathbf{B}$ and $\boldsymbol{G}$ are continuous and bounded, the maximum change of state vector $\xi(t)$, and the maximum change of distances from the sliding mode manifold $\varsigma(t)$ inside interval $t \in [kT, kT+T]$ are of $O(T)$ order. The change of the time derivative of the sliding mode function can be determined as

$$\frac{\mathrm{d}\sigma(kT+t)}{\mathrm{d}t} = \mathbf{Gf}(kT+t) + \mathbf{GBu}(kT+t) + \frac{\partial\sigma(kT+t)}{\partial t}. \tag{11}$$

By assumption $\frac{\partial\sigma}{\partial t}$ is continuous over the interval $t \in [kT, kT+T]$. Control input $\mathbf{u}(kT+t) = \mathbf{u}(kT)$ is constant over the same interval, and consequently (11) can be expressed as

$$\frac{\mathrm{d}\sigma(kT+t)}{\mathrm{d}t} = \frac{\partial\sigma(kT+t)}{\partial t} + \mathbf{Gf}(kT+t) + \mathbf{GBu}(kT+1) =$$
$$= \left\{\left[\frac{\partial\sigma(kT)}{\partial t} + \mathbf{Gf}(kT) + \mathbf{GBu}(kT)\right] - \frac{\mathrm{d}\sigma(kT)}{\mathrm{d}t}\right\} -$$
$$- \mathbf{D}\sigma(kT) + \zeta(t),$$
$$\frac{\mathrm{d}\sigma(kT+t)}{\mathrm{d}t} = -\mathbf{D}\sigma(kT) + \zeta(t),$$
$$\zeta(t) = \left[\frac{\partial\sigma(kT+t)}{\partial t} - \frac{\partial\sigma(kT)}{\partial t}\right] + \mathbf{G}\left[\mathbf{f}(kT+t) - \mathbf{f}(kT)\right]. \tag{12}$$

The maximum change of $\zeta(t)$ inside interval $t \in$ $\in [kT, kT+T]$ is of $O(T)$ order. Both $\varsigma(t)$ and $\xi(t)$ tend to zero if function $\sigma(kT)$ tends to zero or if the sampling interval tends to zero.

The change of the candidate Lyapunov function time derivative

$$\frac{\mathrm{d}v(kT+t)}{\mathrm{d}t} = \sigma(kT+t)^T \frac{\mathrm{d}\sigma(kT+t)}{\mathrm{d}t}$$

becomes

$$\frac{\mathrm{d}v(kT+t)}{\mathrm{d}t} = \left[\sigma^T(kT) + \varsigma^T(t)\right]\left[\zeta(t) - \mathbf{D}\sigma(kT)\right] =$$
$$= -\sigma^T(kT)\mathbf{D}\sigma(kT) + \sigma^T(kT)\zeta(t) + \varsigma^T(t)\left[\zeta(t) - \mathbf{D}\sigma(kT)\right],$$
$$\frac{\mathrm{d}v(kT+t)}{\mathrm{d}t} = -\sigma^T(kT)\mathbf{D}\sigma(kT) + \upsilon(t),$$
$$\upsilon(t) = \sigma^T(kT)\zeta(t) + \varsigma^T(t)\left[\zeta(t) - \mathbf{D}\sigma(kT)\right], \tag{13}$$

where change of

$$\upsilon(t) = \sigma^T(kT)\zeta(t) + \varsigma^T(t)[\zeta(t) - \mathbf{D}\sigma(kT)]$$

is of the $O(T^2)$ order.

To satisfy Lyapunov stability criteria inside the sampling interval $t \in [kT, kT+T]$ the following condition should hold for $\sigma(kT) \neq 0$

$$\left|\upsilon(t) = \sigma^T(kT)\zeta(t) + \varsigma^T(t)\left[\zeta(t) - \mathbf{D}\sigma(kT)\right]\right| \langle$$
$$\langle \left|\sigma^T(kT)\mathbf{D}\sigma(kT)\right|. \tag{14}$$

In order to determine the sampling interval $T, \upsilon(t)$ should be estimated for $t = T$ and T should be calculated from (14) at the end of a sampling interval. The value $\frac{\mathrm{d}\sigma(kT+T)}{\mathrm{d}t}$ can be expressed as

$$\frac{\mathrm{d}\sigma(kT+T)}{\mathrm{d}t} = \frac{\partial\sigma(kT+T)}{\partial t} + \mathbf{G}(\mathbf{f}(kT+T) + \mathbf{Bu}(kT^+)) =$$
$$= -\mathbf{D}\sigma(kT) + \zeta(T);$$
$$\zeta(T) = \mathbf{G}(\mathbf{f}(kT+T) - \mathbf{f}(kT)) + \left[\frac{\partial\sigma(kT+T)}{\partial t} - \frac{\partial\sigma(kT)}{\partial t}\right]. \tag{15}$$

All elements of vector $\Delta\mathbf{f}(kT) = \mathbf{f}(kT+T) - \mathbf{f}(kT)$ are continuous and bounded and consequently vector $\chi(kT) = \mathbf{G}\Delta\mathbf{f}(kT)$ is continuous and bounded. Since is of $O(T)$ order (function $\frac{\partial\sigma}{\mathrm{d}t}$ is continuous), then

$$\zeta(T) = \mathbf{G}(\mathbf{f}(kT+T) - \mathbf{f}(kT)) + \left[\frac{\partial\sigma(kT+T)}{\partial t} - \frac{\partial\sigma(kT)}{\partial t}\right],$$

is of the $O(T)$ order too. The time derivative of Lyapunov function at the end of the sampling interval can be expressed as

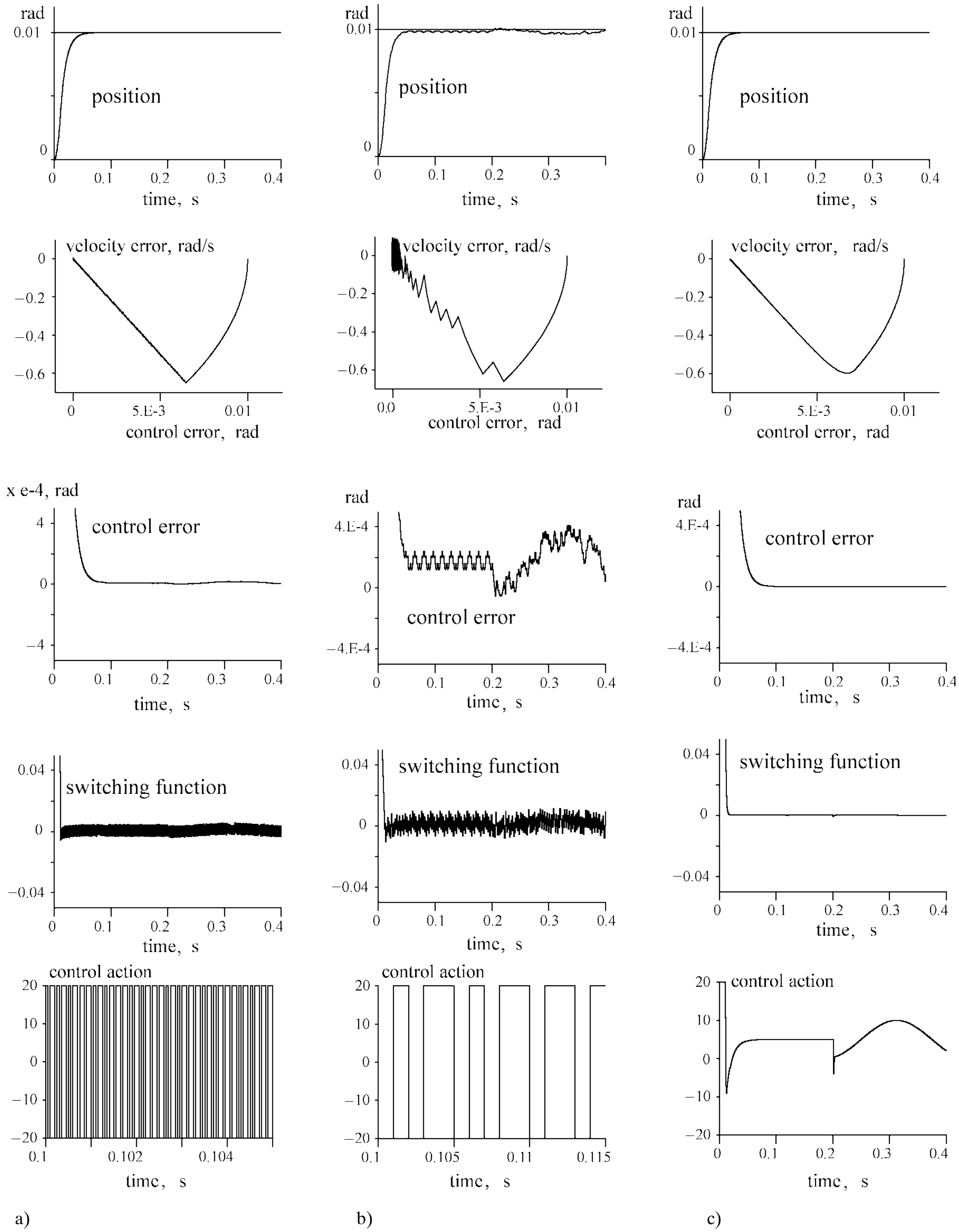

Fig. 2 Transients in the second order system with discontinuous (a), (b) us and continuous control (c).
1) $u(kT) = 20\,\mathrm{sign}(\sigma(kT))$ in the case of discontinuous control action (diagrams (a) and (b));
2) $u(kT) = sat(u(kT - T) + (1/4)(d_{11}\sigma(kT) + \mathrm{d}\sigma(kT)/\mathrm{d}t)$ in the case of the proposed algorithm (18), with $d_{11} = 800$ (c)

$$\frac{\mathrm{d}v(kT+T)}{\mathrm{d}t} = \left[\sigma^T(kT)+\varsigma^T(T)\right]\left[\zeta(T)-\mathbf{D}\sigma(kT)\right]$$
$$= -\sigma^T(kT)\mathbf{D}\sigma(kT)+\sigma^T(kT)\zeta(T)+$$
$$+\varsigma^T(T)\left[\zeta(T)-\mathbf{D}\sigma(kT)\right], \quad (16)$$
$$\frac{\mathrm{d}v(kT+T)}{\mathrm{d}t} = -\sigma^T(kT)\mathbf{D}\sigma(kT)+\varphi(T),$$
$$v(T)=\sigma^T(kT)\zeta(T)+\varsigma^T(T)\left[\zeta(T)-\mathbf{D}\sigma(kT)\right].$$

At the end of the sampling interval the Lyapunov stability conditions will be satisfied if (17) holds for

$$\sigma(kT)\neq 0$$

$$\left|v(T)=\sigma^T(kT)\zeta(T)+\varsigma^T(T)\left[\zeta(T)-\mathbf{D}\sigma(kT)\right]\right| \langle$$
$$\langle \left|\sigma^T(kT)\mathbf{D}\sigma(kT)\right|. \quad (17)$$

With the selection of the sampling interval from (17) the control input (6) provides that there exist $k>N$ or which $\sigma(kT)=0$. The selected control ensures that the motion of the system remains in the ε-vicinity of sliding manifold S after reaching it.

In application the limits of the control input should be introduced. The simplest way to do this is to apply saturation function on expression (6) with $\min(sat(\bullet))=u_{\min}$ and $\max(sat(\bullet))=u_{\min}$. Then the control is calculated as

$$\mathbf{u}(kT^+)=\mathbf{sat}\left(\mathbf{u}(kT^-)-(\mathbf{GB})^{-1}\left(\mathbf{D}\sigma(kT^-)+\frac{\mathrm{d}\sigma(kT^-)}{\mathrm{d}t}\right)\right). \quad (18)$$

To illustrate the behavior of the system with the proposed control strategy simulation results for a second order system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_2; \quad \frac{\mathrm{d}x_2}{\mathrm{d}t}=4(u-i_L(t)); \quad u\in\left[-20,20\right] \quad (19)$$

$$i_L(t)=\begin{cases} 5 & \text{for} \quad 0<t<.2\,\mathrm{s} \\ 5+5\sin(25.16t) & \text{for} \quad t>.2\,\mathrm{s}\end{cases}$$

are depicted in Figure 2 (a), (b) and (c). The sliding mode manifold is selected as the function of the control error and its time derivative

$$\mathbf{S}=\left\{x_1,x_2 : C(x_1^{ref}-x_1)+(x_2^{ref}-x_2)=\sigma=0,\ C=100\right\}.$$

A step transient in $x_1^{ref}=0.01$, rad at $t=0$ is simulated. The control input is calculated according to the following expressions.

Transients with discontinuous control are presented in Figure 2 (a) and (b). In order to show influence of the sampling interval on the behavior of the closed loop system, sampling interval is selected to be $T=10^{-6}$, s in Figure 2 (a), and $T=10^{-3}$, s in Figure 2 (b). It can be observed that control error in Figure 2 (b) is higher and that the chattering clearly rises with the rise of the sampling interval. The activity of the control input is lower. Transients with control algorithm (6) are depicted in Figure 2 (c). In this case all simulations are done with a sampling interval $T=10^{-3}$, s. All other conditions are the same as for the discontinuous control case. As it can be observed, all variables, including the control are smooth and chattering is indeed eliminated. The accuracy of the closed loop system is the same as for systems with discontinuous control and 1000 times shorter sampling interval. The closed loop system exhibits the motion on the selected manifold as can be confirmed from the transients depicted in the phase plane. As it is expected, transients of the system with discontinuous control substantially depend on the selection of the sampling interval. For properly selected sampling interval (Figure 2 (a)) transients are as it is theoretically predicted. Longer step size cause the low frequency chattering (Figure 2 (b)). It can be observed that the sliding mode function (and consequently x_2) has a ripple (the chattering) when discontinuous control is applied. With proposed control all transients are smooth and good load rejection and accuracy are achieved.

In these examples no computational delay has be assumed and the calculation of the time derivative of function $\sigma(t)$ has been required. In the following section first systems with computational delay will be analyzed. Later systems with a backward approximation of the time derivative will be presented. That will allow to use only the information about the value of the sliding mode function (the distance from the sliding mode manifold) in the calculation of the control input, and thus will give a considerable saving in the computational time in comparison with algorithm (18) and other algorithms based on the discrete-time equivalent control calculation.

3.2 Systems with computational delay

In a discrete-time system with computational delay the measurements are taken at $t=kT$, the control is computed during interval $t\in[kT,kT+h]$, and new control input is applied to the system at $t=$ $=kT+h$. The time relation between measurements, computation of control and the computation of other variables is depicted in the Figure 3.

For stability analysis the same procedure as before will be applied. The derivative of selected Lyapunov function should be calculated after new control has been applied e.g. at $t=kT+h$, the intersampling changes should be estimated and the va-

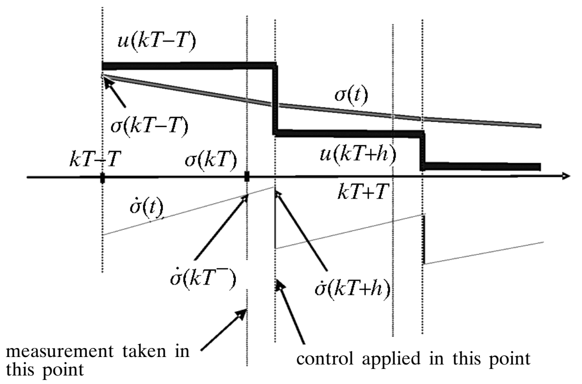

Fig. 3 The relation between measurements, computation and the application of control in systems with computational delay

lues at the end of the sampling interval should be calculated. Both $\sigma(kT+h)$ and its derivative should be estimated based on the values at $t=kT$. Function $\sigma(\mathbf{x},t)$ is continuous and its value $\sigma(kT+h)$ can be estimated as

$$\begin{aligned}&\sigma(kT+h)=\sigma(kT)+\varsigma(h);\\&\|\varsigma(h)\|\le h\max_i(\mathrm{d}\sigma_j(kT)/\mathrm{d}t)\end{aligned}\tag{20}$$

where $v(h)$ is of $O(h)$ order. From (12) the time derivative $\frac{\mathrm{d}\sigma(kT+h)}{\mathrm{d}t}$ can be determined as

$$\begin{aligned}\frac{\mathrm{d}\sigma(kT+h)}{\mathrm{d}t}&=\\&=\frac{\partial\sigma(kT+h)}{\partial t}+\mathbf{Gf}(kT+h)+\mathbf{GBu}(kT+h)=\\&=-\mathbf{D}\sigma(kT)+\xi(h)\end{aligned}\tag{21}$$

$$\xi(h)=\left[\frac{\partial\sigma(kT+h)}{\partial t}-\frac{\partial\sigma(kT)}{\partial t}\right]+\mathbf{G}\left[\mathbf{f}(kT+h)-\mathbf{f}(kT)\right]$$

where $\xi(h)$ is of $O(h)$ order. Now the change of the candidate Lyapunov function time derivative becomes

$$\begin{aligned}&\frac{\mathrm{d}v(kT+h)}{\mathrm{d}t}=-\sigma^T(kT)\mathbf{D}\sigma(kT)+v(h)\\&v(h)=\sigma^T(kT)\xi(h)+\varsigma^T(h)\left[\xi(h)-\mathbf{D}\sigma(kT)\right]\end{aligned}\tag{22}$$

where $v(h)$ is of the $O(h)$ order.

In order to satisfy Lyapunov stability criteria at $t=kT+h$ the permissible calculation delay h can be determined from (23).

$$\left|v(h)=\sigma^T(kT)\xi(h)+\varsigma^T(h)\left[\xi(h)-\mathbf{D}\sigma(kT)\right]\right|\langle$$
$$\langle\left|\sigma^T(kT)\mathbf{D}\sigma(kT)\right|.\tag{23}$$

Selection of the sampling interval T is based on the evaluation of (21), (22) and (23) for $t=kT+T+h$. In this case $\varsigma(T+h)$ and $\xi(T+h)$ are of $O(T+h)$ and $v(T+h)$ is of $\xi((T+h)^2)$. If the calculation delay is equal to the sampling interval e.g. if the measurement is taken at $t=kT$, the control is applied at $t=kT+T$ then sampling interval should be determined from (23) for $h=T$.

3.3 Calculation of control input with approximation of time derivative

To avoid the use of the time derivative in the calculation of control input, the backward approximation of the derivative may be applied by expressing

$$\frac{\mathrm{d}\sigma(kT)}{\mathrm{d}t}=\frac{\sigma(kT)-\sigma(kT-T)}{T}+o(kT),$$

where $o(kT)$ is of $O(T)$ order. Time derivative can be approximated by

$$\frac{\mathrm{d}\sigma(kT)}{\mathrm{d}t}\cong\frac{\sigma(kT)-\sigma(kT-T)}{T}$$

and control algorithm (18), for discrete-time systems with no computational delay, can be expressed as

$$\begin{aligned}&\mathbf{u}(kT)=sat(\mathbf{u}(kT-T)-\\&\quad-(\mathbf{GB}T)^{-1}((\mathbf{E}+T\mathbf{D})\sigma(kT)-\sigma(kT-T)).\end{aligned}\tag{24}$$

Following the same reasoning as in the previous cases, the proof of stability for approximated algorithm (24) is straightforward. Lyapunov function and its derivative should be calculated at $t=kT$. Function $\sigma(kT)$ is measured and its derivative is at the measurement point (before the renewed control is applied) given as

$$\frac{\mathrm{d}\sigma(kT)}{\mathrm{d}t}\cong\frac{\sigma(kT)-\sigma(kT-T)}{T}+o\big(kT\big).$$

The time derivative of sliding mode function $\sigma(kT)$ after renewed control is applied can be calculated as

$$\begin{aligned}\frac{\mathrm{d}\sigma(kT^+)}{\mathrm{d}t}&=\frac{\partial\sigma(kT^+)}{\partial t}+\mathbf{Gf}(kT^+)+\mathbf{GBu}(kT^+)=\\&=\frac{\partial\sigma(kT^+)}{\partial t}+\mathbf{Gf}(kT^-)+\mathbf{GB}\Big[\mathbf{u}(kT-T)\;-\\&\quad-(\mathbf{GB}T)^{-1}\Big[(\mathbf{E}+\mathbf{D}T)\sigma(kT^-)-\sigma(kT-T)\Big]\Big];\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\sigma(kT^+)}{\mathrm{d}t}&=\left[\frac{\partial\sigma(kT^-)}{\partial t}+\mathbf{Gf}(kT^-)+\mathbf{GBu}(kT-T)\right]-\\&\quad-\mathbf{D}\sigma(kT^-)-\frac{\sigma(kT)-\sigma(kT-T)}{T};\end{aligned}$$

$$\frac{d\sigma(kT^+)}{dt} = \frac{\sigma(kT^-)}{dt} - \mathbf{D}\sigma(kT^-) - \frac{\sigma(kT)-\sigma(kT-T)}{T} = \\ = \frac{\sigma(kT)-\sigma(kT-T)}{T} + o(kT) - \\ - \mathbf{D}\sigma(kT^-) - \frac{\sigma(kT)-\sigma(kT-T)}{T}.$$

Taking into account that $\sigma(\mathbf{x},t)$ is continuous one can write

$$\frac{d\sigma(kT^+)}{dt} = o(kT) - \mathbf{D}\sigma(kT^-) = o(kT) - \mathbf{D}\sigma(kT^+). \quad (25)$$

Now $\frac{dv(kT^+)}{dt} = \sigma^T(kT^-)\left[o(kT) - \mathbf{D}\sigma(kT^-)\right]$ or

$$\frac{dv(kT^+)}{dt} = \sigma^T(kT^-)o(kT) - \sigma^T(kT^-)\mathbf{D}\sigma(kT^-). \quad (26)$$

For T small enough

$$\left|\sigma^T(kT^-)o(kT)\right| < \left|\sigma^T(kT^-)\mathbf{D}\sigma(kT^-)\right| \text{ for } \sigma(kT^-) \neq 0 \quad (27)$$

and stability conditions are satisfied at the beginning of a sampling interval. The conditions at the end of a sampling interval can be calculated following the same procedure. The time derivative can be expressed as

$$\frac{d\sigma(kT+T)}{dt} = \frac{\partial\sigma(kT+T)}{\partial t} + \mathbf{G}(\mathbf{f}(kT+T) + \\ + \mathbf{Bu}(kT^+)) = -\mathbf{D}\sigma(kT) + \zeta(T); \\ \zeta(T) = \mathbf{G}(\mathbf{f}(kT+T) - \mathbf{f}(kT)) + \\ + \left[\frac{\partial\sigma(kT+T)}{\partial t} - \frac{\partial\sigma(kT)}{\partial t}\right] + o(kT) \quad (28)$$

where $\zeta(T)$, is of the $O(T)$ order. From

$$\sigma(kT+T) = \sigma(kT) + \varsigma(T); \left\|\varsigma(T)\right\| \le T \max_j \frac{d\sigma_j(kT)}{dt}$$

and (28) the time derivative of Lyapunov function at the end of the sampling interval can be expressed as

$$\frac{dv(kT+T)}{dt} = \left[\sigma^T(kT) + \varsigma^T(T)\right]\left[\zeta(T) - \mathbf{D}\sigma(kT)\right] = \\ = -\sigma^T(kT)\mathbf{D}\sigma(kT) + \upsilon(T)$$

$$\upsilon(T) = \sigma^T(kT)\zeta(T) + \varsigma^T(T)\left[\zeta(T) - \mathbf{D}\sigma(kT)\right].$$

The Lyapunov stability conditions will be satisfied at the end of the sampling interval if the following inequality holds for $\sigma(kT) \neq 0$

$$\left|\upsilon(T) = \sigma^T(kT)\zeta(T) + \varsigma^T(T)\left[\zeta(T) - \mathbf{D}\sigma(kT)\right]\right| \langle \\ \langle \left|\sigma^T(kT)\mathbf{D}\sigma(kT)\right|. \quad (29)$$

Like in the previous case sampling interval T should be calculated from (27) and (29) (at the beginning and at the end of a sampling interval) and the smaller value should be selected.

To show the applicability of the approximated algorithm (24), in Figure 4 transients for system (19) with $i_L = 5\sin(25.16t)$ and reference $x_1^{ref} = 0.01$, rad, in (a) and $x_1^{ref} = (0.01 + 0.01\sin(25.16t))$, rad, in (b) are depicted. Switching manifold is selected as

$$\mathbf{S} = \left\{ x_1, x_2 : C(x_1^{ref} - x_1) + (x_2^{ref} - x_2) = \sigma(t) = 0,\ C = 100 \right\}.$$

The sampling interval is kept $T = 0.001$, s, for all simulations. The control input is calculated according to the approximated algorithm (24) as

$$u(k) = sat\left(u(k-1) + \left(\frac{1}{4T}\right)((1 + Td_{11})\sigma(k) - \sigma(k-1))\right) \quad (30)$$

with $d_{11} = 800$. The simulation results confirm all predictions regarding the implementation of the approximated algorithm. By comparing diagrams (c) in Figure 4. and diagrams (a) in Figure 4. very small differences can be observed. The variation of the external disturbance is rejected without its measurement or estimation. Diagrams (b) in Figure 4. show the tracking of the time varying reference for system under influence of the unknown external disturbance. It shows that the approximation in the control input calculation is applicable in these systems. The information needed for the control input calculation according to (30) is the same as for discontinuous control case, only knowledge about the value of sliding function is required.

In Figure 5 the changes in exact and approximated values of the switching function its derivative and in Lyapunov function and its derivative are depicted for different values of the sampling interval. The control plant is as given in (19) and the control is calculated according to (30) for step change in the reference $x_1^{ref} = 0.5$, rad. The period of 20 milliseconds is depicted in diagrams (a), (b) and (c) and full transient is depicted in diagram (d). The sampling interval is selected to be $T = 0.0005$, s, in (a), $T = 0.001$, s, in (b) and $T = 0.002$, s, in (c) and (d). The scales on horizontal and vertical coordinates are kept the same for all corresponding coordinates to allow simple comparison. The intersampling change of both the switching function and the Lyapunov function in all three cases is very small while the intersampling change of the derivatives for both functions are considerable but proportio-

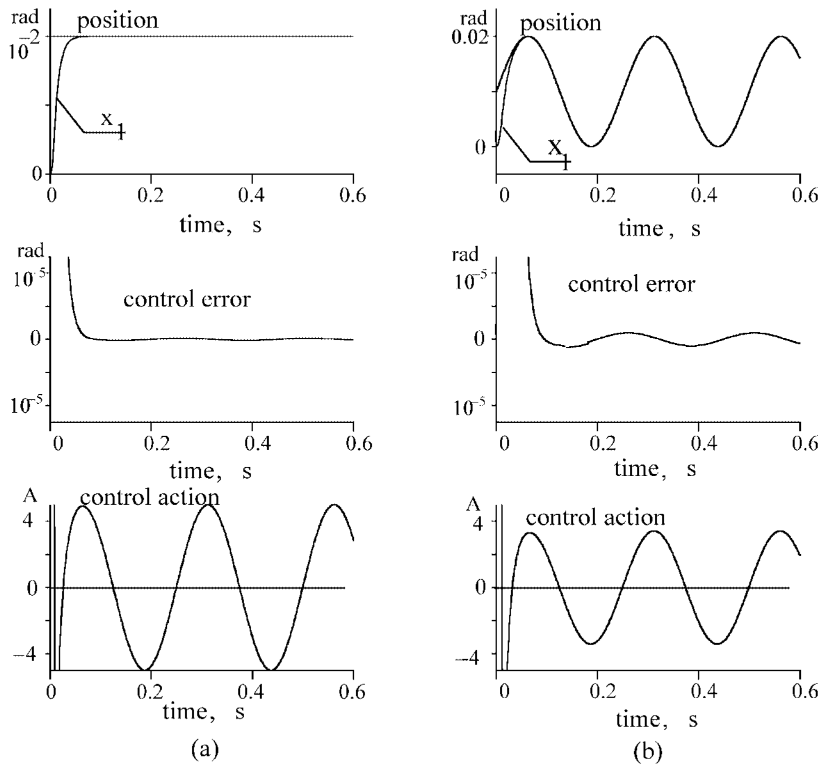

Fig. 4 Transients in the system (19) under approximated control (24)

nal to the sampling interval as it was predicted. The approximated values for the time derivatives of the switching function and the Lyapunov function calculated using backward approximation are, as expected much smoother and constant inside the intersampling interval. Presented behavior of the system confirms the predicted results.

4 THE EQUATIONS OF MOTION OF CLOSED LOOP SYSTEM

It has been shown in the previous analysis that the system motion can be stabilized in the ε-vicinity of the sliding mode manifold by applying the proposed control algorithm. The trajectories of the system motion are close to that of the system with ideal sliding mode, but equations of the closed loop system motion are not derived. In this section the equations of motion for the system with control input expressed as in (5) and its discrete-time realization (18) and (24) will be derived.

The equation of motion for system (1) under proposed control will be derived using so called regular form representation [17]

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{f}_1(\mathbf{x}_1,\mathbf{x}_2); \quad \frac{d\mathbf{x}_2}{dt} = \mathbf{f}_2(\mathbf{x}_1,\mathbf{x}_2) + \mathbf{B}_2\mathbf{u};$$

$$\begin{aligned} &\mathbf{x}_1 \in R^{n-m}, \quad \mathbf{x}_2 \in R^m, \\ &\mathbf{u} \in R^m, \quad rank(\mathbf{B}_2) = m. \end{aligned} \tag{31}$$

Sliding mode manifold can be transformed to the form

$$\begin{aligned} &\mathbf{S} = \{\mathbf{x}_1,\mathbf{x}_2 : \sigma_t(t) - (\sigma_1(\mathbf{x}_1) + \sigma_2(\mathbf{x}_2)) = \\ &= \sigma(\mathbf{x}_1,\mathbf{x}_2,t) = \mathbf{0}\} \\ &\frac{\partial\sigma_1(\mathbf{x}_1)}{\partial\mathbf{x}_1} = \mathbf{G}_1; \quad \frac{\partial\sigma_2(\mathbf{x}_2)}{\partial\mathbf{x}_2} = \mathbf{G}_2; \\ &rank\mathbf{G}_2 = m. \end{aligned} \tag{32}$$

In this case the control input (5) can be calculated as

$$\mathbf{u}(t^+) \equiv \mathbf{sat}\left\{\mathbf{u}(t^-) - (\mathbf{G}_2\mathbf{B}_2)^{-1}\left(\mathbf{D}\sigma(t^-) + \frac{d\sigma(t^-)}{dt}\right)\right\}. \tag{33}$$

In the discrete-time form without approximation control input can be expressed as

$$\mathbf{u}(kT) \equiv \mathbf{sat}\left\{\mathbf{u}(kT-T) - (\mathbf{G}_2\mathbf{B}_2)^{-1}\left(\mathbf{D}\sigma(kT) + \frac{d\sigma(kT)}{dt}\right)\right\}. \tag{34}$$

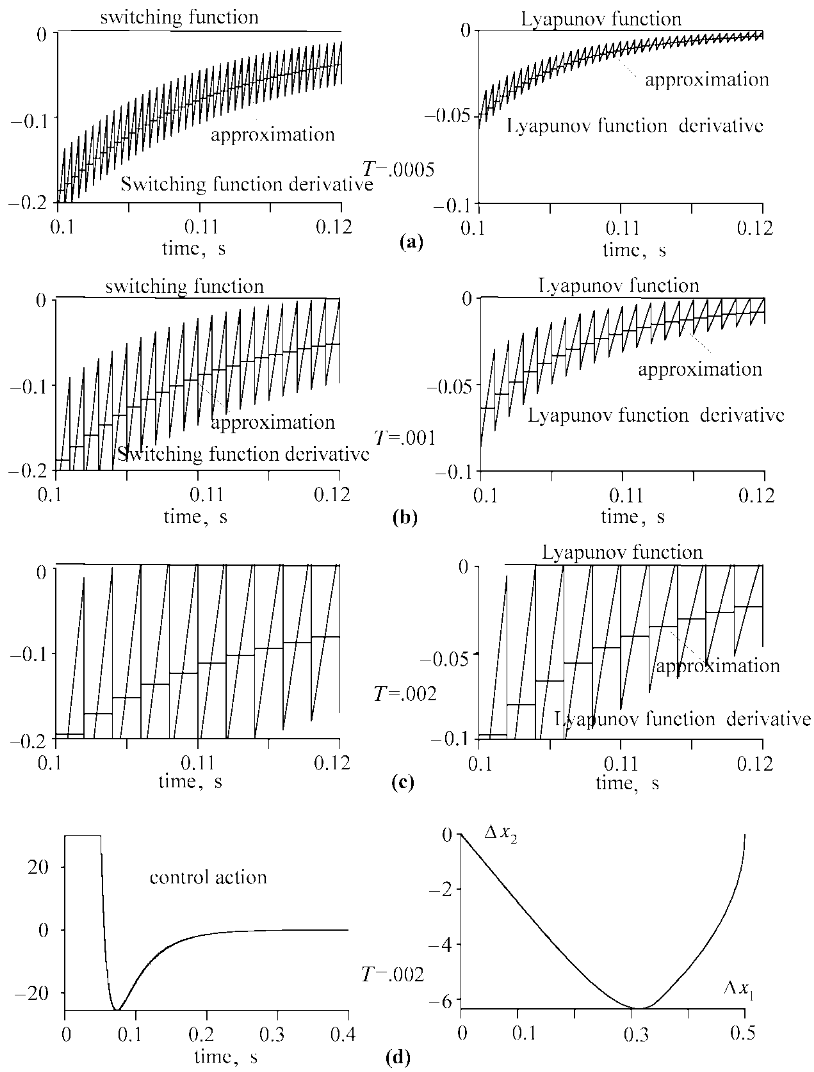

Fig. 5 The changes of the switching function its derivative and the Lyapunov function and its derivative for system (19) and control (30)

and approximated control input can be expressed as

$$\mathbf{u}(kT) \equiv \mathbf{sat}\{\mathbf{u}(kT-T) - \\ -(\mathbf{G}_2\mathbf{B}_2T)^{-1}((\mathbf{E}+T\mathbf{D})\sigma(kT)-\sigma(kT-T))\}. \quad (35)$$

The closed loop dynamics is described by three sets of the differential equations.

– the first stage is outside of the sliding mode manifold $\mathbf{x}_1,\mathbf{x}_2 \notin \mathbf{S}$ where control has limit value (max or min) and the motion is governed along the trajectories defined by $u=u_{\min}$ or $u=u_{\max}$;
– the second stage is outside of the sliding mode manifold $\mathbf{x}_1,\mathbf{x}_2 \notin \mathbf{S}$ where control input has values $u_{\min}<sat(\bullet)<u_{\max}$;
– the third stage is on the sliding mode manifold $\mathbf{x}_1,\mathbf{x}_2 \in \mathbf{S}$.

The first and the second stages are the reaching stages. The first stage is the same as for the system with discontinuous control – the trajectories are determined by the parameters of the system and maximum value of the control input.

The second stage is along the trajectories $u_{\min}<sat(\bullet)<u_{\max}$. By substituting (33) into (31) the equations of motion can be calculated as

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{f}_1(\mathbf{x}_1,\mathbf{x}_2), \\ \frac{d\sigma(\mathbf{x}_1,\mathbf{x}_2,t)}{dt} + \mathbf{D}\sigma(\mathbf{x}_1,\mathbf{x}_2,t) = \mathbf{0}. \quad (36)$$

The second equation describes the change of the distances from the sliding mode manifold. These distances decay with a rate determined by the elements of matrix **D**. During the second stage of motion the system dynamics is constrained by (36), e.g. the change of the distances from the manifold is controlled to satisfy desired dynamics, and after finite time system motion will enter ε-vicinity of the manifold S and will stay in this vicinity. From the second equation it is easy to calculate

$$\frac{d\sigma(\mathbf{x}_1,\mathbf{x}_2,t)}{dt}=\frac{d\sigma_t(t)}{dt}-\frac{\partial\sigma_1(\mathbf{x}_1)}{\partial\mathbf{x}_1}\frac{d\mathbf{x}_1}{dt}-\frac{\partial\sigma_2(\mathbf{x}_2)}{\partial\mathbf{x}_2}\frac{d\mathbf{x}_2}{dt}=$$
$$=\frac{d\sigma_t(t)}{dt}-\mathbf{G}_1\mathbf{f}_1-\mathbf{G}_2\frac{d\mathbf{x}_2}{dt}$$
$$\frac{d\mathbf{x}_2}{dt}=\mathbf{G}_2^{-1}\left(\mathbf{D}\sigma(\mathbf{x}_1,\mathbf{x}_2,t)+\frac{d\sigma_t(t)}{dt}-\mathbf{G}_1\mathbf{f}_1(\mathbf{x}_1,\mathbf{x}_2,t)\right) \tag{37}$$

and finally

$$\begin{aligned}\frac{d\mathbf{x}_1}{dt}&=\mathbf{f}_1(\mathbf{x}_1,\mathbf{x}_2),\\ \frac{d\mathbf{x}_2}{dt}&=\mathbf{G}_2^{-1}\left(\mathbf{D}\sigma+\frac{d\sigma_t(t)}{dt}-\mathbf{G}_1\mathbf{f}_1\right).\end{aligned} \tag{38}$$

The equations of motion (38) describe the n-th order system. During this stage the motion is independent of the control input and all parameters and nonlinearities contained in the second equation in (31).

For a system with discrete-time computation of control (34) the second equation in (36), just after control input is applied, becomes

$$\frac{d\sigma(kT^+)}{dt}+\mathbf{D}\sigma(kT^+)=\mathbf{0} \tag{39}$$

and consequently the value of the x_2 time derivative at the same instant of time becomes

$$\frac{d\mathbf{x}_2(kT^+)}{dt}=\mathbf{G}_2^{-1}\left(\mathbf{D}\sigma(kT^-)+\frac{d\sigma_t(kT^-)}{dt}-\mathbf{G}_1\mathbf{f}_1(kT^-)\right).$$

As it has been demonstrated earlier the intersampling changes of sliding mode function and its derivative are of $O(T)$ order. The same is true for the vector x_2. That means equation (39) holds between two sampling intervals with an accuracy of $O(T)$ order and is reset to the zero value at the beginning of every sampling interval. That enables us to accept (39) as the equation of motion for systems without computational delay.

For systems with computational delay the second equation in (36) becomes

$$\frac{d\sigma(kT+h)}{dt}+\mathbf{D}\sigma(kT+h)=\zeta(h)$$
$$\zeta(h)=\left[\frac{d\sigma_t(kT+h)}{\partial t}-\frac{d\sigma_t(kT)}{\partial t}\right]+$$
$$+\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}\begin{bmatrix}\Delta\mathbf{f}_1\\ \Delta\mathbf{f}_2\end{bmatrix}+\mathbf{D}\Delta\sigma;$$
$$\Delta\mathbf{f}_1=\mathbf{f}_1(kT+h)-\mathbf{f}_1(kT);$$
$$\Delta\mathbf{f}_2=\mathbf{f}_2(kT+h)-\mathbf{f}_2(kT);$$
$$\Delta\sigma=\sigma(kT+h)-\sigma(kT). \tag{40}$$

This equation corresponds to the expression (21). The intersampling changes of $\zeta(h)$ are of $O(T)$ order and if $\sigma(kT)\to 0$ then $\zeta(h)\to 0$. Equation (40) can be accepted as the equation of motion.

For systems with approximated algorithm (35) the second equation in (36) becomes

$$\begin{aligned}&\frac{d\sigma(kT+h)}{dt}+\mathbf{D}\sigma(kT+h)=\zeta(h)\\ &\zeta(h)=\left[\frac{d\sigma_t(kT+h)}{\partial t}-\frac{d\sigma_t(kT)}{\partial t}\right]+\\ &\quad+\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}\begin{bmatrix}\Delta\mathbf{f}_1\\ \Delta\mathbf{f}_2\end{bmatrix}+\mathbf{D}\Delta\sigma+o(kT)\\ &\Delta\mathbf{f}_1=\mathbf{f}_1(kT+h)-\mathbf{f}_1(kT);\\ &\Delta\mathbf{f}_2=\mathbf{f}_2(kT+h)-\mathbf{f}_2(kT);\\ &\Delta\sigma=\sigma(kT+h)-\sigma(kT)\end{aligned} \tag{41}$$

The same observation as for (40), regarding the intersampling change holds, and equation (41) can be accepted as the equation of motion.

It has been demonstrated that selected control assures that there exists $k>N$ and some value of the sampling interval T for which the stability of the solution $\sigma(kT)=0$ is guarantied. The motion in ε-vicinity of the manifold S, for ε small enough can be approximated as the motion on the manifold S. The description of the system motion on the manifold S will remain the same as for ideal sliding mode. During this stage the second equation in (38) can be replaced by the algebraic one $\sigma(\mathbf{x}_1,\mathbf{x}_2,t)=\mathbf{0}$ and the equations of motion becomes

$$\begin{aligned}&\frac{d\mathbf{x}_1}{dt}=\mathbf{f}_1(\mathbf{x}_1,\mathbf{x}_2),\\ &\sigma_t(t)-(\sigma_1(\mathbf{x}_1)+\sigma_2(\mathbf{x}_2))=\mathbf{0}.\end{aligned} \tag{43}$$

These equations are the same as for ideal sliding mode on manifold (32) and they posses all features of the systems with sliding mode. Proposed algorithm ensures the motion of the system in ε-vicinity of manifold (32) but without discontinuity of the control input. It is important to notice that only the

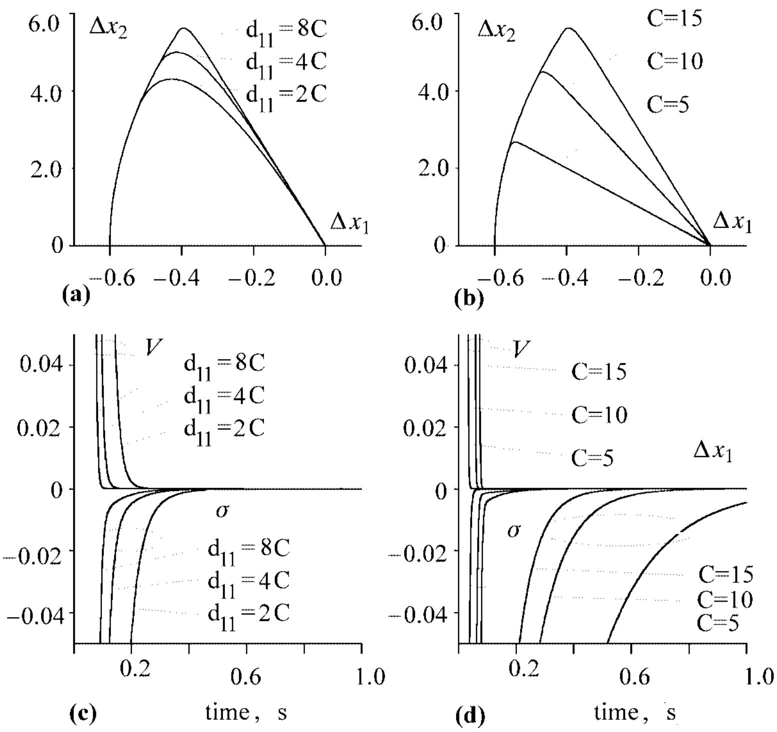

Fig. 6 The influence of the design parameters on the closed system behavior

information about state vector and the gain matrix B of the plant are needed to construct the control system.

To illustrate the validity of the derived equations of motion and the influence of the design parameters (the parameters of the sliding mode manifold (C) and the decay of the Lyapunov function (d_{11})) are presented in Figure 6. The behavior of the system (19) is depicted for step change in reference $x_1^{ref}=0.6$, rad, with sliding mode manifold

$$\mathbf{S}=\left\{x_1,x_2 : C(x_1^{ref}-x_1)+(x_2^{ref}-x_2)=\sigma(t)=0, C>0\right\}$$

and approximated algorithm (35). The phase plane diagrams, with control error on the horizontal axis and its derivative on the vertical axis, the time change in the Lyapunov function (v), the distance from sliding mode manifold (σ) and control error (Δx_1) are illustrated. In diagram (a) the change of coefficient d_{11} is selected to be $d_{11}=2C$, 4C and 8C respectively with $C=15$. For diagrams (b) the value $d_{11}=8C$ is kept while C is changed to be $C=5$, 10 and 15. It can be observed that the reaching stage changes with the change of d_{11} as it was predicted. The reaching stage is determined by the selection of the slope of the Lyapunov function and by its change the transients for both systems become very close. That can be seen better from the transients depicted in Figure 6 (c) and Figure 6 (d) where the change of the Lyapunov function and the control error are depicted for the same conditions as those in Figure 6 (a) and Figure 6 (b), respectively. The influence of the selection of the Lyapunov function decay is as it was theoretically predicted.

The dependence of the motion on the sliding manifold parameters (C in this case) and of the rate of change of the time derivative of the Lyapunov function (d_{11} in this case) is clearly demonstrated to be in agreement with the theoretical results.

5 THE ISSUE OF ROBUSTNESS

The issue of robustness is very important for the analysis of the closed loop system behavior. In the following analysis the equations of motion will be derived for the system (1) with uncertainties in the vector function **f** and in both the function f and the gain matrix **B**.

Assume that the state vector measurement is correct, that nominal value of function f is known along with the bounds of the uncertain continuous vector function $\Delta\mathbf{f}$, and that gain matrix **B** is known. Then the actual system can be represented by

$$\begin{aligned}\frac{d\mathbf{x}}{dt}&=\bar{\mathbf{f}}(\mathbf{x},t)+\mathbf{B}(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\\ \bar{\mathbf{f}}(\mathbf{x},t)&=\mathbf{f}(\mathbf{x},t)+\Delta\mathbf{f}(\mathbf{x},t).\end{aligned} \tag{44}$$

Assume the control vector (18). The time derivative of sliding mode function, for a system without computational delay, just after renewed control input has been applied can be expressed as

$$\frac{d\sigma(kT^+)}{dt} = \mathbf{G}(\bar{\mathbf{f}}(kT^+) + \mathbf{Bu}(kT^+)) + \frac{\partial\sigma(kT^+)}{\partial t}$$
$$\frac{d\sigma(kT^+)}{dt} = \mathbf{G}(\bar{\mathbf{f}}(kT^+) + \frac{\partial\sigma}{\partial t} + \mathbf{Bu}(kT^-)) - -\mathbf{GB}(\mathbf{GB})^{-1}\left(\mathbf{D}\sigma(kT^-) + \frac{d\sigma(kT^-)}{dt}\right) \tag{45}$$
$$\frac{d\sigma(kT^+)}{dt} = \frac{d\sigma(kT^-)}{dt} - \left(\mathbf{D}\sigma(kT^+) + \frac{d\sigma(kT^-)}{dt}\right)$$

and finally $\frac{d\sigma(kT^+)}{dt} + \mathbf{D}\sigma(kT^+) = \mathbf{0}$. This result is the same as in the case of the system with full information about function $\bar{\mathbf{f}}$. At the beginning of every sampling interval the $\frac{d\sigma(kT^+)}{dt} + \mathbf{D}\sigma(kT^+)$ assumes zero value with an intersampling change of the $O(T)$ order. If computational delay exists then (45) has a form

$$\begin{aligned}
&\frac{d\sigma(kT+h)}{dt} + \mathbf{D}\sigma(kT+h) = \zeta(h)\\
&\zeta(h) = \left[\frac{\partial\sigma(kT+h)}{\partial t} - \frac{\partial\sigma(kT)}{\partial t}\right] + \mathbf{G}\Delta\bar{\mathbf{f}} + \mathbf{D}\Delta\sigma\\
&\Delta\bar{\mathbf{f}} = \bar{\mathbf{f}}(kT+h) - \bar{\mathbf{f}}(kT)\\
&\Delta\sigma = \sigma(kT+h) - \sigma(kT)
\end{aligned} \tag{46}$$

with an intersampling change of ζ being $O(T)$ order.

For systems with approximated algorithm and with computational delay the following relation holds

$$\begin{aligned}
&\frac{d\sigma(kT+h)}{dt} + \mathbf{D}\sigma(kT+h) = \zeta(h)\\
&\zeta(h) = \left[\frac{\partial\sigma(kT+h)}{\partial t} - \frac{\partial\sigma(kT)}{\partial t}\right] +\\
&\quad + \mathbf{G}\Delta\bar{\mathbf{f}} + \mathbf{D}\Delta\sigma + o(kT)\\
&\Delta\bar{\mathbf{f}} = \bar{\mathbf{f}}(kT+h) - \bar{\mathbf{f}}(kT)\\
&\Delta\sigma = \sigma(kT+h) - \sigma(kT).
\end{aligned} \tag{47}$$

For a system without computational delay (47) reduces to

$$\frac{d\sigma(kT^+)}{dt} + \mathbf{D}\sigma(kT^+) = o(kT). \tag{48}$$

By using the same reasoning as for the systems without uncertainties one can prove stability and consequently the fact that $\sigma(kT) \to 0$ and last stage of the motion is taking place on the sliding mode manifold S under equivalent control governed by the following equation

$$\begin{aligned}
&\frac{d\mathbf{x}}{dt} = (\mathbf{E} - \mathbf{B}(\mathbf{GB})^{-1}\mathbf{G})\mathbf{f} - \mathbf{B}(\mathbf{GB})^{-1}\frac{\partial\sigma}{\partial t} +\\
&\quad +(\mathbf{E} - \mathbf{B}(\mathbf{GB})^{-1}\mathbf{G})\Delta\mathbf{f}\\
&\sigma(\mathbf{x},t) = \mathbf{0}.
\end{aligned} \tag{49}$$

These equations are the same as the ideal sliding mode equations. If uncertain vector satisfy matching conditions $\Delta\mathbf{f} = \mathbf{B}\lambda$ the equations of motion will not depend on $\Delta\mathbf{f}$.

Assume now the system (44) but with additional uncertainty in the plant gain matrix. Suppose that only nominal value B of the gain matrix $\bar{\mathbf{B}} = \mathbf{B} - \Delta\mathbf{B}$ is known, $rank\bar{\mathbf{B}} = m$ and that control input is calculated without computational delay. In this case one can write

$$\begin{aligned}
&\frac{d\sigma(kT^+)}{dt} + \mathbf{D}\sigma(kT^+) =\\
&= \mathbf{G}\Delta\mathbf{B}(\mathbf{GB})^{-1}\left(\mathbf{D}\sigma(kT^-) + \frac{d\sigma(kT^-)}{dt}\right).
\end{aligned} \tag{50}$$

To ensure the convergence of $\frac{d\sigma(kT)}{dt} + \mathbf{D}\sigma(kT)$ to zero it is enough to satisfy inequality

$$\left\|\sigma^T(kT^+)\frac{d\sigma(kT^+)}{dt} + \sigma^T(kT^+)\mathbf{D}\sigma(kT^+)\right\| \langle$$
$$\langle \left\|\sigma^T(kT^-)\mathbf{D}\sigma(kT^-) + \sigma^T(kT^-)\frac{d\sigma(kT^-)}{dt}\right\|.$$

With a fact that inter-sampling change of the $\frac{d\sigma(kT)}{dt} + \mathbf{D}\sigma(kT)$ is of $O(T)$ order, matrix ΔB should satisfy $\|\mathbf{G}\Delta\mathbf{B}\| \le \|\mathbf{GB}\|$ and consequently, for some $k>N$ it becomes $\frac{d\sigma(kT^+)}{dt} + \mathbf{D}\sigma(kT^+) = 0$, and according to the previous analysis $\sigma(kT) \to 0$.

Further we shall analyze the behavior of the system with approximated. First for systems without computational delay in which the measurement is taking place at $t = kT^-$ and control is delivered at $t = kT^+$, one can find

$$\begin{aligned}
&\frac{d\sigma(kT^+)}{dt} = \frac{\partial\sigma}{\partial t} + \mathbf{G}\bar{\mathbf{f}}(kT^+) + \mathbf{G}\bar{\mathbf{B}}\mathbf{u}(kT^+) = \frac{\partial\sigma}{\partial t} +\\
&\quad + \mathbf{G}\bar{\mathbf{f}}(kT^+) + \mathbf{G}\bar{\mathbf{B}}\big[\mathbf{u}(kT-T) -\\
&\quad - (\mathbf{GB}T)^{-1}\left[(\mathbf{E} + \mathbf{D}T)\sigma(kT) - \sigma(kT-T)\right]\big]
\end{aligned}$$

$$\frac{\mathrm{d}\sigma(kT^+)}{\mathrm{d}t} = \left[\frac{\partial\sigma(kT^+)}{\partial t} + \mathbf{G}\bar{\mathbf{f}}(kT^-) + \mathbf{G}\bar{\mathbf{B}}\mathbf{u}(kT-T)\right] -$$
$$-(\mathbf{G}\bar{\mathbf{B}})(\mathbf{GB})^{-1}\left[\mathbf{D}\sigma(kT) + \frac{\sigma(kT)-\sigma(kT-T)}{T}\right]$$
$$\frac{\mathrm{d}\sigma(kT)}{\mathrm{d}t} = \frac{\sigma(kT)}{\mathrm{d}t} - \mathbf{D}\sigma(kT) - \frac{\sigma(kT)-\sigma(kT-T)}{T} +$$
$$+(\mathbf{G}\Delta\bar{\mathbf{B}})(\mathbf{GB})^{-1}\left[\mathbf{D}\sigma(kT) + \frac{\sigma(kT)-\sigma(kT-T)}{T}\right].$$

After some algebra one can finally obtain

$$\frac{\mathrm{d}\sigma(kT^+)}{\mathrm{d}t} = \frac{\sigma(kT)-\sigma(kT-T)}{T} + o(kT) -$$
$$-\mathbf{D}\sigma(kT) - \frac{\sigma(kT)-\sigma(kT-T)}{T} +$$
$$+(\mathbf{G}\Delta\mathbf{B})(\mathbf{GB})^{-1}\left[\mathbf{D}\sigma(kT) + \frac{\sigma(kT)-\sigma(kT-T)}{T}\right]$$
$$\frac{\mathrm{d}\sigma(kT^+)}{\mathrm{d}t} + \mathbf{D}\sigma(kT^+) =$$
$$= \mathbf{G}\Delta\mathbf{B}(\mathbf{GB})^{-1}\left[\mathbf{D}\sigma(kT^-) + \frac{\sigma(kT^-)-\sigma(kT-T)}{T}\right] +$$
$$+o(kT) = \mathbf{G}\Delta\mathbf{B}(\mathbf{GB})^{-1}\left[\mathbf{D}\sigma(kT^-) + \frac{\mathrm{d}\sigma(kT^-)}{\mathrm{d}t}\right] +$$
$$+(\mathbf{E} - \mathbf{G}\Delta\mathbf{B}(\mathbf{GB})^{-1})o(kT). \tag{51}$$

By restricting $\Delta\mathbf{B}$ to satisfy inequality $\|\mathbf{G}\Delta\mathbf{B}\| \le \|\mathbf{GB}\|$, and from the fact that intersampling change of a continuous function is of $O(T)$ order, the sampling period should be selected so that

$$\left\|\sigma^T(kT^+)\frac{\mathrm{d}\sigma(kT^+)}{\mathrm{d}t} + \sigma^T(kT^+)\mathbf{D}\sigma(kT^+)\right\| \langle$$
$$\langle \left\|\sigma^T(kT^-)\mathbf{D}\sigma(kT^-) + \sigma^T(kT^-)\frac{\mathrm{d}\sigma(kT^-)}{\mathrm{d}t}\right\| \tag{52}$$

which ensures that for some $k > N$ the motion of the system will be described by

$$\frac{\mathrm{d}\sigma(kT^+)}{\mathrm{d}t} + \mathbf{D}\sigma(kT^+) = 0. \tag{53}$$

This equation is the same as for the system without uncertainties. The motion on the manifold S is described by the equations of a system with an ideal sliding mode on manifold (2).

To illustrate the sensitivity of the system with respect to the combined uncertainties (dynamical and the change of the plant gain) the load and gain in system (19) has been changed according to the following expressions

the load:

$$i_L = \begin{cases} 5 - 15\cdot x_2 & \text{if} \quad 0 < t < .2 \\ 5 + 5\cos(12.56t) & \text{if} \quad t \ge .2 \end{cases} \tag{54}$$

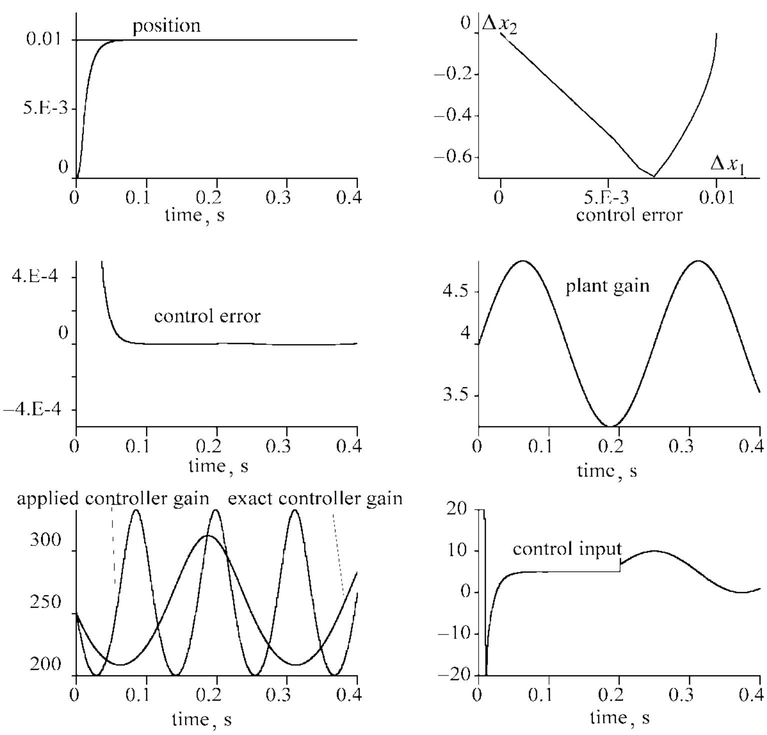

Fig. 7 Transients for the system with variable dynamics, time varying plant gain and time varying controller gain

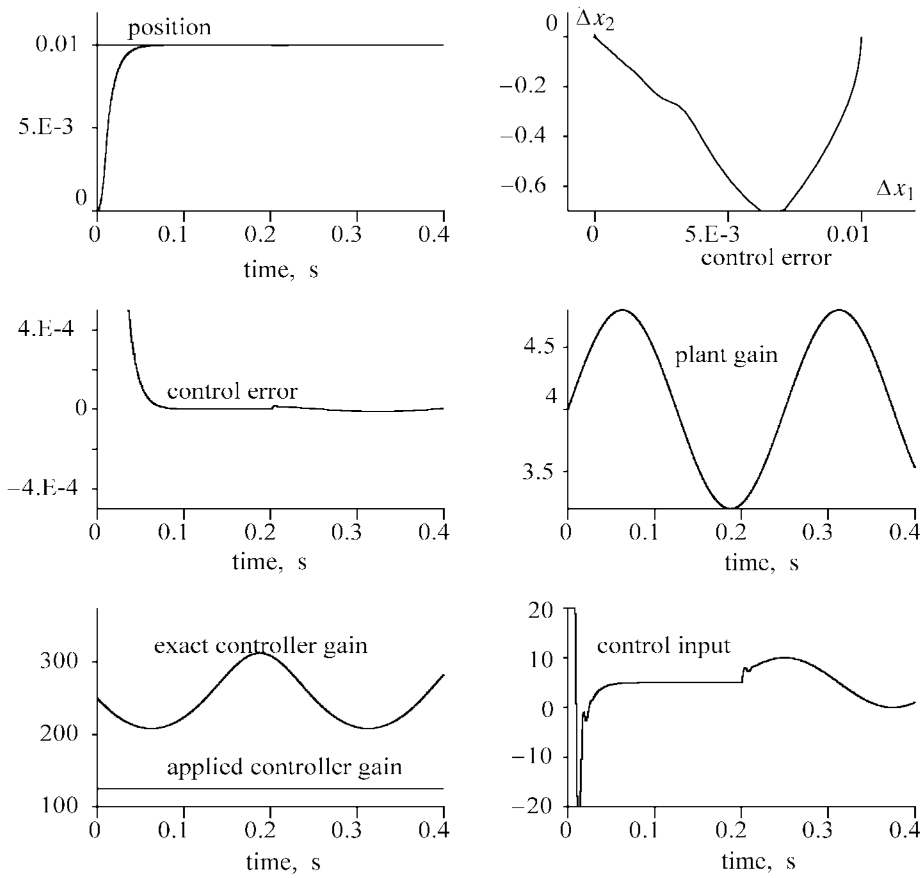

Fig. 8 Transients for the system (19) with variable dynamics, time varying plant gain and constant controller gain

the plant gain

$$\lambda = 4 + 0.8 \cdot \sin(25.16\,t). \tag{55}$$

In the experiments the approximated value of the controller gain

$$\frac{1}{\lambda T} \approx \frac{1}{(4+1.0 \cdot \sin(55.6t)) \cdot T} \tag{56}$$

has been used, instead the correct value of the controller gain $(1/\lambda T)$. Both the value and the frequency are changed. The simulation results are depicted in Figure 7 for a step change in reference 0.01, rad, at $t = 0$. Sliding line is determined by $C = 100$ and the Lyapunov function decay coefficient is $d_{11} = 800$. The controller is defined as in (30) with gain (56). The time change of the output, the phase plane transient, the control error, control input, the plant gain and the controller gain are depicted. By comparison with diagrams (a) in Figure 4 it can be concluded that all properties of the system motion are preserved. No visible influence in the change of the system dynamics and the system gain on the closed loop behavior can be observed. The transients confirm theoretical prediction and rejection of the uncertainties is confirmed.

In Figure 8 the step change in the position reference for plant (19) with load (54) and gain (55) is depicted under controller (30) with constant gain equal to $(1/8T)$. Sliding line is determined by $C = 100$ and the Lyapunov function decay coefficient is $d_{11} = 800$. The time transients (the position change the control error and the control input) do not show big changes in comparison with results shown in Figure 7 or Figure 2 (c). The biggest change is visible in the phase plane transient with small oscillation around the sliding line. This show the capability of the proposed controller to reject the matched dynamical and parameter uncertainties, and confirm theoretically predicted behavior.

In the previous analysis the exact information about state vector has been assumed. The measurement of the state vector is sometimes to complicated and state observers are used to restore not measurable part of the state vector. The behavior of the proposed algorithm in the systems with state observers will be analyzed in the following section.

6 SYSTEMS WITH STATE OBSERVERS

Assume that the sliding mode manifold is constructed using output $\bar{\mathbf{x}}$ of a stable state observer. Then (2) can be expressed as

$$\mathbf{S} = \left\{ \mathbf{x} : \sigma(\bar{\mathbf{x}}, t) = \mathbf{0}, \bar{\mathbf{x}} = \mathbf{x} + \varepsilon, \mathbf{G} = \frac{\partial \sigma}{\partial \mathbf{x}} \right\}, \tag{57}$$

where ε is the error in the state vector estimation. Assume that observation error is governed by the stable dynamics $\frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} = \mathbf{f}_\varepsilon(\boldsymbol{\varepsilon}, t)$ such that $\boldsymbol{\varepsilon}$ and $\mathbf{f}_\varepsilon(\boldsymbol{\varepsilon}, t)$ tend to zero. Further we will assume that $\sigma(\mathbf{x}, t)$ is linear transformation with respect to $\mathbf{x}$. Then time derivative $\frac{\sigma(\mathbf{x}, t)}{\mathrm{d}t}$ can be calculated as

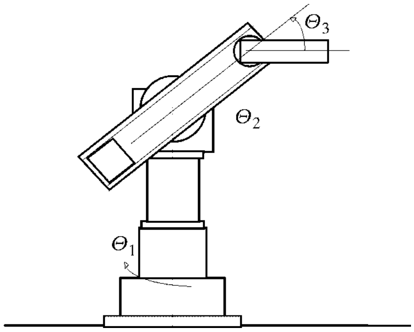

Fig. 9.a Experimental robotic manipulator

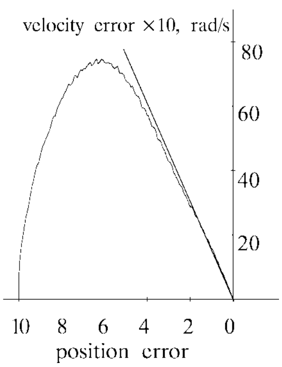

Fig. 9.b Phase plane transient for a step change in reference

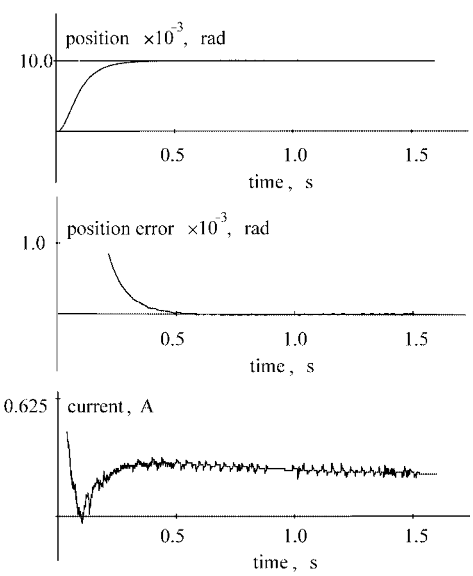

Fig. 9.c Transients for a step change in reference position for the second link

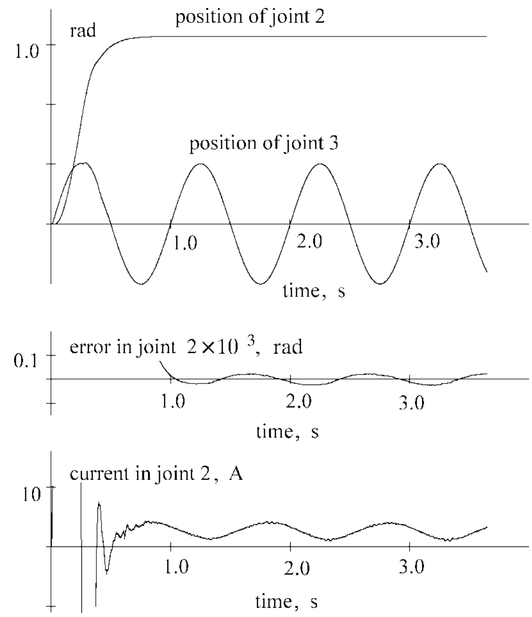

Fig. 9.d Transients for a step change in reference for second link and sinusoidal changes of the third link position

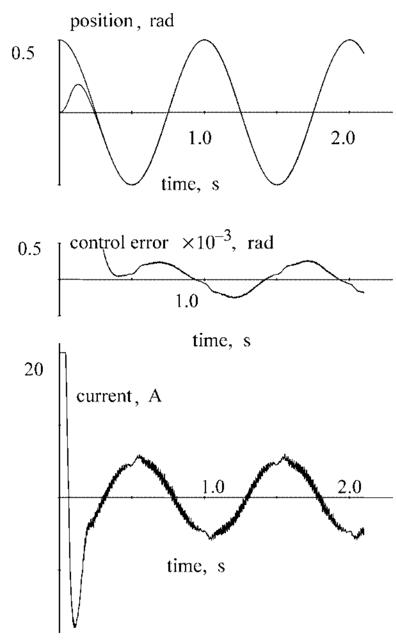

Fig. 9.e The tracking of the cosine reference for the second link

$$\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^+)}{\mathrm{d}t}=\mathbf{G}(\mathbf{f}(kT^+)+\mathbf{Bu}(kT^+))+\frac{\partial\sigma}{\partial t};$$

$$\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^+)}{\mathrm{d}t}=\mathbf{Gf}(kT^+)+\frac{\partial\sigma}{\partial t}+\mathbf{GB}(\mathbf{u}(kT^-)-$$
$$-(\mathbf{GB})^{-1}\left(\mathbf{D}\sigma(\bar{\mathbf{x}},kT^-)+\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^-)}{\mathrm{d}t}\right);$$

$$\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^+)}{\mathrm{d}t}=\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^-)}{\mathrm{d}t}-\left(\mathbf{D}\sigma(\bar{\mathbf{x}},kT^+)+\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^-)}{\mathrm{d}t}\right)-$$
$$-\left(\mathbf{D}\sigma(\varepsilon,kT^+)+\frac{\mathrm{d}\sigma(\varepsilon,kT^-)}{\mathrm{d}t}\right);$$

$$\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^+)}{\mathrm{d}t}+\mathbf{D}\sigma(\bar{\mathbf{x}},kT^+)=$$
$$=-\left(\mathbf{D}\sigma(\varepsilon,kT^+)+\frac{\mathrm{d}\sigma(\varepsilon,kT^-)}{\mathrm{d}t}\right)=$$
$$=\frac{\mathrm{d}\sigma(\bar{\mathbf{x}},kT^+)}{\mathrm{d}t}+\mathbf{D}\sigma(\bar{\mathbf{x}},kT^+)=$$
$$=-\left(\mathbf{D}\sigma(\varepsilon,kT^+)+\mathbf{G}\frac{\mathrm{d}\varepsilon(kT)}{\mathrm{d}t}\right)\underset{\varepsilon\to 0}{\to}0.$$

It follows that reaching stage is influenced by the observer dynamics, adding some restrictions on the rate of change of the distances from the sliding mode manifold. After the transient in observer recedes, the dynamics of the closed loop system is the same as for the system without state observer.

In Figures 9 a,b,c,d and e the experimental results for direct drive robot control are presented. The joint position is measured and velocities are estimated using stable reduced order Luenberger state observer. The transient for a step change and the sinusoidal change in the second link reference are depicted along with experimental robot. For all experiments the controller has the structure defined by (30) with controller gain ($1/4T$), the slope of the sliding line $\mathrm{C}=20$, the Lyapunov function decay $\mathrm{d}_{11}=100$, and the sampling interval $T=0.001$, sec. The phase plane transient for a step change in the second link reference position is depicted, the corresponding time transients in position, control error and the torque component of the motor current are depicted in at the time diagram. All transients are showing predicted behavior. The controller, with estimated velocity, is capable of stabilizing the motion of the system on the selected manifold. Very good external disturbance rejection capabilities can be observed at the diagrams showing step change in the second link reference while the third link reference is selected to provide oscillation in third link position for a half of radian. The tracking capabilities are illustrated at the diagram with cosine change of the second link reference. High accuracy of the system can be observed in all experiments. The depicted current is proportional to the joint torque. As it can be observed chattering is indeed eliminated in the system.

7 CONCLUSIONS

A new method of discrete-time sliding mode control system is presented with the aim to avoid calculation of the equivalent control, to alleviate the effect of uncertainties and to remove the chattering in the sampled-data systems with sliding mode. A design procedure is developed witch allows the controller design without transformation of the control plant description to the sampling-data form. The upper bound for the sampling interval is determined. The stability of the system with and without computational delay is proven, the robustness of the system with respect to the plant parameters change and the behavior of the system with state observers is discussed in details.

Unlike other approaches to discrete-time sliding modes proposed design method does not require the calculation of the discrete-time equivalent control. Like in continues time sliding mode design, proposed approach is based on the information about the value of sliding mode function. In addition to this the structure of the plant gain matrix is required to properly select the controller gain. It has been demonstrated that system is robust against the changes of the value of the plant gain.

Two design parameters are important, the structure of the switching function (represented by matrix G) and the rate of change of the candidate Lyapunov function (represented by matrix D). The influence of these parameters is discussed and verified by simulation. From the robustness analysis follows that the time change of the design parameters is acceptable. The proposed algorithm is very suitable for the application of some sort of self-tuning procedures (Fuzzy of NN) to these two parameters. More work in this direction has to be done in order to clarify all mathematical stability conditions.

The experimental and simulation results are presented to clarify the design procedure and the features of the proposed algorithm.

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Klizni režimi u diskretnim sustavima upravljanja. Primjena algoritama kliznih režima rada, koji spadaju u grupu algoritama s promjenljivom strukturom upravljanja, može u diskretnim sustavima upravljanja rezultirati neželjenim oscilacijama regulirane varijable. U svrhu sprječavanja ovih oscilacija u radu je predložen jedan novi pristup u sustavu upravljanja s kliznim režimom rada. Predloženim postupkom eliminira se računanje ekvivalentnog upravljanja, smanjuje utjecaj neodređenosti sustava i značajno smanjuju oscilacije izlazne regulirane varijable. Postupak je jednako primjenljiv na linearne i na nelinearne sustave. On omogućuje sintezu sustava bez transformacije u diskretnu formu (z-područje). Gornja granica vrijednosti vremena uzorkovanja određena je iz zahtjeva za ograničenjem promjene funkcije klizanja unutar vremena intervala uzorkovanja. Analiziran je sustav s estimatorom stanja. Kao pokazatelj kvalitete predloženog algoritma prikazani su simulacijski i eksperimentalni rezultati ispitivanja.

Ključne riječi: vremenski diskretni sustav, linearni sustav, sinteza po Ljapunovu, klizni režim, sustavi s promjenljivom strukturom

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