ELABORATION OF A MATHEMATICAL MODEL FOR DESIGNING ATTRIBUTE ACCEPTANCE PLANS

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In this article, attribute acceptance plans are explained and elaborated, together with their design by using standards and nomograms. The elaboration of a mathematical model for designing specific acceptance plans is also presented as well as the implementation of these models in a software application. Their possible application in specific cases is shown on several examples. The results obtained by using a software application are compared with the standards and are presented with adequate curves.

Keywords: AQL, attribute acceptance plans, HRN ISO 2859-1:1989, Larson nomogram, operating characteristic curve of the acceptance plan, sampling, statistical quality control

1 Introduction

Quality is a notion that has always been present in nature and can be simply described as the survival of the fittest. In today’s sense, quality control has existed since the day men started making things. It evolved with time in order to satisfy larger production of goods, to cut expenses, and to satisfy market demand for higher quality. One of the definitions of quality is: “Quality is the level at which a set of distinctive characteristics satisfies requirements.” [1]

In order to satisfy the above mentioned definition, it is necessary to determine the mode and control parameters.

Two basic types of control offer themselves:

- 100 % control
- Statistical control.

100 % control is a type of inspection applied to certain characteristics of all products or materials in a group to determine whether a product meets imposed requirements. The main disadvantage of the 100 % method is that each element needs to be inspected separately, which results in high costs of this inspection type. Moreover, it is impossible to apply control during which the object of control gets destroyed.

Statistical control presents a set of methods and procedures for gathering, processing, analysis, and interpretation of data to ensure the quality of a product, process and service. It can be divided into control during a process (control charts) and control after the finished process (acceptance plans). Acceptance plans can be further divided depending on the controlled characteristics into variable acceptance plans and attribute acceptance plans. The advantage of statistical control presents significantly smaller control volume in the form of element numbers which need to be controlled. Tests are simpler and a large number of objects can be controlled in a relatively short time. Acceptance plans present an only possibility in case the control is destroying an object.

2 Attribute acceptance plans

An acceptance plan (sampling plan) presents a set of regulations which determine the size of a sample (or samples, when it is a matter of double or multiple systems), which needs to select acceptance or rejection numbers from the basic lot.[2, 8]

Within the system of sample selection, one or more samples are selected from the delivery by using any procedure which ensures randomness or adequate representative quality of the whole delivery. If the subject are smaller parts or parts packaged in a loose condition (e.g. screws, nuts, and similar), it is enough to randomly select from the container as many pieces as necessary according to the received size of the sample. In case the subject are parts which are separately packaged or placed in separate containers, after labeling each piece in any numerical order, random numbers are used for choosing representative pieces [7].

Acceptance plans can be further divided into single (Fig. 1), double (Fig. 2), and multiple (Fig. 3). The decision on the choice of an acceptance plan is usually based on economic and administrative factors. Double and multiple acceptance plans enable the use of smaller sample size than the size used with single acceptance plans. However, administrative difficulties and the price per unit in a sample are usually smaller with single acceptance plans. With multiple acceptance plans, the procedure is repeated until the right condition for acceptance or rejection of the shipment is achieved. The main advantage here is that the lot is sorted out during the inspection process.

Acceptance number (Ac or c) is the maximum number of defective units which can be tolerated in a sample to
consider the basic lot satisfying.

2.1 Operating characteristic curve of an acceptance plan

Operating characteristic curve represents the probability of accepting the sample \( Pa \) depending on the fraction of defects in sample \( p \) (Fig. 4).

When it comes to 100 % control, operating characteristic curve represents an ideal case (Fig. 4, left). This means that each lot that has a lower quality level (the quality level represents the percentage of defectiveness, what means lower is better from the acceptable level \( < \) ) will be accepted with the probability of 100 %, while every shipment with a quality level higher than the acceptable \( > \) will be rejected. The borderline case \( = \) requires an agreed upon procedure.

When it comes to sampling (Fig. 4, right), the situation changes since there is no more 100 % certainty for acceptance or rejection of the shipment, depending on the fraction of defects in the sample. Operating characteristic curve in that case presents the probability that a lot of certain quality level is accepted or rejected. Depending on the acceptance plan parameters there is a certain probability for a good lot to be rejected or for a bad lot to be accepted. These values are defined by using \( \alpha \) and \( \beta \) risks for the predetermined percentage of defective parts [9, 10].
Standard HRN ISO 2859-1:1989 specifies sampling plans and procedures for the inspection of discrete elements according to the attributes. The sampling plans in this standard are indexed according to the values of acceptable quality level AQL. [3]

The purpose of this standard is to affect the supplier by putting economic and psychological pressure on him with not accepting the delivery in order to maintain the process average at least on the level of specified quality of AQL, at the same time giving the consumer upper border for the risk of occasionally accepting a defective lot.

Application:
- Final products
- Components and raw materials
- Procedures
- Materials in process
- Warehouse stocks
- Maintenance procedures
- Data or records
- Administrative procedures.

Fig. 7 shows the process of acceptance plan design by using the standard.

3 Statistical bases
Statističke osnove

Standard HRN ISO 2859-1:1989 is focused on the protection of a producer and the acceptance of shipments whose quality level is the same as AQL, while the buyer's need to identify a bad shipment is put in a position of secondary importance (alpha risk is always higher than beta risk). In order to develop a mathematical model which enables arbitrary choice of sampling parameters, it is necessary to study data distribution.

For the elaboration of attribute acceptance plans discontinuous distributions are used (Hypergeometric, Binomial and Poisson), while for the making of variable acceptance plans continuous distributions are used (Normal, Student’s-1, Fisher) [11].
3.1 Hypergeometric distribution

The hypergeometric distribution presents the probability of appearance of certain elements with attribute A in a random sample selected from the basic lot size N.

The distribution function is given in equation (3) and it presents the probability of exactly d elements with attribute A in a representative sample of size n which is randomly selected from lot N.

\[ H(d, N, M, n) = \binom{M}{d} \binom{N-M}{n-d} \binom{N}{n} \]  

Cumulative distribution function is given in equation (4) and it presents the probability of selecting or less elements with attribute A in a representative sample of size n which is randomly selected from lot N.

\[ P_c(d \leq c) = \sum_{i=0}^{c} \binom{M}{i} \binom{N-M}{n-i} \binom{N}{n} \]  

Hypergeometric distribution is applied for isolated shipments when the basic lot is small, and by selecting samples proportions of good and bad parts in the basic lot are changed.

3.2 Binomial distribution

For easier understanding of binomial distribution it is necessary to explain the binomial experiment which satisfies these conditions:

- The experiment consists of n trials
- Each trial can result in only two possible outcomes which can be called success or failure
- The probability of success p is the same in each trial (the probability of failure q=1−p)

- The trials are independent, which means the outcome of one trial does not affect the outcome of the other.

In case all the abovementioned points are satisfied, the probability of success is given in equation (5) which presents the distribution function of binomial distribution.

\[ P_c(d \leq c) = \sum_{i=0}^{c} \binom{n}{i} p^i q^{n-i} \]  

Cumulative distribution function of binomial distribution is given in equation (6) and it presents the probability c or less success realized in n trials, when in the sample there is a fraction of defective parts given in p.

\[ P_c(d \leq c) = \sum_{i=0}^{c} \binom{N}{i} p^i q^{N-i} \]  

In case the basic lot N is big in relation to the sample n, so that selecting the sample from the basic lot does not significantly change the rest of the basic lot, independently on how many defects there are in the basic lot, the defect distribution can be considered binomial with parameters n and p [7, 8].

3.2.1 Larson nomogram

Nomograms are considered to be an out-of-date technology and are mostly replaced with standards and software, but there are cases where they present a fast and elegant solution. Fig. 9 shows the Larson nomogram which presents a graphic illustration of cumulative distribution function of binomial distribution.

On the basis of the following example, the way of designing an acceptance plan by using the Larson nomogram is explained:

It is necessary to create such a sampling plan which accepts 95 % of samples containing 2 % of defective lots and accepts only 5 % of samples containing 9 % of defective lots.

1. Determine the allowed fraction of defective lots p_a to consider the shipment satisfying: p_a = 2 % (0,02)
2. Determine the probability needed for the acceptance of a good shipment: P_a = 1 – p_a = 95 % (0,95)
3. Determine the size of fraction p_b of defective lots to consider the shipment unsatisfying: p_b = 9 % (0,09)
4. Determine the probability for accepting a bad shipment with P_b which can be tolerated: P_b = 5 % (0,05)
5. Connect the points p_a and P_a = 1 – p_a with a straight line
6. Connect the points p_b and P_b = β with a straight line
7. Read the values n and c in the nomogram.

It can be seen in Fig. 10 that the solution of this problem presents acceptance plan given with parameters n = 100 and c = 4.
The cumulative distribution function of the Poisson distribution is given in equation (8).

\[ P_r(d \leq c; \lambda) = \sum_{i=0}^{c} \frac{\lambda^i}{i!} e^{-\lambda}. \] (8)

The Poisson distribution is used for approximating the binomial with parameters \( n \) and \( p \) when \( n \) is big and \( p \) is small, e.g. \( n \geq 20 \) and \( p \leq 5 \) and excellent overlap for \( n \geq 100 \) and \( np \leq 10 \).

4 The elaboration of mathematical model

For an easier understanding of mathematical formulation for the sampling plans design, it is necessary to remember operating characteristic of an acceptance plan. The graph of operating characteristic curve presents the probability of accepting a delivery with a certain percentage of defective parts for the mentioned sampling plan.

Given that the size of delivery \( N \) is big in relation to the sample size \( n \), and that selecting a sample from the delivery does not significantly change the average structure of the rest of the delivery (independently on how many defective lots there are in the sample), the distribution of defects \( d \) in a random sample \( n \) is binomial with parameters \( n \) and \( p \), where \( p \) presents the fraction of defects per delivery.

The probability of observing exactly \( d \) effects according to the binomial formula is given in equation (5).

\[ p(d) = \binom{n}{d} p^d (1-p)^{n-d}, \] (5)

The probability of acceptance is the probability at which the number of defects found in a sample is smaller or equal to the acceptance number and for the binomial distribution is given in equation (6).

\[ P_A = P(d \leq c; n, p), \] (6)

In order to design a sampling plan with associated operating characteristic curve two points are needed. The first point presents the probability of acceptance \( P_A = 1 - \alpha \) for deliveries with the fraction of defects \( p = AQL \). The other point can be called \( P_R \), where the fraction of defects is \( p = RQL \). [5]

Given that the binomial distribution is adequate, parameters \( n \) and \( c \) can be obtained by solving nonlinear system of equations obtained in (6):

\[ 1 - \alpha = B(d; c; n, p_1) = \sum_{d=0}^{c} \binom{n}{d} p_1^d (1-p_1)^{n-d}, \] (9)

\[ \beta = B(d; c; n, p_2) = \sum_{d=0}^{c} \binom{n}{d} p_2^d (1-p_2)^{n-d}. \] (10)

Equations (9) (binomial acceptance probability of a good shipment) and (10) (binomial acceptance probability of a shipment of significantly worse quality) present nonlinear system of equations, which means a direct solution does not exist. There are several iterative methods for solving this system, thus for the design of the acceptance plan a computer is essential. Equivalently to the equation systems (9) and (10) it is possible, with minor changes, to use the systems obtained by using equation (4) both for hypergeometric or by using equation (8) for the Poisson distribution.
4.1 Integration of the mathematical model in software application

Integracija matematičkog modela u računalnu aplikaciju

In 1969, W. C. Guenthier found appropriate algorithm for searching the solution for the system of equations (9) and (10). One of the forms of this algorithm applicable in programming in software applications is given in the flow chart shown in Fig. 11 [6]. The example of integration of the mathematical model for binomial distribution by using m-file in MATLAB is shown in Fig. 12. On the basis of the code in Fig. 12, a user interface is created for easier input and printout of results and curves, which is shown in Fig. 13.

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Figure 11 Chart flow of the algorithm for iterative solution finding

Slika 11. Dijagram toka algoritma za iterativno traženje rješenja

Figure 12 The example of algorithm adaptation for implementation in Matlab

Slika 12. Primjer prilagodbe algoritma za izvođenje u Matlabu

Figure 13 The example of user interface designed in Matlab

Slika 13. Primjer korisničkog sučelja izrađenog u Matlabu

5 Application

Primjena

In this chapter, on the basis of examples, classic examples of usage and comparison with the Larson nomogram and standard are presented. Since the copying of Croatian standards or their parts is prohibited, tables from the cancelled American military standard MIL STD-105E will be used for comparison, together with the name of analogue table from the HRN ISO 2859-1:1989 standard (standard HRN ISO 2859-1:1989 derived from the American standard MIL STD-105, with cosmetic changes so the tables and results correspond).
5.1 Comparison with the Larson Nomogram

It is necessary to create such a sampling plan which accepts 95% of samples containing 2% of defective lots and accepts only 5% of samples containing 9% of defective lots.

Since the task is analogue to the former example solved with the Larson nomogram, following conclusions can be drawn, based on Fig. 13:

- Operating characteristic curve derived from the data in the Larson nomogram (curve marked with circles in Fig. 13) is placed below the curve derived by using the mathematical model for the input data (curve marked with squares in Fig. 13), which means that it will result in stricter criteria for acceptance with the same percentage of defective lots
- The average outgoing quality curve for the data from the Larson nomogram can also be found below the outgoing quality curve for the input data, which means that after a number of shipments big enough at the entrance warehouse would be less defective lots for the acceptance plan derived from the Larson nomogram.

5.2 Comparison with the standard ISO HRN 2859-1:1989

In order to make the comparison with standards, it is necessary to find input data for the software application first. Since in the standard only AQL and the level of review are defined, it is necessary to inspect carefully the tables and operating characteristic curves for the appropriate code letter and take out the needed data.

As an example, a code letter F is chosen:
HRN ISO 2859-1:1989, graph F, Table X-F-1,
MIL STD-105E, graph F, table X-F-1,
Sample size 91-150, AQL = 4, n=20, c=2,
shown in Fig. 14.

The following data can be read from the diagram and the table in Fig. 14:
\[ AQL = 4 \text{%; } \alpha \approx 5\% \]
\[ RQL = 24.5\% \text{ for } \beta = 10\% . \]

When this data is inserted into the application, obtained results are shown in Fig. 15. The sample size \( N \) when using binomial distribution does not play any part during the design of sampling plans, but it is necessary to insert the number which describes the size of the basic lot for the calculation necessary for the printout of the AOQ curve.

Obtained sampling plan has parameters \( n = 20 \) and \( c = 2 \) which correspond to the data in the standard (Fig. 15).
When the operating characteristic curve from Fig. 15 is copied onto the graph in Fig. 14, it is visible that the overlap of the curves is more than satisfying (Fig. 16).

Based on the procedure presented in the previous example, it is necessary to design sampling plans for the data given in Tab. 1.

### Conclusion

It is visible from Tab. 2 that the application gives results which correspond to the standard. It is also visible that the Poisson distribution approximates well the binomial for large samples and smaller probabilities.

### 6 Conclusion

Standard HRN ISO 2859–1:1989 is created with the aim to enable a simple and reliable system for the design of sampling plans. The main problem presents the fact that it is formed around the acceptable level of quality AQL and is stern when it comes to the choice of other parameters for the design of acceptance plans.

Larson nomogram presents a fast and simple way for the design of acceptance plans, which is with parameters adapted to the specific case. However, because of the network distribution, some discrepancy of parameters is possible, which results in insufficiently critical or overly critical acceptance plan.

As opposed to the mentioned standard, statistical approach enables the design of mathematical models which correspond to specific situations, but it presents a problem since there is extensive and slow calculation of cumulative distribution functions in relation to data distribution.

This paper presents an overview of one of the possible ways of development and implementation of mathematical model in a software application which is not limited with sampling parameters or with the volume of mathematical calculation and it presents a simple and fast way for the design of acceptance plans adapted for the use in concrete situations. The emphasis is put on design of optimal acceptance plans which ensure that the obtained parameters and operating characteristic curves faithfully follow requirements (AQL, RQL, alpha and beta risks, and sample sizes) and by that favor both the producer and the consumer.

### Table 1: Input data for comparison

<table>
<thead>
<tr>
<th>Ord. numb.</th>
<th>Code letter</th>
<th>Sample size</th>
<th>AQL</th>
<th>RQL</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>D</td>
<td>26-50</td>
<td>10</td>
<td>4</td>
<td>58</td>
</tr>
<tr>
<td>2</td>
<td>G</td>
<td>151-280</td>
<td>2,5</td>
<td>5</td>
<td>15,8</td>
</tr>
<tr>
<td>3</td>
<td>J</td>
<td>501-1200</td>
<td>1</td>
<td>5</td>
<td>6,25</td>
</tr>
<tr>
<td>4</td>
<td>L</td>
<td>3201-10000</td>
<td>0,25</td>
<td>10</td>
<td>1,94</td>
</tr>
</tbody>
</table>

**Table 2: Compared solutions by using the standard and software application**

<table>
<thead>
<tr>
<th>Ord. numb.</th>
<th>Standard</th>
<th>Application</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>n=8, c=2</td>
<td>B: n=8, c=2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>P: n=12, c=3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>H: n=7, c=2</td>
</tr>
<tr>
<td>2</td>
<td>n=32, c=2</td>
<td>B: n=32, c=2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>P: n=43, c=3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>H: n=32, c=2</td>
</tr>
<tr>
<td>3</td>
<td>n=80, c=2</td>
<td>B: n=80, c=2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>P: n=103, c=3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>H: n=79, c=2</td>
</tr>
<tr>
<td>4</td>
<td>n=200, c=1</td>
<td>B: n=200, c=1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>P: n=201, c=1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>H: n=197, c=1</td>
</tr>
</tbody>
</table>

B – Binomial distribution, P – Poisson distribution, H – Hypergeometric distribution

### Literatura