Zb. Rad. - Sveuč. u Rij., Ekon. fak., god. 22. Sv. 2 (2004), str. 117-137

Saša Žiković[1]

Preliminary communication
UDK 336.763

OPTIMAL TRADING QUANTITY INTEGRATION AS A BASIS FOR OPTIMAL PORTFOLIO MANAGEMENT[2]

ABSTRACT

The author in this paper points out the reason behind calculating and using optimal trading quantity in conjunction with Markowitz's Modern portfolio theory. In the opening part the author presents an example of calculating optimal weights using Markowitz's Mean-Variance approach, followed by an explanation of basic logic behind optimal trading quantity. The use of optimal trading quantity is not limited to systems with Bernoulli outcome, but can also be used when trading shares, futures, options etc. Optimal trading quantity points out two often-overlooked axioms: (1) a system with negative mathematical expectancy can never be transformed in a system with positive mathematical expectancy, (2) by missing the optimal trading quantity an investor can turn a system with positive expectancy into a negative one. Optimal trading quantity is that quantity which maximizes geometric mean (growth function) of a particular system. To determine the optimal trading quantity for simpler systems, with a very limited number of outcomes, a set of Kelly's formulas is appropriate. In the conclusion the summary of the paper is presented.

Key words: *optimal portfolio, trading, optimal trading quantity, game theory*

1. Introduction

Investors that trade in capital markets, especially in transitional economies, such as Croatia; need to have a certain appetite for gambling. Investing in transitional economies has a huge potential for incredible gains, but it also bears a level of risk that is rare in the developed markets.

All experienced investors know that stock market crashes do occur, companies file for bankruptcy, natural disasters happen and technology changes, but seldom do they

[1] M.Sc., Assistant of the Faculty of Economics, University of Rijeka

[2] Article received: 17-01-2005; Article accepted: 06-05-2005

prepare for worst case scenarios, because of its remote probability of occurring. Cases like Enron, Worldcom, Barings bank and Daiwa, from real life show that in recent years these remote probabilities were not so remote after all.

Worst-case scenario should be the starting point of portfolio selection process and every trading technique, to prepare for, or even take advantage of situations that include the realization of the worst case. The fact that risk management and preparation for worst case event is not some fictional concept brought about by state administration and conservative bankers is best expressed in the following axiom: "If you play a game with unlimited liability, you will go bankrupted with a probability that approaches certainty as the length of the game approaches infinity"[3]. The unlimited liability refers to securities whose owner may loose more then he invested, such as the case with futures or taking a short position in shares or options[4]. On the other hand when investing in limited liability securities such as long position in shares, bonds and options an investor can loose only the amount of money he invested[5]. Although the probability of going bankrupt is low, if an investor trades long enough, this probability will be realized, regardless of the diversification of his investments.

These facts of life are not intended to drive away potential investors but to prepare them for not only the good times (that currently dominate Croatian market[6]) but also for the prolonged losing streaks that will inevitably occur. That is why every rational investor should think about worst-case scenario and make contingency plans for such an event.

There are three possible courses an investor can take to avoid going broke:

1) Trading only in securities that have limited liability
2) Setting up a reasonable, prespecified target return, upon reaching it getting out of trading unlimited liability securities.
3) Not trade in unlimited liability securities for infinitely long period of time.

Based on the previous work by Vince Ralph, Fred Gehm and Florence Bobeck[7] this paper will try to combine the classical Modern portfolio theory with the concept of optimal trading quantity. This combination has the potential to maximize the effect of diversification that stems from the modern portfolio theory by combining it with the basic principles from the game theory.

---

3 Vince R.; The mathematics of money management: risk analysis techniques for traders, John Wiley & Sons, New York, 1992.

4 For more on the subject: Kolb W. R.; Futures, options and swaps, fourth edition, Blackwell Publishing, Cornwall, 2003.

5 At present options do not exist on Croatian stock exchanges, and citizens of Croatia are restricted to trading only in the domestic market; individuals can trade on foreign exchanges only via a licensed intermediary and are restricted to investing only in high grade securities (Odluka o načinu i uvjetima pod kojima rezidenti ulažu u strane vrijednosne papire i udjele u stranim investicijskim fondovima, Narodne novine, broj 146/2003.).

6 VIN index – the official index of Varaždin stock exchange jumped by 75,11% from the beginning of the year 2004. till 10.09.2004., making it the fastest growing stock exchange index in the region (www.vse.hr/news, 11.09.2004)

7 Vince R.; Portfolio Management Formulas, John Wiley & Sons, New York, 1992.

2. Optimal portfolio

Markowitz's Modern portfolio theory[8] laid groundwork for a wide range of theories and applications not only in portfolio management but also in many other different areas, such as risk management. Although the theory was developed to be practical and ease to implement, its implementation was painfully slow because of four main difficulties[9]:

1) the difficulty in estimating the necessary input data (especially correlation matrices);
2) the time and cost necessary to generate efficient portfolios (solving a quadratic equation); and
3) the difficulty of educating portfolio managers to relate risk return tradeoffs expressed in terms of covariances as well as returns and standard deviation.
4) Many investors do not act in accordance with their utility curve, meaning that they do not behave consistently when faced with simple (such as investing in a single security) and complicated decisions (investing in a portfolio). Markowitz himself admits that there isn't a person in a world who knows his utility curve[10].

Modern portfolio theory is based on idea that an optimal portfolio minimizes the portfolio's variance, for a set rate of return[11]. The expected rate of return of the portfolio is defined as the weighted sum of individual securities forming the portfolio, with each security's participation in the portfolio serving as a weight:

$$E(R_{port}) = \sum_{i=1}^{N} X_i E(R_i)$$

where:

$E(R_{port})$ – expected rate of return of a portfolio

X_i – share of security (i) in the portfolio

$E(R_i)$ – expected rate of return of a security (i)

The portfolio variance is defined as:

$$\sigma^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_i \sigma_j \rho_{ij}$$

where:

σ_i^2 – variance of the i-th security

σ_i – standard deviation of the i-th security

8 Markowitz, H.M.; Portfolio Selection, Journal of Finance, Vol. 7, No. 1, 1952.

9 Elton, E.J., Gruber, M.J., Padberg, M.W.; Simple criteria for optimal portfolio selection, , Journal of Finance, 1976.

10 Markowitz H.M.; Foundations of Portfolio Theory, The Journal of Finance, Vol. 44, No. 2, 1991.

11 Standard deviation is obtained by extracting a square root of variance

ρ_{ij} – coefficient of correlation between the rates of return of the two securities (i) and (j)

The last part of the equation ($\sigma_i \sigma_j \rho_{ij}$) can substituted by the term $Cov_{i,j}$ (covariance). This is done to simplify the equation, since[12]:

$$\rho_{ij} = \frac{Cov_{ij}}{\sigma_i \sigma_j}$$

To calculate the efficient frontier it is necessary to minimize the risk for every level of the expected return. In case of a portfolio, when the investor is not short selling, portfolios on the efficient frontier are obtained by minimizing the portfolio variance, while obtaining the pre-set target return, under the following three restrictions:

1) $\sum_{i=1}^{N} X_i = 1$
   – sum of shares in the portfolio must equal 1 (100%)

2) $X_i \geq 0$ i = 1,..., N
   – quantity of a each security must be equal or greater than 0

3) $\sum_{i=1}^{N} X_i E(R_i) \geq E(R_{port})$
   – expected rate of return of an efficient portfolio must be equal or greater than a pre-set target return

By changing portfolio's expected rate of return $E(R_{port})$ while minimizing the variance, a set of efficient portfolios is calculated. The presented set of equations represent a smooth non-linear (quadratic) programming problem[13] and can be solved by using specialized portfolio optimization programs or Excel solver. The investor can choose between a given set of efficient portfolios, and the choice of a portfolio depends sonly on investor's willingness to bear additional risk or his risk aversion.

3. Optimal trading quantity

Markowitz's modern portfolio theory determines the optimal weights of individual securities in an optimal portfolio[14], and represents an indispensable tool of modern financial management, but it does not say anything about optimal quantities that should be used when trading with a particular system, a particular security or portfolio.

---

12 See Elton, J.E., Gruber, J.M.; Modern Portfolio Theory and Investment Analysis, John Wiley & Sons, New York, 1991.

13 Gupta, V.; Financial Analysis using Excel, VJ Books, Canada, 2002.

14 Finding a portfolio with the smallest standard deviation for a set level of return, or finding a portfolio with the highest level of return for set level of standard deviation (Markowitz, H.M.; Portfolio Selection, Journal of Finance, Vol. 7, No. 1, 1952.)

For every trading system an investor applies to a particular market, a critical factor to consider is the quantity of money to trade on a particular security or contract given his current wealth. A truly optimal portfolio can only be achieved by combining Modern portfolio theory (provides the investor with optimal weights) and function of trading quantity (provides investor with optimal trading quantity). Such well-rounded trading system should allow traders to take better advantages of any market imperfections.

In a nonleveraged situation, as in a portfolio that consists only of long position in shares without margin credit, weight and quantity represent the same thing, but in a leveraged portfolio consisting of shares bought on margin contracts, futures contracts and short positions in options, weighting and quantity have quite different meanings.

3.1. Definition and basic concept behind optimal trading quantity

Optimal trading quantity represents a regulator of growth in an environment of geometric consequences[15] and it also reveals a great deal of information about the growth rate of a particular environment.

A basic decision in investing, besides choosing the right security and the decision to go long or short, is how much money will one invest. This decision of quantity to invest is ultimately a function of investor's wealth. If an investor holds 5.000 HRK and invests in 40 shares, each priced at 100 HRK he is taking a big gamble, on the other hand investor holding 1 mil HRK and investing in 5 such shares is taking it to light and would have been smarter depositing that money in a bank. This simple example clearly shows that quantity a person is going to invest in a given trade is inseparable from the level of his wealth.

Investors' wealth will grow fastest when a constant fraction of investors account is traded on each trade, in other words when the investor trades a constant quantity relative to the size of his account. Besides being the function of investor's wealth, the quantity decision among other things depends upon following functions:

- investor's possible worst case loss on the next trade,
- speed at which the investor wants his account to grow, and
- dependency to past trades.

These and many other variables are used to solve the investor's problem of, taking all the above stated variables in consideration, how many shares should he invest in. Usually this decision is made subjectively (rule of thumb) and often erroneously with serious consequences. Some traders even think that this decision is somewhat arbitrary in that it does not matter how much they have invested, as long as the optimal portfolio form is satisfied and as long as they are right about the direction of the trade[16]. An even

[15] Environment where a quantity that an investor has at his disposal today is a function of prior outcomes (Wilmott, P., Howison, S., Dewynne J.; The Mathematics of Financial Derivatives, University of Cambridge, 1995.)

[16] Vince, R.; Portfolio management Formulas, John Wiley & Sons, New York, 1992.

more dangerous misconception is that of presuming a linear relationship between how many shares an investor buys and how much he stands to win or loose in the long run.

Unfortunately this relationship is not a straight line but rather a curve. This curve has one peak, a point where maximum growth and profit are obtained, and on both sides of the curve there are steep slopes. Contrary to the wide spread opinion, being correct about the direction of the market, does not take precedence over investing the right quantity on a particular asset. Choosing the wrong quantity may in fact turn a positive shift in the market into a loss for the investor. In a developed market a single investor has no influence or control over the direction of the market[17]. What every investor has control over is what percentage of his account he will invest in a particular trade, and since market movement does not take precedence over optimal quantity, investor is better of spending his time and effort on finding the optimal quantity to trade (a factor he can influence) instead of trying to determine whether the next trade will be profitable (a factor he can not influence).

Returning to the functions that determine the optimal quantity to trade, the first and probably the most important one is the worst-case loss. However safe a particular investment may seem at the present moment the investor is aware that he is exposed to a potential loss. The probability of worst-case loss influences the investors decision of how much money (more precisely what fraction of his account) should he invest in a particular asset. To determine how many contracts to trade the investor can use a divisor of the largest expected loss, a number between 0 and 1[18].

$$LD_i = \frac{N_i \cdot WL_i}{A_t} \qquad\qquad N_i = \frac{A_t \cdot LD_i}{WL_i}$$

Where:

LD_i – loss divisor of asset (i)

N_i – number of contracts on asset (i)

WL_i – worst-case loss on asset (i)

A_t – investor's account at time (t)

To explain why trading the right quantity is equally important as getting right the direction of the market, the basic principles of mathematical expectation have to be explained. Mathematical expectation represents the amount an investor expects to make or loose, on average, each time he trades. The basic formula for mathematical expectation is:

$$\text{Mathematical expectation} = \sum_{i=1}^{N} (P_i \cdot A_i)$$

---

17 In developing countries such as Croatia there is evidence that wealthy individuals with their actions can influence the direction of the market.

18 Divisor of 0 means that investor will not invest in that asset, divisor of 1 means he invested his entire account in one trade.

Where:

P – probability of winning or loosing

A – amount gained or lost

N – number of possible outcomes

Multiplying each possible gain or loss by the probability associated with it and then summing the products gives the mathematical expectation of each system. To explain the matters further, a simple example of positive mathematical expectation[19] is given.

Example: Investor is faced with a game where he has a 50% chance of wining 2 HRK, and 50% chance of loosing 1 HRK. According to the presented formula his mathematical expectation is:

$$\text{Mathematical expectation} = (0{,}5 \times 2) + (0{,}5 \times (-1)) = 1 + (-0{,}5) = 0{,}5$$

In such a game, investors mathematical expectation is winning on average 0,5 HRK per each bet.

3.2. Geometric growth

The ultimate goal of optimal trading quantity is finding the optimal share of wealth the investor should invest in each trade to maximize his profits. Finding an optimal trading quantity is not a complicated matter to solve when dealing with Bernoulli outcomes[20]. When dealing with stock markets almost infinite number of outcomes is possible, and the computations get very demanding even for a modern PC. One of the methods used to find optimal trading quantity is a system of Kelly formulas[21]. Kelly criterion states that the investor should invest a fixed fraction of his account that maximizes the growth function (G).

$$G = P \cdot \ln(1 + B \cdot q) + (1 - P) \cdot \ln(1 - q)$$

Where:

P – probability of a winning trade

B – ratio of amount won on a wining trade to amount lost on a losing trade

q – optimal fixed fraction (optimal trading quantity)

ln – natural logarithm

In the case of a simple problem as the one in the previous example, a simple formula can be used to find the optimal trading quantity:

[19] A game with positive mathematical expectation is the game where the investor has an advantage over the market.

[20] Bernoulli outcome – only two discrete outcomes are possible (See Chiang A.C. Osnovne metode matematičke ekonomije, treće izdanje, Mate, Zagreb, 1996.)

[21] Vince, R.; Portfolio management Formulas, John Wiley & Sons, New York, 1992.

$$q = \frac{((B+1)\cdot P - 1)}{B}$$

In the example of a game, where a player can win 2 HRK or lose 1 HRK:

q = ((2 + 1) · 0,5 – 1)/2
= 0,25

The optimal amount to invest in this game is 25% of investor's wealth. This formula will give a correct optimal trading quantity under the condition that all wins are always of the same amount, and all losses are also always of the same amount. After determining that the optimal fraction of investor's account to invest in the trade is 25%, every investor will be interested in knowing how much will he profit from such a decision.

When trading in markets, gains and losses differ significantly by amounts, and therefore Kelly's formula does not yield a correct trading quantity, and a different approach is required[22]. First of all, the trading quantity must be included in calculating the expected rate of return from a single trade:

$$1 + R_i = 1 + q \cdot (-Trade_i \,/\, Biggest\ \ loss)$$

Where:

R_i – rate of return
q – trading quantity
$Trade_i$ – profit or loss on the i-th trade
Biggest loss – biggest loss occurred in trading with a particular asset

Multiplication of all the rates of return obtained from trading in a particular asset results in the value of TWR. TWR stands for Terminal wealth relative and represents the return on the initial stake as a multiple, in other words, it is a simple geometric product of the rates of return.

$$TWR = \prod_{i=1}^{N} 1 + R_i$$

It can also be written as:

$$TWR = \frac{Investment + Profit}{Investment}$$

TWR is a multiplicative function yielding two important implications:

[22] Taking the averages of experienced profits and losses and using them in Kelly formulas will not result in a correct optimal trading quantity since Kelly formula must be specific to a single trade.

(1) if investor suffers a $1+R_i = 0$, his investment will disappear, which is logical since any number multiplied by 0 equals 0.

(2) Any trade resulting in a big loss or a big gain will have a very strong effect on the value of TWR because of its' multiplicative, rather than additive nature.

To get the distribution of trading quantities following formulas are needed:

$$TWR = \prod_{i=1}^{N} \left(1 + q \cdot (-Trade_i \;/\; Biggest\ \ loss)\right)$$

$$G = \prod_{i=1}^{N} \left(1 + q \cdot (-Trade_i \;/\; Biggest\ \ loss)\right)^{1/N}$$

Where:

G – geometric mean of (1 + rate of return)

N – number of trades

Geometric mean[23] can also be calculated as the N-th root of the TWR

$$G = TWR^{(1/N)}$$

By looping through values of optimal trading quantity between 0 and 1, the value of trading quantity can be found that yields the highest TWR. This is the value of trading quantity that results in the highest growth potential (maximum return) for the investment, using fixed fraction investing. Put in another way, this is the way to find the value of trading quantity that results in the highest geometrical mean. Logarithm of geometric mean of a numerical variable is the arithmetic mean of logarithms of its values[24].

$$\log G = \frac{1}{N} \sum_{i=1}^{N} \log X_i$$

X_i – numerical variable

Logarithm of geometric mean is used as a good estimate of average values for asymmetric groups of data, and that is why it is very useful in determining the average return of securities. Logarithmic value of asymmetrically distributed variables is closer to a symmetric distribution so the arithmetic mean of logarithms is a representative measure of average value. Simply put, geometric mean represents the growth factor of any investment.

The market or a security with the highest geometric mean is the most profitable market or security provided that the investor reinvests the returns. A geometric mean greater

[23] Geometric mean is used to calculate the average rate of change.

[24] Šošić, I., Serdar, V.; Uvod u statistiku, Školska knjiga-Zagreb, Zagreb, 1994.

than 1 represents a positive mathematical expectation meaning that the market or security is profitable for investor. A geometric mean less than 1 represents a negative mathematical expectancy meaning that the investor will lose money by trading. Geometric mean can be used as a risk adjusted measure of profitability, much in the same way as Sharpe ratio, Traynor measure, Jensen measure and other. Unlike the mentioned measures, geometric mean measures investment performance relative to dispersion in the same mathematical form as that in which the balance in investor's bank account is affected.

Using the stated formulas to calculate the values of TWR for the previous game where there is a 50% chance a player will win 2 HRK, and a 50% change that he will loose 1 HRK is a straightforward process using a mathematical software package like Excel. A distribution of TWR dependent on trading quantities is given in table 1 and figure 1:

Table 1

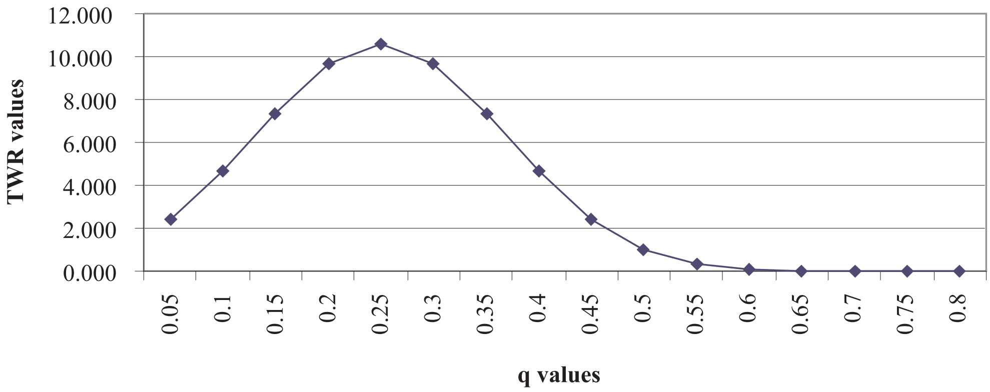
Value of TWR for different trading quantities

0.05	2.412
1	4.661
0.15	7.366
0.2	9.646
0.25	10.545
0.3	9.646
0.35	7.366
0.4	4.661
0.45	2.412
0.5	1.000
0.55	0.323
0.6	0.078
0.65	0.013
0.7	0.001
0.75	0.000
0.8	0.000

When the optimal trading quantity of 25% is reached after 40 trades TWR reaches its' maximum at 10,55, meaning that the investor earned 10.55 times his original stake, or a profit of 955%. If the investor missed the optimal trading quantity by 15 percentage points (0.10 or 0.40) he would have earned only 366.1% - more than 2.5 times less than with the optimal trading quantity. If the investor invested 50% of his account he would have barely come even. Any quantity larger than that would result in a loose for the investor. This example clearly shows that the simple rule "the more you invest the more you earn" does not apply here.

Figure 1

Trading quantity distribution curve after 40 trades



Source: Table 1

A similar principle applies to the systems with negative mathematical expectancy. Investor will always loose in the long run if the system, or market, or portfolio he is investing in has a negative mathematical expectancy, because the sum of investments made with the negative mathematical expectation will always be a negative number, regardless of the initial amount invested.

The same principle applies to even-money systems, which leaves the investor with only one logical choice – finding a market (portfolio) with a positive mathematical expectancy. By investing properly any positive mathematical expectancy can be turned into an exponential growth function. How tiny this positive mathematical expectation is, has no bearing on the final result, and using that expectancy will allow investor's profits to grow geometrically. At the end of the day it does not matter how profitable a portfolio or a share has been in the past but rather how certain can the investor be that it will continue to be profitable in the future[25]. Instead of spending money, time and effort on finding the share or portfolio with highest profits, investors should focus on finding the share or a portfolio with maximum certainty of a marginal profit.

4. Combining optimal trading quantity with the optimal portfolio

To investigate the effects of using the optimal trading quantity on a portfolio of securities a simple example is given. In table 2, three investment opportunities are presented; a bond paying 1% interest monthly and two portfolios consisting of shares, all with the starting nominal amount of 1.000 HRK. The investment opportunities are analyzed during a five-month period with and without reinvestment.

[25] Trough proper investing a share that constantly produces 10 HRK of profit per trade can be much more profitable than the share that *on average* produces 1.000 HRK of profit per trade.

Table 2

Returns and profit and loss results with and without reinvestment during five consecutive periods for three investment opportunities

Portfolio A - in HRK

Time	Without reinvestment		With reinvestment	
	Return (%)	Total	Return (%)	Total
0		1,000		1,000.00
1	60%	1,600	60%	1,600.00
2	-50%	1,100	-50%	800.00
3	-5%	1,050	-5%	760.00
4	15%	1,200	15%	874.00
5	10%	1,300	10%	961.40

Arithmetic mean	1,208.33	999.23
Average trade	60.00	-7.72
Standard dev.	220.04	308.21
CV	18.21%	30.84%

Portfolio B - in HRK

Time	Without reinvestment		With reinvestment	
	Return (%)	Total	Return (%)	Total
0		1,000		1,000.00
1	25%	1,250	25%	1,250.00
2	-10%	1,150	-10%	1,125.00
3	-2%	1,130	-2%	1,102.50
4	-1%	1,120	-1%	1,091.48
5	-1%	1,110	-1%	1,080.56

Arithmetic mean	1,126.67	1,108.26
Average trade	22.00	16.11
Standard dev.	80.17	81.45
CV	7.12%	7.35%

Bond - in HRK

Time	Without reinvestment		With reinvestment	
	Return %	Total	Return %	Total
0		1,000		1,000.00
1	1%	1,010	1%	1,010.00
2	1%	1,020	1%	1,020.10
3	1%	1,030	1%	1,030.30
4	1%	1,040	1%	1,040.60
5	1%	1,050	1%	1,051.01

Arithmetic mean	1,025.00	1,025.34
Average trade	10.00	10.20
Standard dev.	18.71	19.09
CV	1.83%	1.86%

From this example a couple of very important conclusions can be made:

Ad 1 Performance at the end of the investment period is not affected by the order in which the profits and looses occur.

Ad 2 A winning portfolio can be turned into a losing portfolio if earned profits are reinvested, when the returns from trades are inconsistent (portfolio A).

Ad 3 Trading on reinvestment (fixed fraction) basis has a positive side effect of serving as an accelerator in times of prosperity (bull market), and as a buffer in times of depression (bear market)[26].

Ad 4 Finding a system[27] with positive mathematical expectancy and trading on reinvestment basis transforms a linear relationship between investment and profit into a growth function, and the difference between the two grows exponentially by the passing of time.

Different investors analyzing the presented investment opportunities would come to different conclusions about the appeal of each portfolio.

- A bullish investor would search for investment opportunity with the highest total amount, highest arithmetic mean, highest average trade or a system with highest number of winning trades, and he would probably choose portfolio A.
- A cautious, risk averse investor would search for consistency and lowest standard deviation and coefficient of variance in the investment and would choose to buy a bond.
- For an investor looking to maximize his investments by looking for a system with highest growth potential, both of these choices are sub optimal, and such an investor would choose portfolio B.

By selecting the portfolio B the investor would choose a portfolio with highest geometric mean and that is the investment that yields the highest profit by trading on fixed fraction basis. The process of finding the system with highest geometric mean is as follows:

1) TWR is calculated by multiplying all the trading returns from each system

$$TWR = \prod_{i=1}^{N} 1 + R_i$$

26 By investing on fixed fraction basis, as the investor profits from his trades, in nominal amount he invests more and more wealth. Vice versa, when the investor starts to loose on his trades by investing on fixed fractions basis, in nominal amount he invests less and less each time he trades.

27 Security, portfolio, trading system or market

2) Geometric mean is calculated by taking the N-th root of TWR[28] and since five periods were observed in the example this means that 5-th root must by taken from calculated value of TWR, the results are presented in table 3:

$$G = TWR^{1/N}$$

Table 3

Calculated values of TWR and geometric mean for possible investment opportunities

	TWR	Geometric mean
Portfolio A	0.9614	0.99216
Portfolio B	1.08056	1.01562
Bond	1.05101	1.01000

Source: Calculated from table 2

There are two investment with geometric mean greater than 1, portfolio B and the bond. Portfolio B is the system with the highest geometric mean (1,01562) and as such is the optimal choice for the investor. Portfolio A has a geometric mean smaller than 1 and as such should be avoided or at least not be traded on a reinvestment basis.

Although a portfolio or a particular share might have a geometric mean significantly higher than 1, this does not mean that the investor can not experience serious loosing streaks during certain periods, in fact it can be stated with great certainty that this is a very likely scenario. Shares and/or portfolios with high growth potentials[29] will have a high optimal trading quantity so as to take advantage of their embedded capability of experiencing high exponential growth and because of that, they are more exposed to serious losses than shares/portfolios with marginal growth potential.

Incurring a loosing streak while trading in a portfolio with high geometric mean and thus high optimal trading quantity (for example 50%) will result in more serious consequences than when trading in a marginal growth portfolio[30]. A very practical way to minimize the effects of loosing streaks, particularly when dealing with shares/portfolios with high optimal trading quantities, is to concentrate on their mutual correlations and by using the basic principles of modern portfolio theory diversify by investing into shares/portfolios that have low or negative coefficient of correlation.

[28] Any system with geometric mean greater than 1 yield a positive profit from trades on a reinvestment basis and holds the potential to grow exponentially. A system with geometric mean less than 1 will continually loose money on the reinvestment basis.

[29] Geometric mean much higher than 1

[30] During the winning streaks the portfolio with high geometric mean and high optimal trading quantity will experience substantially higher profits.

Diversification combined with fixed fraction investing provides an investor with a powerful investment tool for taking advantage of the bull market and serves as a double buffer in times of a bear market.

In a bear market:

– nominal amounts invested become smaller since the investor is investing on a fixed fraction basis and thus buffering the losses;
– diversification in an ideal situation enables the investor to cover his loses from one share/portfolio/market as another share/portfolio/market surges provided they are negatively correlated or in a real life situation covers at least a part of his loses.

In a bull market:

– nominal amounts invested become bigger because the investor is investing on a fixed fraction basis which enables him to reap higher profits;
– after a bear market if the investor has diversified his investments he will still have a substantial amount of his wealth left to invest in the prosperous markets.

In the developed markets like US, correlations between markets and sectors as well as optimal trading quantities tend to remain constant over prolonged periods of time and investors trading on such markets can profit substantially from this attribute that other markets, such as those in the transitional countries[31], do not have the luxury of. Obtaining and preserving an optimal portfolio in the developed markets is much less time consuming and accompanied by much smaller transaction costs than it is in the transitional countries.

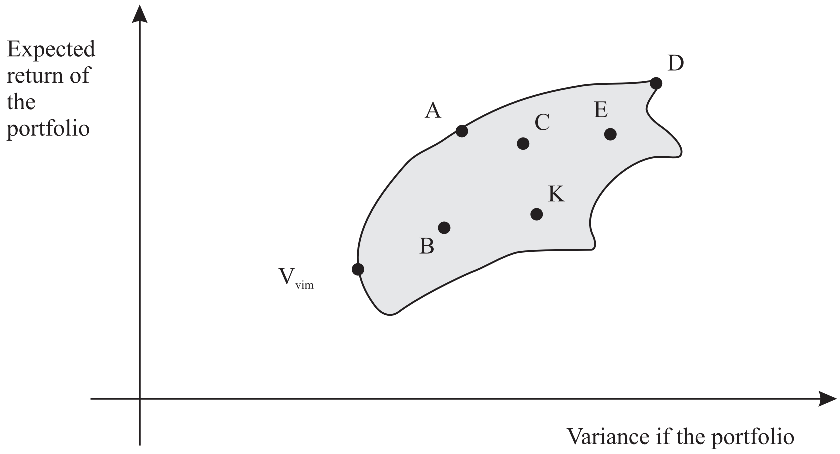
Provided that the investor has diversified properly his portfolio or even hedged his open positions with derivatives, it still does not guarantee him that he will not incur serious losses. As recent history shows, very often in times of crises different securities and sectors tend to converge toward correlation coefficient of 1. This situation is especially obvious in the developing markets partially because of low level of knowledge and trust regarding the securities. In case of bigger crises (Russian, Brazilian, Japanese etc.) the trend of correlations converging to 1, seems to attain global proportions[32]. Resulting conclusion is that diversification does provide buffering and additional security in cases of smaller crises and local events, but quickly disappears when confronted with more serious crisis.

[31] Developing markets, like Croatian market, are generally characterised by continuous changes of volatilities and correlations between the shares and sectors. Modern portfolio theory in its' original form does not provide a good guidance for investors in these situations, since with the outbreak of crises diversification effects disappear quickly (for example: Russian crisis, Croatian market crash of 1997.)

[32] Compare: - Crouhy M., Galai D., Mark R.; Risk Management, McGraw Hill, New York, 2001.
- Bank for International settlements, Bank failures in Mature Economies, Working paper No. 13, April 2004.
- Jorion P.; Value at Risk, The New Benchmark for Managing Financial Risk, 2nd edition, McGraw Hill, New York, 2001.

When choosing from all the possible portfolios in which to invest, modern portfolio theory starts by constructing the efficient frontier[33] of portfolios. Portfolios that are found on the efficient frontier (portfolios from V_{min} to D) dominate over all the other portfolios in the shaded area because they either have a lower variance or a higher return than any other portfolio. Efficient frontier is shown in figure 2.

Figure 2
Efficient frontier for given portfolios



Source: Elton, J.E., Gruber, J.M.; Modern Portfolio Theory and Investment Analysis, John Wiley & Sons, New York, 1991.

A rational investor will always choose a portfolio lying on the efficient frontier. What portfolio of the ones lying on the efficient frontier the investor chooses depends on his risk preference. A conservative investor might choose a portfolio V_{min} with minimum variance, and an aggressive investor might choose the portfolio D.

An important rule of thumb to remember is that corner portfolios[34] are usually much less diversified than the ones lying in the middle of the efficient frontier. These middle portfolios would be appropriate for use with the optimal trading quantity and they are the ones that would provide the best defense against market shocks. Although the choice of the efficient portfolio to invest in is a matter or investors personal taste and convictions, looking from the optimal trading quantity point of view there is only one optimal solution for a given set of efficient portfolios.

To illustrate the basic idea behind modern portfolio theory a simple example is given;

[33] The efficient frontier represents a set of portfolio that yield maximum possible return for a given level of risk, and minimum risk for a given level of return.

[34] Portfolios situated high to the right and low to the left of efficient frontier

Investor cannot decide between three shares and is not sure what percentage of each share to include in his new portfolio. Shares and their varying returns[35] are presented in table 4:

Table 4

Share returns and states of economy

- in percent

Share	A	B	C
State of the economy			
Recession	5.00	0.00	21.00
Normal	15.00	18.00	16.00
Prosperity	25.00	29.00	1.00

Source: Author

With the use of a software package "Modern Investment Theory"[36] and under the assumption that no short selling is allowed, following results are obtained:

	A	B	C
Expected return	15.00%	15.67%	12.67%
Standard deviation	10.00%	14.64%	10.41%

Correlation matrix

1.00000	0.95066	-0.91481
0.95066	1.00000	-0.74437
-0.91481	-0.74437	1.00000

The first three optimal portfolios from the efficient frontier, sorted by minimal variance are presented in table 5:

Table 5

Optimal portfolios

- in percent

	Portfolio 1	Portfolio 2	Portfolio 3
Expected return	13.86	14.06	14.26
Standard deviation	1.43	2.25	3.72
Weights			
A	51.02	59.64	43.73
B	0.00	0.00	19.07
C	48.98	40.36	37.20

Source: Author

35 It is assumed that each state of the economy has equal chance of occurring (33.333%)

36 Haugen, R.A.; Modern Investment Theory, Prentice Hall, 1989.

The effect of diversification is obvious, since the return on Portfolio 1 is only 1,81 percentage points below the highest possible return – 15,67%[37], but at the same time it has a standard deviation that is 7 times smaller than the standard deviation of the safest security (share A).

Following the logic of optimal trading quantity and geometric mean the correct portfolio to choose from the set of efficient portfolios is the one that has the highest geometrical mean. Portfolio with the highest geometric mean can ensure the maximal growth of the investors' wealth over longer periods. To take into account the risk factor, the portfolios that the investor inspects as candidates for applying the optimal trading quantity should be efficient portfolios located in the central part of efficient frontier to provide for the best possible diversification. A reliable approximation of geometric mean of each portfolio can be calculated form the arithmetic mean of the returns and their standard deviations. Both of these parameters can be easily calculated by using an optimization software package modern. The presented formula calculates an estimation of geometric mean that is very close to the real geometric mean and can be used without any serious negative consequences.

$$\text{Estimated geometric mean} = \sqrt{\left(\left(1+\mu_r\right)^2 - \sigma_r^{\,2}\right)}$$

Where:

μ_r – arithmetic mean of portfolio returns

σ_r – variance of portfolio returns

Applying this formula on shares presented in table 4, and calculated optimal portfolios in table 5, a solution for optimal trading portfolio according to optimal trading quantity is reached. The results are presented in table 6.

Table 6

Calculated geometric means for efficient portfolios

-in percent

	A	B	C
Return	13.86	14.06	14.26
Standard deviation	1.43	2.25	3.72
EGM[38]	1.138510	1.140378	1.141994

Source: Calculated from table 5

From the three efficient portfolios analyzed, portfolio C has the highest geometric mean and represents the optimal portfolio according to geometric mean criterion. Since

[37] A portfolio that would yield 15,67% would be achieved if investor formed a portfolio consisting sonly of security B

[38] Estimated geometric mean

the daily prices are used to determine the return and standard deviation, to preserve the optimal portfolio, in theory the investor should adjust his portfolio on daily basis. In real life this sort of adjusting is impractical and very costly since the transaction costs would cost a lot more than any potential benefit an investor could have from continuously holding the optimal portfolio. In portfolios that are constructed to be held for longer periods of time, readjusting can be performed on weekly or even monthly basis. Although, this means that for certain time periods investors' portfolio will be out of balance and will not lie on the efficient frontier the savings made by avoiding daily transaction costs will outweigh the negative side effects.

Another important question that finds no clear answer is what time period should be taken into account when calculating the efficient frontier. The same unanswered question applies to determining the optimal trading quantity for a particular system. It is common opinion that the longer time period will give better results. This approach can prove to be quite faulty for three basic reasons[39]:

1) Volatilities of securities do change over time, and sometimes they do so abruptly.
2) Correlations between securities tend to change, especially in times of crisis, and in the transitional countries.
3) Using long time series can result in slow adjustment to new information, the fact that could be crucial in times of crises.

On the other hand, using short time series tends to neglect the general trend in return and volatility developments of individual securities. It is obvious that there exist a trade-off between long and short time series for constructing the optimal portfolio and very often the decision must be based on pure common sense.

6. Conclusion

Although many have criticized the basic assumptions of modern portfolio theory and the notion of calculating expected future volatilities and correlations from past volatilities and correlations, the modern portfolio theory still represents the cornerstone of modern financial management. Even when neglecting the well founded criticism and accepting all of the problematic assumptions the modern portfolio theory, it still does not provide the investor with any insight about the optimal quantities that should be used when trading with a particular portfolio or a particular security.

Contrary to the wide spread opinion, being correct about the direction of the market, does not take precedence over investing the right quantity on a particular asset. Choosing the wrong quantity may in fact turn a positive shift in the market into a loss for the investor.

[39] See: - Dowd, K.; Measuring market risk, John Wiley & Sons, Chichester, 2002.
- Jorion P.; Value at Risk, The New Benchmark for Managing Financial Risk, 2nd edition, McGraw Hill, New York, 2001.
- Crouhy M., Galai D., Mark R.; Risk Management, McGraw Hill, New York, 2001.

Investors wealth will grow fastest when a constant optimal fraction of investors account is traded on each trade, in other words when the investor trades a constant optimal quantity relative to the size of his account. Besides being the function of investor's wealth, the quantity decision among other things depends upon: (1) investor's possible worst case loss on the next trade, (2) speed at which the investor wants his account to grow, and (3) dependency to past trades.

A truly optimal portfolio can only be achieved by combining the modern portfolio theory (provides the investor with optimal weights) and the function of trading quantity (provides investor with optimal trading quantity). Such a well-rounded trading system should allow traders to take better advantages of any market imperfections and additionally secure them from market risk.

The core philosophy of modern portfolio theory – the concept of diversification combined with fixed fraction investing provides an investor with a powerful investment tool for taking advantage of a bull market and serves as a double buffer in times of a bear market.

Following the logic of optimal trading quantity and the geometric mean the correct portfolio to choose from the set of efficient portfolios is the one that has the highest geometrical mean. Portfolio with the highest geometric mean can ensure the maximum growth of the investors' wealth over longer periods. To take into account the risk factor, the portfolios that the investor inspects as candidates for applying the optimal trading quantity should be efficient portfolios located in the central part of efficient frontier. These portfolios provide the best possible diversification opportunities.

Literature

1. Bank for International settlements; "Bank failures in Mature Economies", *Working paper* No. 13, April 2004.
2. Chiang A.C.; *Osnovne metode matematičke ekonomije*, treće izdanje, Mate, Zagreb, 1996.
3. Crouhy M., Galai D., Mark R.; *Risk Management*, McGraw Hill, New York, 2001.
4. Dowd, K.; *Measuring market risk*, John Wiley & Sons, Chichester, 2002.
5. Elton, E.J., Gruber, M.J., Padberg, M.W.; "Simple criteria for optimal portfolio selection", *Journal of Finance*, 1976.
6. Elton, J.E., Gruber, J.M.; *Modern Portfolio Theory and Investment Analysis*, John Wiley & Sons, New York, 1991.
7. Gupta, V.; *Financial Analysis using Excel*, VJ Books, Canada, 2002.
8. Haugen, R.A.; *Modern Investment Theory*, Prentice Hall, 1989.
9. Jorion P.; *Value at Risk, The New Benchmark for Managing Financial Risk*, 2nd edition, McGraw Hill, New York, 2001.

10. Kolb W. R.; *Futures, options and swaps*, fourth edition, Blackwell Publishing, Cornwall, 2003.
11. Lewis, E.E.; *Methods of Statistical Analysis in Economics and Business*, Houghton Miflin Company, Boston, 1973.
12. Markowitz, H.M.; “Portfolio Selection”, *Journal of Finance*, Vol. 7, No. 1, 1952.
13. Markowitz H.M.; “Foundations of Portfolio Theory”, *The Journal of Finance*, Vol. 44, No. 2, 1991.
14. Šošić, I., Serdar, V.; *Uvod u statistiku*, Školska knjiga-Zagreb, Zagreb, 1994.
15. Vince, R.; *Portfolio Management Formulas*, John Wiley & Sons, New York, 1992.
16. Vince, R.; *The mathematics of money management: risk analysis techniques for traders*, John Wiley & Sons, New York, 1992.
17. Wilmott, P., Howison, S., Dewynne J.; *The Mathematics of Financial Derivatives*, University of Cambridge, 1995.
18. Thomas P. Fitch; *Dictionary of Banking Terms*, 4th edition, Barron’s, New York, 2000.
19. “Odluka o načinu i uvjetima pod kojima rezidenti ulažu u strane vrijednosne papire i udjele u stranim investicijskim fondovima”, *Narodne novine*, broj 146/2003.

Saša Žiković[1]

INTEGRACIJA OPTIMALNE KOLIČINE TRGOVANJA KAO OSNOVNE SASTAVNICE ZA UPRAVLJANJE OPTIMIZIRANIM PORTFOLIJOM

SAŽETAK

Autor u ovom radu obrađuje glavne razloge za korištenje optimalne količine trgovanja u kombinaciji s Markowitz-evom modernom portfolio teorijom. U uvodnom dijelu rada, autor prikazuje primjer izračuna optimalnih udjela vrijednosnica unutar portfolija koristeći se nelinearnim programiranjem, nakon čega slijedi objašnjenje osnova koncepta optimalne količine trgovanja. Korištenje optimalne količine trgovanja nije ograničeno na Bernoullijeve sustave, nego se može koristiti i pri trgovanju dionicama, futuresima, opcijama itd. Optimalna količina trgovanja naglašava dva često zanemarena aksioma: (1) sustav s negativnom matematičkom nadom ne može se transformirati u sustav s pozitivnom matematičkom nadom, (2) korištenjem količina trgovanja različitih od optimalne, investitor može sustav s pozitivnom matematičkom nadom pretvoriti u sustav s negativnom matematičkom nadom. Optimalna količina trgovanja predstavlja onu količinu koja maksimizira geometrijsku sredinu (funkciju rasta) pojedinog sustava. Pri utvrđivanju optimalne količine za jednostavne sustave, s malim brojem mogućih rezultata, mogu se koristiti Kelly formule. Za složenije sustave, potrebno je koristiti se maksimizacijom vjerojatnosti.

Ključne riječi: *optimalni portfolio, trgovanje, optimalna količina trgovanja, teorija igara*

[1] Mr.sc., asistent Ekonomskog fakulteta Sveučilišta u Rijeci