FINITE $p$-GROUPS ALL OF WHOSE PROPER SUBGROUPS HAVE ITS DERIVED SUBGROUP OF ORDER AT MOST $p$

Zvonimir Janko
University of Heidelberg, Germany

Abstract. We give in Theorem 7 a complete characterization of the title groups.

Here we give a complete characterization of the title groups. This result is important for the structure theory of finite $p$-groups and also it solves the Problem 39 stated by Y. Berkovich in [1]. In the proofs we use partly some ideas of J. Q. Zhang and X. H. Li ([5, Proposition 3]) and V. Čepulić and O. Pylyavska ([4, Proposition 5]). To facilitate the proof of the main result (Theorem 7), we shall first prove some auxiliary results.

Our notation is standard (see [1]) and we consider here only finite $p$-groups.

Proposition 1. Let $G$ be a title group. Then for all $x, y \in G$ such that $\langle x, y \rangle < G$ we have $o([x, y]) \leq p$ and $[x, y] \in Z(G)$.

Proof. Suppose that $[x, y] \neq 1$. Let $X$ be a maximal subgroup of $G$ containing $\langle x, y \rangle$. Then $X' = \langle [x, y] \rangle \leq G$ with $o([x, y]) = p$ and so $[x, y] \in Z(G)$.

Proposition 2. If $G$ is a title group, then $G'$ is abelian of order $\leq p^3$.

Proof. We may assume that $G$ is nonabelian. Let $X \neq Y$ be two maximal subgroups of $G$. Then $|X'| \leq p$ and $|Y'| \leq p$. By a result of A. Mann (Exercise 1.69(a) in [1]), $|G' : (X'Y')| \leq p$ and so $|G'| \leq p^3$. If $G'$ would be nonabelian, then $|G'| = p^3$ and $Z(G')$ (being of order $p$) is cyclic and so (by an elementary result of W. Burnside, see Lemma 1.4 in [1]) $G'$ is also cyclic, a contradiction. Hence $G'$ is abelian.

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Proposition 3 (Zhang and Li). If $G$ is a title group and $|G'| \geq p^2$, then $d(G) \leq 3$.

Proof. Assume that $|G'| \geq p^2$. Then $G$ is not minimal nonabelian and so there exists a maximal subgroup $A$ with $|A'| = p$ and we have $A' \triangleleft G$. Suppose that $M' \leq A'$ for each maximal subgroup $M$ of $G$. Then $G/A'$ is minimal nonabelian. But then $d(G/A') = 2$, $A' \leq \Phi(G)$ and so $d(G) = 2$ and we are done in this case.

We may assume that $G$ has a maximal subgroup $B$ such that $B' \not\leq A'$. We get $|A'| = |B'| = p$ and $A' \cap B' = \{1\}$. Let $a_1, a_2 \in A$ and $a_3, a_4 \in B$ be such that $A' = \langle [a_1, a_2] \rangle$ and $B' = \langle [a_3, a_4] \rangle$. Since $|[a_1, a_2, a_3, a_4]|' \geq p^2$, we get $[a_1, a_2, a_3, a_4] = G$ and so $d(G) \leq 4$.

We assume, by a way of contradiction, that $d(G) = 4$. By Proposition 1, for any $x, y \in G$ we have $o([x, y]) \leq p$ and $[x, y] \in Z(G)$. This implies that $G'$ is elementary abelian and $G' \leq Z(G)$. In particular, $G$ is of class 2.

For any $k \in \{1, 2\}$ and $l \in \{3, 4\}$, we have $(a_1, a_2, al) < G$ and $(a_k, a_3, a_4) < G$ and so $(a_1, a_2, al)' = \langle [a_1, a_2] \rangle$ and $(a_k, a_3, a_4)' = \langle [a_3, a_4] \rangle$. It follows that $[a_k, a_l] \in \langle [a_1, a_2] \rangle \cap \langle [a_3, a_4] \rangle = \{1\}.$

This implies $[a_1, a_2, a_3] = [a_1, a_2][a_1, a_3] = [a_1, a_2]$ and $[a_2, a_3, a_4] = [a_2, a_4][a_3, a_4] = [a_3, a_4]$.

But then $(a_1, a_2, a_3, a_4)$ is a proper subgroup of $G$ and we have $|(a_1, a_2, a_3, a_4)'| \geq p^2$, a contradiction. Our proposition is proved. 

Proposition 4 (Y. Berkovich). Suppose that $G$ is a nonabelian $p$-group. If $d(G) = 2$, then $H' < G'$ for each $H < G$.

Proof. Let $R < G'$ be a $G$-invariant subgroup of index $p$ in $G'$. Then $|G/R'| = p$ and $d(G/R) = 2$. This implies that $G/R$ is minimal nonabelian. For each maximal subgroup $H$ of $G$, $H' \leq R < G'$ and we are done.

Proposition 5 (Čepulić and Pylyavskaya). Let $G$ be a title $p$-group with $p > 2$. Then for any $a, b \in G$, we have $[a^p, b] = [a, b]^p = [a, b]^p$.

Proof. We set $g = [a, b]$. If $g$ commutes with $a$, then for each $n \geq 1$ we prove by induction that $[a^n, b] = [a, b]^n$. Indeed, for $n > 1$,

$$[a^n, b] = [a^{n-1}, b] = [a, b]^{n-1} [a^{n-1}, b] = [a, b][a^{n-1}, b] = [a, b][a, b]^{n-1} = [a, b]^n.$$ 

In particular, we have $[a^p, b] = [a, b]^p$.

We assume now that $[a, b] = z \neq 1$. Since $\langle g, a \rangle < G$, Proposition 1 implies that $o(z) = p$ and $z \in Z(G)$. We note that

$$g^a = a^{-1} a^g = g (g^{-1} a^{-1} g a) = g[a, a] = g z$$ and so $g^n = g z^n.$
for all \( i \geq 1 \). We have

\[
[a^p, b] = [a \cdot a^{-1}, b] = [a, b]^{a^{-1}} [a^{p-1}, b] = [a, b]^{a^{-1}} [a \cdot a^{p-2}, b]
\]

and so continuing we get finally:

\[
[a^p, b] = [a, b]^{a^{-1}} [a, b]^{a^{p-2}} \cdots [a, b]^2 [a, b]^3
\]

of which we may use Lemma 1.1 in [1] since \( \Phi(G) \) is cyclic and therefore \( K_3(G) \cong E_{p^2} \). We have \( \mathcal{U}_1(G) \leq Z(G) \) and so \( \Phi(G) = \mathcal{U}_1(G)G' \) is abelian and \( G'/\Phi(G) \cong E_{p^2} \). Also, \( \mathcal{U}_1(G)K_3(G) \leq Z(G) \) and in fact \( \mathcal{U}_1(G)K_3(G) = Z(G) \). Indeed, if \( \mathcal{U}_1(G)K_3(G) < Z(G) \), then \( G/Z(G) \cong E_{p^2} \). But in that case \( G \) has \( p+1 \) abelian maximal subgroups and this implies (Exercise P1 in [3]) \( |G'| = p \), a contradiction. Let \( M \) be any maximal subgroup of \( G \) such that \( |M: \Phi(G)| = p \). Then \( M \) is either abelian or \( Z(G) = Z(M) \) and \( M/Z(M) \cong E_{p^2} \). In the second case we may use Lemma 1.1 in [1] since \( \Phi(G) \) is an abelian maximal subgroup of \( M \). From \( |M| = p[Z(M)]/|M'| \), we get \( |M'| = p \). We have proved that in this case \( G \) has the title property.

Suppose that \( G \) is a \( p \)-group in (d). For any \( x, y \in G \) we have \([x^p, y] = [x, y]^p = 1 \) and so \( \mathcal{U}_1(G) \leq Z(G) \). It follows that \( \Phi(G) = \mathcal{U}_1(G)G' \leq Z(G) \) and \( G'/\Phi(G) \cong E_{p^2} \). Let \( M \) be any maximal subgroup of \( G \) such that \( |M: \Phi(G)| = E_{p^2} \). It follows that \( p+1 \) maximal subgroups of \( M \) which contain \( \Phi(G) \) are abelian. This implies that \( |M'| \leq p \) and we are done. \( \Box \)
Theorem 7. A p-group \( G \) has the property that each proper subgroup of \( G \) has its derived subgroup of order at most \( p \) if and only if one of the following holds:

(a) \( |G'| \leq p \);
(b) \( d(G) = 2, |G'| = p^2 \);
(c) \( p > 2, d(G) = 2, c(G) = 3, G' \cong E_{p^3}, \mathcal{U}_1(G) \leq Z(G) \)

(note that such \( p \)-groups exist. See for example \( \Lambda_2 \)-groups of order \( p^5 \), \( p > 2 \), in Proposition 71.5(b) in [2] );
(d) \( d(G) = 3, c(G) = 2, G' \cong E_{p^3} \) or \( E_{p^2} \). Here we have \( \Phi(G) = Z(G) \).

Proof. If \( G \) is a \( p \)-group in (a), (b), (c) or (d), then Proposition 6 implies \( |H'| \leq p \) for each subgroup \( H < G \).

Suppose that \( G \) is a \( p \)-group all of whose proper subgroups have its derived subgroup of order \( \leq p \). If \( |G'| \leq p \), then we have the groups in part (a) of our theorem. In what follows we assume that \( |G'| \geq p^2 \). By Proposition 2, \( G' \) is abelian of order \( p^2 \) or \( p^3 \). By Proposition 3, we have \( d(G) \leq 3 \).

(i) First assume \( d(G) = 2 \). If \( |G'| = p^2 \), then we have obtained the groups in part (b) of our theorem. In the sequel we shall assume here \( |G'| = p^3 \). By a result of A. Mann (Exercise 1.69(a) in [1]), all \( p + 1 \) maximal subgroups \( M_i \) \((i = 1, 2, \ldots, p + 1)\) of \( G \) are nonabelian, \( |M_i'| = p \) and for any \( i \neq j \) we have \( M_i' \cap M_j' = \{1\} \) so that \( M_i' \times M_j' \cong E_{p^2} \) and \( M_i' \times M_j' \leq Z(G) \). If \( c(G) = 2 \), then \( d(G) = 2 \) would imply that \( G' \) is cyclic, contrary to the existence of the subgroup \( M_i' \times M_j' \cong E_{p^2} \). Hence \( c(G) \geq 3 \). But \( \{1\} \neq K_3(G) = [G, G'] \leq M_i' \times M_j' \leq Z(G) \) and so \( c(G) = 3 \). We set \( E = M_i' \times M_j' = G' \cap Z(G) \cong E_{p^2} \).

Whenever \( a, b \in G \) are such that \( (a, b) = G \), then \( [a, b] \in G' \). For any \( x \in G \) we have \( g^x = ge \) with some \( e \in E \). Then \( g^{x^2} = ge^i \) and so \( g^{x^p} = g \). It follows that \( \mathcal{U}_1(G) \) centralizes \( G' \) and so \( \Phi(G) = \mathcal{U}_1(G)G' \) centralizes \( G' \).

(ii) Now assume \( p > 2 \). Suppose in addition that \( G' \) is not elementary abelian. Then \( E = \mathcal{U}_1(G') \) and set \( \{1\} \neq \mathcal{U}_3(G') = \langle s \rangle < E \) so that \( G'/\langle s \rangle \cong E_{p^2} \). If \( K_3(G) = [G, G'] \cong \langle s \rangle \), then \( G'/\langle s \rangle \) is of class 2 so that \( d(G'/\langle s \rangle) = 2 \) would imply that \( G'/\langle s \rangle \cong (G/\langle s \rangle)' = (G/\langle s \rangle)'' \) is cyclic, a contradiction. Hence there is an element \( c \in G - \Phi(G) \) such that \( g^c = gl \) with \( l = [g, c] \in E - \langle s \rangle \). Let \( d \in G - \Phi(G) \) such that \( [c, d] = G \) so that \( [c, d] = G'' \). By Proposition 5, \( [c, dp] = [c, dp] \equiv s^j \) where \( j \neq 0 \) (mod \( p \)). Consider the maximal subgroup \( C = \langle \Phi(G), c \rangle \). Since \( g, c, dp \in C \), we have \( C' \geq \langle g, c \rangle, [c, dp] = \langle l, s^j \rangle = E \cong E_{p^2} \), a contradiction. We have proved that \( G' \cong E_{p^2} \). For any \( x, y \in G \) we get by Proposition 5, \( [x^p, y] = [x, y]^p = 1 \) and so \( \mathcal{U}_1(G) \leq Z(G) \). We have obtained the groups given in part (c) of our theorem.

(2) It remains to consider the case \( p = 2 \). Assume in addition that \( \{1\} \neq K_3(G) = [G, G'] < E \) and set \( [G, G'] = \langle u \rangle \), where \( u \) is an involution
in $E \leq Z(G)$. Note that $\Phi(G)$ centralizes $G'$ and for each $x \in G - \Phi(G)$ and $y \in G' - E$ we have $y^x = yu'$ with $u' \in \langle u \rangle$. Set $G_0 = C_G(G')$ so that we have $|G : G_0| = |G_0 : \Phi(G)| = 2$. Since $G'/\langle u \rangle$ is of class 2 and $d(G'/\langle u \rangle) = 2$, we have $G'/\langle u \rangle$ is cyclic. Hence if $g \in G' - E$, then $g^2 = v$ is an involution in $E - \langle u \rangle$ and therefore $E = \Omega_1(G') = \langle u, v \rangle$ and $U_1(G') = \langle v \rangle$. Take some elements $a \in G_0 - \Phi(G)$ and $b \in G - G_0$. Then $\langle a, b \rangle = G$ and therefore $[a, b] = h \in G' - E$ with $h^2 = v$, $h^a = h$ and $h^b = hu$. Consider the maximal subgroup $H = \langle \Phi(G), b \rangle$. Since

$$[a^2, b] = [a, b]^a[a, b] = h^a h = h^2 = v \text{ and } [h, b] = u,$$

we get $H' \geq \langle u, v \rangle = E \cong E_4$, a contradiction.

We have proved that $K_3(G) = [G, G'] = E = G' \cap Z(G) \cong E_4$. Let $a, b \in G - \Phi(G)$ be such that $\langle a, b \rangle = G$. Then $g = [a, b] \in G' - E$, $[g, a] = c_1$, $[g, b] = c_2$, where $\langle c_1, c_2 \rangle = E = K_3(G)$. We set $c_3 = c_1c_2$ and get

$$[g, ab] = [g, b][a, g]^b = [g, b][a, g] = c_2 c_1 = c_3.$$

We compute the commutator subgroups of our three nonabelian maximal subgroups $X_1 = \langle \Phi(G), a \rangle$, $X_2 = \langle \Phi(G), b \rangle$ and $X_3 = \langle \Phi(G), ab \rangle$, where we note that we must have $|X_i'| = 2$ for $i = 1, 2, 3$.

Since $[g, a] = c_1$ and

$$[a^2, b] = [a, b]^a[a, b]^b = gg^b = g \cdot gc_2 = g^2 c_2,$$

we have $X_1' = \langle c_1 \rangle$ and so we must have $g^2 c_2 \in \langle c_1 \rangle$. This forces either $g^2 = c_2$ or $g^2 = c_3$.

Since $[g, b] = c_2$ and

$$[a^2, b] = [a, b]^a[a, b] = g^a g = gc_1 \cdot g = g^2 c_1,$$

we have $X_2' = \langle c_2 \rangle$ and so we must have $g^2 c_1 \in \langle c_2 \rangle$. This forces either $g^2 = c_1$ or $g^2 = c_3$. With the above we get exactly $g^2 = c_3$.

Since $[g, ab] = c_3$ and

$$[a^2, ab] = [a, ab]^a[a, ab] = g^a g = gc_1 \cdot g = g^2 c_1,$$

(where we have used the fact that $[a, ab] = [a, b]$) we have $X_3' = \langle c_3 \rangle$ and so we must have $g^2 c_1 \in \langle c_3 \rangle$. But we know that $g^2 = c_3$ and so $g^2 c_1 = c_3 c_1 = c_2 \in \langle c_3 \rangle$, a contradiction. We have proved that such 2-groups do not exist!

(ii) Finally, assume that $d(G) = 3$. For any $x, y \in G$ we have $\langle x, y \rangle < G$ and so Proposition 1 implies that $o([x, y]) \leq p$ and $[x, y] \in Z(G)$. But then $G'$ is elementary abelian (of order $p^2$ or $p^3$) and $G' \leq Z(G)$ and so we have obtained the groups from part (d) of our theorem. For any $a, b \in G$, $[a^p, b] = [a, b]^p = 1$ and so $\Phi(G) \leq Z(G)$. If $Z(G) \not\leq \Phi(G)$, then there is a maximal subgroup $M$ of $G$ such that $G = \langle M, x \rangle$, where $x \in Z(G)$. But then $G' = M'$ and so $G' = 2$, a contradiction. Hence $\Phi(G) = Z(G)$. Theorem 7 is completely proved.
References

[4] V. Čepulíč and O. Pylyavska, Determination of $p$-groups all of whose proper subgroups have a commutator subgroup of order equal or less than $p$ ($p \geq 3$), Naukovi zapysky, Kyjevo 39 (2005), 28–34.

Z. Janko
Mathematical Institute
University of Heidelberg
69120 Heidelberg
Germany
E-mail: janko@mathi.uni-heidelberg.de
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