Characteristic classes of vector bundles over $CP(j) \times HP(k)$ and involutions fixing $CP(2m + 1) \times HP(k)^*$

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Abstract. In this paper, we determine the total Stiefel-Whitney classes of vector bundles over the product of the complex projective space $CP(j)$ with the quaternionic projective space $HP(k)$. Moreover, we show that every involution fixing $CP(2m + 1) \times HP(k)$ bounds.

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1. Introduction

In 1962, Steenrod raised to Conner the following questions:

Given a smooth closed manifold $F$ (not necessarily connected), does there exist a non-trivial smooth involution $T$ on a smooth closed manifold $M$ with $F$ as its fixed point set? Can we determine all involutions $(M, T)$ up to bordism for the manifold $F$?

When $F$ is the disjoint union of some spaces, there have been many results, see [3, 4, 7, 8, 11, 12]. But there are few results for the case that $F$ is the product of some spaces, see [6, 10, 13]. We shall particularly be concerned with the case in which $F = CP(2m + 1) \times HP(k)$, where $CP(2m + 1)$ and $HP(k)$ we denote a $(2m + 1)$-dimensional complex projective space and a $k$-dimensional quaternionic projective space, respectively.

From [1], we know that the bordism class of an involution $(M, T)$ with $F$ as its fixed point set is determined by the bordism class of the normal bundle over $F$. To calculate characteristic numbers of the normal bundle over $F = CP(2m+1) \times HP(k)$, we need to know the possible form of the total Stiefel-Whitney classes of vector bundles over it. We have the following theorem:

**Theorem 1.** The total Stiefel-Whitney class of a vector bundle $\xi$ over $CP(j) \times HP(k)$ has the form

$$w(\xi) = (1 + \alpha)^a(1 + \beta)^b(1 + \alpha^2 + \beta)^d(1 + \alpha' \beta^{2^n-2})^e,$$

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where $\alpha \in H^2(CP(j); Z_2)$, $\beta \in H^4(HP(k); Z_2)$ are nonzero classes, $a, b, d$ are non-negative integers, and $\varepsilon = 0$ or 1. When $\varepsilon = 1$, we must have

$$
\begin{align*}
  i &= 2^t(2p + 1), \\
  j &= 2^t(2p + 1) + x, \\
  4k &= 2^s - 2^{t+1}(2p + 1) + y
\end{align*}
$$

By using this result, we prove

**Theorem 2.** Every involution fixing $CP(2m + 1) \times HP(k)$ bounds.

The paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we discuss the existence of involutions fixing $CP(2m + 1) \times HP(k)$ and prove Theorem 2.

### 2. Characteristic classes of vector bundles

Let

$$H^*(CP(j) \times HP(k); Z) = Z[\alpha]/\alpha^{j+1} \times Z[\beta]/\beta^{k+1},$$

where $\alpha \in H^2(CP(j); Z_2)$, $\beta \in H^4(HP(k); Z_2)$ are generators. For convenience, we also denote generators of $H^2(CP(j); Z_2)$, $H^4(HP(k); Z_2)$ by $\alpha$, $\beta$.

Let $P_1 : CP(j) \times HP(k) \rightarrow CP(j)$, $P_2 : CP(j) \times HP(k) \rightarrow HP(k)$ be the projection maps. We have a complex line bundle $P_1^*(L_\alpha)$ over $CP(j) \times HP(k)$, which is the pullback of the canonical complex line bundle $L_\alpha$ over $CP(j)$ with the total Chern class $c(P_1^*(L_\alpha)) = 1 + \alpha$, and a 2-dimensional complex bundle $P_2^*(L_\beta)$ over $CP(j) \times HP(k)$, which is the pullback of the canonical quaternionic line bundle $L_\beta$ over $HP(k)$ with total Chern class $c(P_2^*(L_\beta)) = 1 + \beta$.

**Lemma 1.** The total Chern class of the bundle $P_1^*(L_\alpha) \otimes P_2^*(L_\beta)$ over $CP(j) \times HP(k)$ is $c(P_1^*(L_\alpha) \otimes P_2^*(L_\beta)) = 1 + 2\alpha + \alpha^2 + \beta$.

**Proof.** We define a map $i_1 : CP(j) \rightarrow CP(j) \times HP(k)$ by $i_1(x) = (x, pt_1)$, $x \in CP(j)$ and a map $i_2 : HP(k) \rightarrow CP(j) \times HP(k)$ by $i_2(x) = (pt_2, x)$, $x \in HP(k)$, where $pt_1 \in HP(k)$, $pt_2 \in CP(j)$ are fixed points. Thus

$$
P_1i_1 : CP(j) \rightarrow CP(j) \text{ is the identity on } CP(j),
$$

$$
P_2i_2 : HP(k) \rightarrow HP(k) \text{ is the identity on } HP(k).
$$

So we have

$$
(P_1i_1)^*(L_\alpha) = i_1^*P_1^*(L_\alpha) = L_\alpha
$$

(1)

and

$$
(P_2i_2)^*(L_\beta) = i_2^*P_2^*(L_\beta) = L_\beta.
$$

(2)

From (1), we have

$$
i_1^*(c(P_1^*(L_\alpha) \otimes P_2^*(L_\beta))) = c(i_1^*P_1^*(L_\alpha) \otimes i_1^*P_2^*(L_\beta)) = c(L_\alpha \otimes C^2)
$$

$$
= c(L_\alpha \otimes (C \oplus C)) = c(L_\alpha \oplus L_\alpha) = 1 + 2\alpha + \alpha^2,
$$

(3)
where $C^i$ is an $i$-dimensional trivial complex bundle over $CP(j)$. Similarly, from (2) we have $i_2^*(c(P_1^*(L_\alpha) \otimes P_2^*(L_\beta))) = c(L_\beta) = 1 + \beta$.

Let $c(P_1^*(L_\alpha) \otimes P_2^*(L_\beta)) = 1 + \varepsilon_0 \alpha + \varepsilon_1 \alpha^2 + \varepsilon_2 \beta$. Then

$$i_1^*(c(P_1^*(L_\alpha) \otimes P_2^*(L_\beta))) = i_1^*(1 + \varepsilon_0 \alpha + \varepsilon_1 \alpha^2 + \varepsilon_2 \beta) = 1 + \varepsilon_0 \alpha + \varepsilon_1 \alpha^2.$$

From (3), $\varepsilon_0 = 2$ and $\varepsilon_1 = 1$. Similarly, we have $\varepsilon_2 = 1$.

**Lemma 2.** There is a 4-dimensional real vector bundle $\eta$ over $CP(j) \times HP(k)$ such that the total Stiefel-Whitney class $w(\eta) = 1 + \alpha^2 + \beta$.

**Proof.** Consider the 2-dimensional complex bundle $P_1^*(L_\alpha) \otimes P_2^*(L_\beta)$ as a real bundle. Let $\eta$ be the real bundle. It follows from Lemma 2.1 that

$$w(\eta) = c(P_1^*(L_\alpha) \otimes P_2^*(L_\beta)) \mod 2 = 1 + \alpha^2 + \beta.$$

**Lemma 3.** Let the total Stiefel-Whitney class of a vector bundle $\xi$ be $w(\xi) = 1 + w_2 + \text{higher terms.}$ Then $w_{2^s+1} = 0$ and $Sq^i w_{2^s} = 0$ for $0 < l < 2^{s-1}$, where $Sq^i$ is the Steenrod operation.

**Proof.** If $0 < l < 2^{s-1}$, then $w_{2^s-1+l} = 0$. Using the Wu formula

$$Sq^i w_j = \sum_{t=0}^{j} \left( \begin{array}{c} j - i - 1 + t \end{array} \right) w_{i-t} w_{j+t} \quad \text{for } i < j,$$

we have that for $0 < l < 2^{s-1}$,

$$0 = Sq^{2^s-1} w_{2^s-1+l} = \sum_{t=0}^{2^s-1} \left( \begin{array}{c} l - 1 + l \end{array} \right) w_{2^s-1-t} w_{2^s-1+l+t}$$

$$= \left( \begin{array}{c} 2^{s-1} + l - 1 \end{array} \right) w_0 w_{2^s+l} = w_{2^s+l}.$$

Then $Sq^i w_{2^s} = \left( \begin{array}{c} 2^s - 1 \end{array} \right) w_0 w_{2^s+l} = 0$.

**Proof of Theorem 1.** Let $P_1^*(L_\alpha)$, $P_2^*(L_\beta)$ as above. Consider $P_1^*(L_\alpha)$ and $P_2^*(L_\beta)$ as real bundles. We have $w(P_1^*(L_\alpha)) = 1 + \alpha$ and $w(P_2^*(L_\beta)) = 1 + \beta$. We write $a\xi$ for $\xi \oplus \cdots \oplus \xi$ and $\zeta = \xi - \eta$ for $\xi \oplus \eta = \xi$.

If $w(\xi) = 1 + a_1 \alpha + \text{higher terms}$, then we have $w(\xi - a_1 P_1^*(L_\alpha)) = 1 + a_2 \alpha^2 + b_1 \beta + \text{higher terms}$. Since $w(2P_1^*(L_\alpha)) = 1 + \alpha^2$, $w(\xi - a_1 P_1^*(L_\alpha) - 2a_2 P_1^*(L_\alpha) - b_1 P_2^*(L_\beta)) = 1 + w_s + \text{higher terms}$. We have

$$w(4P_1^*(L_\alpha)) = 1 + \alpha^4, w(2P_2^*(L_\beta)) = 1 + \beta^2, w(\eta) = 1 + \alpha^2 + \beta.$$
and
\[ w(2P_1^*(L_\alpha) + P_2^*(L_\beta) - \eta) = \frac{(1 + \alpha^2)(1 + \beta)}{1 + \alpha^2 + \beta} = 1 + \alpha^2 \beta + \text{higher terms.} \]

By subtracting multiples of these bundles, we may obtain a sum \( \theta \) of vector bundles such that \( w(\xi - \theta) = 1 + w_{16} + \text{higher terms.} \) Proceeding inductively, we may suppose that there is a sum \( \theta' \) of vector bundles such that \( w(\xi - \theta') = 1 + w_{2r} + \text{higher terms.} \)

Since
\[ w(2^{s-1}P_1^*(L_\alpha)) = 1 + \alpha^{2^{s-1}}, \quad w(2^{s-2}P_2^*(L_\beta)) = 1 + \beta^{2^{s-2}} \]
and
\[ w(2^{s-3}(P_1^*(L_\alpha) + P_2^*(L_\beta) - \eta)) = 1 + \alpha^{2^{s-2}} \beta^{2^{s-3}} + \text{higher terms,} \]
we may also suppose that \( w_{2r}(\xi - \theta') \) is a sum of monomials \( \alpha^i \beta^{2^{r-2i}} \) with \( i \neq 0, 2^{r-2}, 2^{r-1} \). Among all such monomials we may suppose that the values of \( i \) are all divisible by \( 2^r (2 \leq 2^r < 2^{s-2}) \) with at least one odd multiple of \( 2^i \) occurring. If a monomial \( \alpha^i \beta^{2^{r-2i}} \) with \( h = 2^i (2p + 1) \) occurs, then we have
\[ Sq^{2^{r+1}}(\alpha^i \beta^{2^{r-2i}}) = \left( \frac{h}{2^i} \right) \alpha^{2^i} \beta^{2^{r-2i}} + \left( \frac{2^{r-2i}}{2^{i-1}} \right) \alpha^i \beta^{2^{r-2i} + 2^{r+i-1}} \]
\[ = \alpha^{2^i} \beta^{2^{r-2i}} + \alpha^i \beta^{2^{r-2i + 2^{r+i-1}}} \]
If \( h \) is an even multiple of \( 2^i \), then \( Sq^{2^{r+1}}(\alpha^i \beta^{2^{r-2i}}) = 0 \). Thus, we have
\[ 0 = Sq^{2^{r+1}} w_{2r} = \sum_h Sq^{2^{r+1}}(\alpha^i \beta^{2^{r-2i}}) = \sum_{h = 2r (\text{odd})} (\alpha^{2^i} \beta^{2^{r-2i}} + \alpha^i \beta^{2^{r-2i + 2^{r+i-1}}}). \]

However if \( h, h' \) are odd multiples of \( 2^i \) and \( h \neq h' \), then
\[ \alpha^{2^i} \beta^{2^{r-2i}} \neq \alpha^{2^i} \beta^{2^{r-2i}} \quad \text{and} \quad \alpha^i \beta^{2^{r-2i + 2^{r+i-1}}} \neq \alpha^i \beta^{2^{r-2i + 2^{r+i-1}}}. \]
\[ i.e. \text{ cancellation does not occur among } Sq^{2^{r+1}}(\alpha^i \beta^{2^{r-2i}}) \text{ and } Sq^{2^{r+1}}(\alpha^i \beta^{2^{r-2i}}). \]
So, if \( w_{2r} \) is nonzero, there must be a monomial \( \alpha^i \beta^{2^{r-2i}} \) with \( i = 2^r (2p + 1) \) for which \( \alpha^{i+2^i} \beta^{2^{r+2^{r+i}}} \) and \( \alpha^i \beta^{2^{r-2i + 2^{r+i-1}}} \) are zero. For \( \alpha^i \beta^{2^{r-2i}} \) to be nonzero, we have \( i \leq j \) and \( 2^{r-2i} \leq k \). We must have \( j < i + 2^i \) and \( k < 2^{r-2i+2^{r+i-1}} \) so that \( \alpha^{i+2^i} \beta^{2^{r+i}} \) and \( \alpha^i \beta^{2^{r-2i + 2^{r+i-1}}} \) are zero. Since every other monomial in \( w_{2r} \) is of the form \( \alpha^i \beta^{2^{r-2i}} \) with \( h \) divisible by \( 2^i \) and \( h \neq i \), then either \( h > i \) or \( \frac{2^{r-2i}}{2^i} > \frac{2^{r-2i}}{2^i} \), and so the monomials are zero. Thus \( w_{2r} = \alpha^i \beta^{2^{r-2i}} \) and
\[
\begin{cases}
i = 2^r (2p + 1), & t \geq 1, \\
j = 2^r (2p + 1) + x, & 0 \leq x < 2^i, \\
k = 2^{r-2i+1} (2p + 1) + y, & 0 \leq y < 2^{r+i}.
\end{cases}
\]
From Lemma 3, we have \( w_{2^l+t} = 0 \) for \( 0 < l < 2^{s-1} \). For \( l \geq 2^{s-1} \), suppose that \( w_{2^l+t} \) contains a monomial \( \alpha^u \beta^v \) with \( 2u + 4v = 2^s + l \geq 2^s + 2^{s-1} \). If \( u \geq i + 2^l \), then \( u > j \). If \( u < i + 2^l \), then

\[
v \geq \frac{2^s + 2^{s-1} - 2u}{4} > \frac{2^s + 2^{s-1} - 2i - 2^{l+1}}{4} \geq \frac{2^s - 2i + 2^{l+1}}{4} > k.
\]

For both cases we have \( \alpha^u \beta^v = 0 \). So \( w_{2^l+t} = 0 \) for \( l > 0 \).

The proof is completed. \( \square \)

**Corollary 1.** If \( \nu \) is a non-bounding vector bundle over \( CP(2m+1) \times HP(k) \) with the total Stiefel-Whitney class \( w(\nu) = (1 + \alpha)^{2m+2}(1 + \beta)^{2^{k+1}} \), then \( a \) is odd.

**Proof.** \( \nu \) has a nonzero characteristic number because it is non-bounding. A nonzero characteristic number must contain the monomial \( \alpha^{2m+1} \beta^k \). Since the total Stiefel-Whitney class of \( CP(2m+1) \times HP(k) \) is of the form \( w = (1 + \alpha)^{2m+2}(1 + \beta)^{2^{k+1}} \) which contains only even powers of \( \alpha \), the class \( w(\nu) \) must involve an odd power of \( \alpha \). By Theorem 1, we know that \( i \) is even. Thus the odd power of \( \alpha \) can only be given by \( (1 + \alpha)^a \) and \( a \) is odd. \( \square \)

### 3. Existence of involutions and their classification

Since \( F = CP(2m+1) \times HP(k) \) bounds, there exists a manifold \( V^{4m+2+4k+1} \) such that \( CP(2m+1) \times HP(k) = \partial V \). Let \( \xi^r \to V \) be the \( r \)-dimensional trivial bundle over \( V \). If \( \nu^r \) is the boundary of \( \xi^r \to V \), then the disc bundle \( D\xi^r \) has the boundary \( D\nu^r \cup S\xi^r \). Multiplying the fibers of \( \xi^r \) by \(-1\) induces an involution on \( D\xi^r \). The restriction on \( S\xi^r \) of the involution is free and on \( D\nu^r \) is to multiply the fibers by \(-1\), so it fixes the zero section, which is \( CP(2m+1) \times HP(k) \). The normal bundle over \( CP(2m+1) \times HP(k) \) is \( \nu^r \). Thus there is a bounding involution \( (M^{4m+2+4k+1}, T) \) fixing \( CP(2m+1) \times HP(k) \) for every \( r \geq 0 \). However, we are interested in whether there is a non-bounding involution fixing \( CP(2m+1) \times HP(k) \).

Let us recall some results about the bordism of involutions. Suppose that \( (M, T) \) is a closed manifold \( M \) with involution \( T \) and the fixed point set of \( T \) is \( F = CP(2m+1) \times HP(k) \). Let \( \nu \) denote the normal bundle of \( F \) in \( M \). From [1] we know that the bordism class of \( (M, T) \) is determined by the bordism class of the bundle \( (F, \nu) \). Further, the real projective space bundle \( RP(\nu) \) bounds in the bordism of \( RP^\infty \), where the map into \( RP^\infty \) classifies the double cover of \( RP(\nu) \) by the sphere bundle \( S(\nu) \).

The mod 2 cohomology of \( CP(2m+1) \times HP(k) \) is

\[
H^*(CP(2m+1) \times HP(k); Z_2) = Z_2[\alpha, \beta]/(\alpha^{2m+2} = \beta^{k+1} = 0),
\]

where \( \alpha \) is the 2-dimensional class coming from \( CP(2m+1) \) and \( \beta \) is the 4-dimensional class coming from \( HP(k) \). The total Stiefel-Whitney class of \( CP(2m+1) \times HP(k) \) is

\[
w = (1 + \alpha)^{2m+2}(1 + \beta)^{k+1}.
\]
Let
\[ u = 1 + u_1 + u_2 + \cdots + u_r \in H^*(CP(2m + 1) \times HP(k); Z_2) \]
denote the total Stiefel-Whitney class of \( \nu^r \). Then the cohomology of \( RP(\nu^r) \) is
\[ Z_2[\alpha, \beta, c]/(\alpha^{2m+2} = \beta^{k+1} = 0; \ c^r + u_1 c^{r-1} + u_2 c^{r-2} + \cdots + u_r = 0) \]
and the total Stiefel-Whitney class of \( RP(\nu^r) \) is
\[
w(RP(\nu^r)) = w\{(1 + c)^r + u_1 (1 + c)^{r-1} + \cdots + u_r\} = (1 + \alpha)^{2m+2} (1 + \beta)^{k+1} \{(1 + c)^r + u_1 (1 + c)^{r-1} + \cdots + u_r\},\]
where \( c \in H^1(RP(\nu^r); Z_2) \) is the Stiefel-Whitney class of the double cover of \( RP(\nu^r) \) by \( S(\nu^r) \) (see [1, p. 75]).
The class of \( RP(\nu^r) \) in the bordism of \( RP^\infty \) is determined by the characteristic numbers
\[ w_{i_1}(RP(\nu)) \cdots w_{i_s}(RP(\nu)) c^l [RP(\nu)], \]
where \( i_1 + \cdots + i_s + t = \dim RP(\nu^r) = 4m + 2 + 4k + r - 1 \). In order to find the value of such numbers, we have a formula of Conner [2, (3.1)]
\[
\alpha^i \beta^j c^l [RP(\nu)] = \alpha^i \beta^j \pi_{4m+2+4k-2i-4j} [CP(2m + 1) \times HP(k)] = \text{coefficient of } \alpha^{2m+1} \beta^k \text{ in } \alpha^i \beta^j \pi_{4m+2+4k-2i-4j},
\]
where \( 2i + 4j + t = 4m + 2 + 4k + r - 1 \) and \( \pi = 1/u \) is the dual Stiefel-Whitney class of \( \nu^r \).

For convenience, we introduce the following characteristic classes which were initially introduced in [9],
\[
w[j] = \frac{w(RP(\nu^r))}{(1 + c)^{r-j}} = w\{(1 + c)^j + u_1 (1 + c)^{j-1} + \cdots + u_j + u_{j+1}(1 + c)^{-1} + \cdots\} = 1 + w[j]_1 + w[j]_2 + \cdots + w[j]_{4m+2+4k+r-1},
\]
for which \( w[j]_1 \) is a polynomial in the classes \( w_i(RP(\nu)) \) and \( c \). These classes satisfy (see [9])
\[
w[i]_{2i} = w_i c^i + \text{terms with smaller powers of } c,
\]
\[
w[i]_{2i+1} = (w_{i+1} + u_{i+1}) c^i + \text{terms with smaller powers of } c,
\]
\[
w[i]_{2i+2} = u_{i+1} c^{i+1} + \text{terms with smaller powers of } c.
\]
In particular,
\[
w[0]_1 = u_1 + w_1,
\]
\[
w[0]_2 = u_1 c + (w_2 + u_1 w_1 + u_2),
\]
\[
w[0]_4 = u_1 c^3 + (w_2 + w_1 u_1) c^2 + (u_3 + w_2 u_1) c + w_4 + w_3 u_1 + w_2 u_2 + w_1 u_3 + u_4.
\]
Suppose that \( (M^{4m+2+4k+r}, T) \) is an involution fixing \( CP(2m + 1) \times HP(k) \). When \( r \geq 4m + 2 + 4k \), from [5] we know that the involution bounds. When \( r=0 \) or
$r=1$, it is not difficult to prove that every involution bounds. Then we assume that $1 < r < 4m + 2 + 4k$.

The proof of Theorem 2 is divided into two cases: (I) $k = 2n$, (II) $k = 2n + 1$.

(I) $k = 2n$

**Proposition 1.** Every involution fixing $CP(2m + 1) \times HP(2n)$ bounds.

**Proof.** If there is a non-bounding involution fixing $CP(2m + 1) \times HP(2n)$, then the normal bundle $\nu^r$ is non-bounding. By Corollary 1, we know that $a$ is odd. Then $u_1 = 0$, $u_2 = \alpha$, $w_2 = 0$ and $w[0]_2 = u_1 c + (w_2 + u_1 w_1 + u_2) = \alpha$. Let $2m + 1 = 2^p(2q + 1) - 1$. By (1), it is not difficult to prove that every involution bounds. Then we assume that $2m + 1 = 2^p(2q + 1) - 1 (p \geq 1, q \geq 0)$. Then

$$w(RP(\nu^r)) = (1 + \alpha)^{2m+1}(1 + \beta)(1 + \beta)^{2n+1}(1 + (1+c)^{r-1} + \cdots + u_r),$$

$$= (1 + \alpha)^{2m+1}(1 + \beta)(1 + (1+c)^{r-1} + \cdots + u_r),$$

where $\alpha^{2p(2q+1)} = 0$ and $\beta^{2n+1} = 0$. Thus $(1 + \alpha)^{2p(2q+1)} = 1 + \alpha^{2p} + \cdots + \alpha^{2p} \beta^2$ and $(1 + \beta)^{2n+1} = 1 + \beta + \cdots + \beta^{2n}$.

If $r$ is odd, then

$$w_{2r+1, 2q+8n+r-1}(RP(\nu^r)) = \alpha^{2p} \beta^{2n}(rc^{r-1} + (r-1)u_1 c^{r-2} + \cdots + u_{r-1})$$

is the top-dimensional class in $w(RP(\nu^r))$, and

$$w[0]_2^{2p-1} w_{2r+1, 2q+8n+r-1}(RP(\nu^r))[RP(\nu^r)] = \alpha^{2m+1} \beta^{2n} (c^{r-1} + (r-1)u_1 c^{r-2} + \cdots + u_{r-1})[RP(\nu^r)]$$

$$= r \alpha^{2m+1} \beta^{2n} c^{r-1}[RP(\nu^r)]$$

$$= r \alpha^{2m+1} \beta^{2n} [CP(2m + 1) \times HP(2n)] = r,$$

which is a nonzero characteristic number. Since we know that $RP(\nu^r)$ bounds, this is a contradiction.

If $r = 4h + 2$, then $\binom{r-2}{2} \equiv 1 \pmod{2}$, $\binom{r-2}{2} \equiv 0 \pmod{2}$ and

$$w[0]_2^{2p-1} w_{2r+1, 2q+8n+r-2}(RP(\nu^r)) c[RP(\nu^r)] = \alpha^{2m+1} \beta^{2n} (c^{r-1} + \cdots + u_{r-2} c)[RP(\nu^r)]$$

$$= \alpha^{2m+1} \beta^{2n} c^{r-1}[RP(\nu^r)]$$

we get a contradiction.

If $r = 4h$, then $\binom{r-2}{2} \equiv 1 \pmod{2}$, $\binom{r-2}{2} \equiv 1 \pmod{2}$ and

$$w[0]_2^{2p-1} w_{2r+1, 2q+8n+r-2}(RP(\nu^r)) c^{3}[RP(\nu^r)] = \alpha^{2m} \beta^{2n} (ac^{r-1} + \cdots + u_{r-2} c^3)[RP(\nu^r)]$$

$$= \alpha^{2m+1} \beta^{2n} c^{r-1}[RP(\nu^r)]$$

we also get a contradiction.

So every involution fixing $CP(2m + 1) \times HP(2n)$ bounds. \qed
(II) $k = 2n + 1$

Suppose that $2m + 1 = 2^p(2q + 1) - 1$ and $2n + 1 = 2^{p'}(2q' + 1) - 1$, where $p \geq 1$, $q \geq 0$, $p' \geq 1$ and $q' \geq 0$. To determine the bordism classification of all involutions fixing $CP(2m + 1) \times HP(2n + 1)$, we explore the conditions under which the bundle with class $u = (1 + \alpha)^n(1 + \beta)^{(1 + \alpha^2 + \beta)^{(1 + \alpha^2 \beta^{2 - 2\alpha})}}$ bounds.

**Lemma 4.** Suppose that $\nu^r$ is the normal bundle of the fixed point set of a non-bounding involution fixing $CP(2m + 1) \times HP(2n + 1)$ with $u = (1 + \alpha)^n(1 + \beta)^{(1 + \alpha^2 + \beta)^{(1 + \alpha^2 \beta^{2 - 2\alpha})}}$.

If $\epsilon = 1$ and $\frac{2\nu - 4i}{4}$ is odd, then $u_{4m + 8n + 4} = \alpha^{2m}\beta^{2n + 1}$ and $u_{4m + 8n + 4} = 0$.

**Proof.** Let $\frac{2\nu - 4i}{4} = 2l - 1$ ($l > 0$). Then $i = 2^{s - 1} - 4l + 2 = 2^{s - 2} - 2l + 1$. By Theorem 1, $2m + 1 = i + 1$ and $8n + 4 = 2^s - 2i$. Thus $i = 2m$ and $\frac{2\nu - 4i}{4} = 2n + 1$. We assert $u'_{4m + 8n + 4} \neq 0$. If $u'_{4m + 8n + 4} = 0$, then $u_{4m + 8n + 4} = \alpha^{2m}\beta^{2n + 1} = 0$. Then $r \geq 4m + 8n + 4$. Since $r < 4m + 2 + 8n + 4$, we have $r = 4m + 8n + 4$ or $r = 4m + 1 + 8n + 4$.

(1) For $r = 4m + 8n + 4$, we have $w = (1 + \alpha)^{2m + 1}(1 + \beta)^{2n + 2}$, $w_1 = w_{r + 1} = w_2 = w_{r + 2} = 0$ and

$$w[r - 1]_{2r} = u_r c^r + u_r w_1 c^{r - 1} + w_{r + 1} c^{r - 1} + u_r w_2 c^{r - 2} + w_{r + 2} c^{r - 2}$$

+ terms with smaller powers of $c$

= $u_r c^r +$ terms with smaller powers of $c$

= $u_r (u_1 c^{r - 1} + u_2 c^{r - 2} + \ldots + u_r)$

+ terms with dimension smaller than $2r$

= $u_r u_2 c^{r - 2} +$ terms with smaller power of $c$

= $\alpha^{2m + 1}\beta^{2n + 1} c^{r - 2} +$ terms with smaller power of $c$.

Then $w[r - 1]_{2r}[RP(\nu^r)] = \alpha^{2m + 1}\beta^{2n + 1} c^{r - 1}[RP(\nu^r)] \neq 0$, which is a contradiction.

(2) For $r = 4m + 1 + 8n + 4$, we have

$$w[r - 2]_{2(r - 1)} = u_{r - 1} c^{r - 1} +$ terms with smaller power of $c$

= $\alpha^{2m}\beta^{2n + 1} c^{r - 1} +$ terms with smaller power of $c$.

So $w[0]_{2} w[r - 2]_{2(r - 1)}[RP(\nu^r)] = \alpha^{2m + 1}\beta^{2n + 1} c^{r - 1}[RP(\nu^r)] \neq 0$, which is a contradiction. Thus $u'_{4m + 8n + 4} \neq 0$, and it contains a monomial $\alpha^i \beta^j$ with $i' \leq 2m + 1$, $j' \leq 2n + 1$ and $2i' + 4j' = 4m + 8n + 4$. Such a monomial must be $\alpha^{2m}\beta^{2n + 1}$. So $u'_{4m + 8n + 4} = \alpha^{2m}\beta^{2n + 1}$ and $u_{4m + 8n + 4} = 0$.

Lemma 4 shows that terms of the form $\alpha^\text{odd}$, $\alpha^\text{odd} \beta^\text{odd}$, $\alpha^\text{odd} \beta^\text{even}$, $\alpha^\text{even} \beta^\text{odd}$ in $u$ can only be given by $u'$.  

**Lemma 5.** If $\nu^r$ is the normal bundle of the fixed point set of a non-bounding involution fixing $CP(2m + 1) \times HP(2n + 1)$ with $u = (1 + \alpha)^n(1 + \beta)^{(1 + \alpha^2 + \beta)^{(1 + \alpha^2 \beta^{2 - 2\alpha})}}$, then $b$ and $d$ are odd.
Proof. If $b$ and $d$ are even, by Lemma 4, $u$ and $w$ contain only even power of $\beta$, where $w$ denotes the total Stiefel-Whitney class of $CP(2m+1) \times HP(2n+1)$. Thus $\nu'$ bounds, which is a contradiction.

By Corollary 1, we know that $a$ is odd. If $b$ is even and $d$ is odd, then
\[
u' = (1 + \alpha)^a(1 + \beta)^b(1 + \alpha^2 + \beta)^d
\]
\[
= (1 + \alpha)^a(1 + \alpha^2 + \beta)(1 + \alpha)^{a-1}(1 + \beta)^b(1 + \alpha^2 + \beta)^{d-1}
\]
\[
= (1 + \alpha + \alpha^2 + \alpha^3 + \beta + \alpha\beta)(\sum \alpha^{\text{even}} \beta^{\text{even}}).
\]

If $b$ is odd and $d$ is even, then
\[
u' = (1 + \alpha)^a(1 + \beta)^b(1 + \alpha^2 + \beta)^d
\]
\[
= (1 + \alpha)(1 + \beta)(1 + \alpha)^{a-1}(1 + \beta)^b(1 + \alpha^2 + \beta)^d
\]
\[
= (1 + \alpha + \beta + \alpha\beta)(\sum \alpha^{\text{even}} \beta^{\text{even}}).
\]

For both cases, we have $w[0]_2 = u_1 c + (u_2 + u_1 w_1 + u_2) = \alpha$ and
\[
w[0]_4 = u_1 c^3 + (u_2 + u_1 w_1)c^2 + (u_3 + w_2 u_1)c + w_4 + w_3 u_1 + w_2 u_2 + w_1 u_3 + u_4
\]
\[
= \alpha c^3 + \epsilon_1 \alpha^2 + \beta,
\]
where $\epsilon_1 = 0$ or 1. Then
\[
w[0]_2^{2m+1} w[0]_4^{2n+1} c^{r-1}[RP(\nu')] = \alpha^{2m+1} \beta^{2n+1} c^{r-1}[RP(\nu')] \neq 0,
\]
which is a contradiction. So $b$ and $d$ are odd.

Lemma 6. Suppose that $\nu'$ is a vector bundle over $CP(2m+1) \times HP(2n+1)$ and the total Stiefel-Whitney class of $\nu'$ has the form $u = (1 + \alpha)^a(1 + \beta)^b(1 + \alpha^2 + \beta)^d(1 + \alpha^i \beta^{2j-2})^c$, for which $a$, $b$ and $d$ are odd. Then for $2m + 1 \geq 5$, $\nu'$ bounds if and only if

(1) $2m + 1 < 2^{p' + 1} - 2$, $2n + 1 = 2^{q'} (2q' + 1) - 1$, where $p' \geq 1$ and $q' \geq 0$,

(2) $2m + 1 < 2^{b' + 1} - 2$, where $b - d = 2^b (2f + 1),$

(3) $\epsilon = 0$ or $\epsilon = 1$ and $2m + 1 \neq 2^{b' + 1} - 1$, where $2^{b'}$ is the largest power of 2 in the common terms of the 2-adic expansions of $2m + 1$ and $8n + 4$.

Proof.
\[
u = (1 + \alpha)^a(1 + \beta)^b(1 + \alpha^2 + \beta)^d(1 + \alpha^i \beta^{2j-2})^c
\]
\[
= (1 + \alpha + \alpha^2 + \alpha^3 + \beta + \alpha\beta)(\sum \alpha^{\text{even}} \beta^{\text{even}})\hat{u},
\]
where $\hat{u} = (1 + \alpha)^{a-1}(1 + \beta)^{b-1}(1 + \alpha^2 + \beta)^{d-1}(1 + \alpha^i \beta^{2j-2})^c$. Since $2m + 1 \geq 5$, we have $i \neq 2$, $u_1 = 0$, $u_2 = \alpha$, $u_3 = 0$, $u_4 = \epsilon_1 \alpha^2$, $u_5 = 0$, $u_6 = \epsilon_3 \alpha^3$, $u_7 = 0$ and $u_8 = \alpha^2 \beta + \epsilon_2 \alpha^4 + \epsilon_5 \beta^2$, where $\epsilon_k = 0$ or 1 ($1 \leq k \leq 2$) and $\epsilon_3 \equiv \binom{b}{3} \binom{d}{3} + bd \equiv \binom{b + d}{2}$ (mod 2).
Since \( w = (1 + \alpha)^{2m+2}(1 + \beta)^{2n+2} = (1 + \alpha)^{2m+2}(1 + \beta^{2^k})^{2n+1} \), we have

\[
w_{2^{2^k}+1} = 0, \quad w_{2^{k}} = \left(\frac{m + 2}{q}ight)^{\zeta} \alpha^{i'} \quad \text{for} \quad i' < 2^{k+1}
\]

\[
w_{2^{2^k}+2} = \beta^{2^k} \left(\frac{m + 2}{2^{k+1}}\right)^{\alpha^{i'+1}}.
\]

Let \( \tilde{w}_{2^{2^k}+2} = w_{2^{2^k}+2} + \left(\frac{m + 2}{2^{k+1}}\right)^{\alpha^{i'+1}} = \beta^{2^k}. \)

If \( 2m + 1 > 2^{k+1} - 2 \), we have

\[
\tilde{w}_{2^{2^k}+2} = (u_8 + \epsilon_2 u_2^2)^{2^{k'}-1} u_2^{2m+1-2(2^{k'}-1)} [CP(2m + 1) \times HP(2n + 1)]
\]

\[
= \beta^{2^{k'}-2^k} \left(\alpha^2 + \epsilon_2 \alpha^4\right)^{2^{k'}-1} u_2^{2m+1-2(2^{k'}-1)} [CP(2m + 1) \times HP(2n + 1)]
\]

which is nonzero. Thus the bundle \( \nu' \) does not bound.

So we suppose \( 2m + 1 < 2^{k+1} - 2 \). The following argument is divided into two cases: (1) \( u_8 = \alpha^2 \beta + \epsilon_2 \alpha^4 \), (2) \( u_8 = \alpha^2 \beta + \epsilon_2 \alpha^4 + \beta^{2^k}. \)

(1) \( u_8 = \alpha^2 \beta + \epsilon_2 \alpha^4 \)

In this case, \( \epsilon_1 \equiv \left(\frac{b + d}{2}\right) \equiv 0 \pmod{2} \), then \( b + d \) is divisible by 4. We write \( b + d = 2^k \). (odd) with \( 2^k \geq 4 \). Then

\[
u'' = \frac{u}{(1 + u_2)^2}
\]

\[
= (1 + \beta)^{b} (1 + \alpha^2 + \beta)^{d} (1 + \alpha^2 \beta^{2^k-1})
\]

\[
= [(1 + \beta)^{2^{k-1}} (1 + \alpha^2 + \beta)]^{d} (1 + \beta)^{b-(2^{k-1})d} (1 + \alpha^2 \beta^{2^k-2})
\]

\[
= [1 + \alpha^2 \beta^{2^k-2} + (\alpha^2 \beta^{2^k-1} + \beta^{2^k})]^{d} (1 + \beta)^{b+d-2^k d}
\]

with \( b + d = 2^k \). (odd) \( -2^k \). (odd) \( \equiv 0 \pmod{2^{k+1}} \).

(i) If \( 2^k > 2^{k'} \), then the characteristic ring of \( \nu'' \) (i.e. the subring of \( H^*(CP(2m + 1) \times HP(2n + 1); \mathbb{Z}_2) \)) generated by the classes \( u_i \) and \( w_i \) contains \( \alpha, \alpha^2 \beta, \cdots, \alpha^2 \beta^{2^{k-1}} \) and \( \beta^{2^{k'}} \). So we have a nonzero characteristic number

\[
(\beta^{2^{k'}} 2^{k'} \alpha^2 \beta^{2^{k-1}}) \alpha^{2m+1-2} [CP(2m + 1) \times HP(2n + 1)].
\]

(ii) If \( 2^k \leq 2^{k'} \), then \( \alpha, \alpha^2 \beta, \cdots, \alpha^2 \beta^{2^{k-1}} + \beta^{2^k} \) are characteristic classes. Let \( 2n + 1 = 2^k + 1 + 2^{k'} \cdot l \). We have a nonzero characteristic number for \( 2m + 1 \geq 5 \)

\[
(\alpha^2 \beta^{2^{k-1}} + \beta^{2^k}) \alpha^2 \beta^{2^k-2} \alpha^2 \beta^{2m+1-4} [CP(2m + 1) \times HP(2n + 1)].
\]
These nonzero characteristic numbers show that the bundle is always non-bounding for \( u_8 = \alpha^2 \beta + \epsilon_2 \alpha^4 \).

(2) \( u_8 = \alpha^2 \beta + \epsilon_2 \alpha^4 + \beta^2 \)

In this case, \( \epsilon_3 \equiv \frac{b+d}{2} \equiv 1 \mod 2 \), so \( b+d \equiv 2 \mod 4 \) and \( b-d \equiv b+d-2d \equiv 2-2 \equiv 0 \mod 4 \). We write \( b-d = 2^t (2f+1) \) with \( 2^t \geq 4 \). Then

\[
u'' = \frac{u}{(1 + u_2)^a} = (1 + \alpha^2 + \alpha \beta + \beta^2) \cdot (1 + \beta \epsilon + (1 + \alpha \beta^{2^t-2}) \epsilon)
\]

\[
u'' = (1 + \alpha^2 + \alpha \beta + \beta^2) \cdot (1 + \beta \epsilon + (1 + \alpha \beta^{2^t-2}) \epsilon).
\]

If \( \epsilon = 0 \) and \( 2^t < 2^p \), the characteristic ring of the bundle is generated by the classes \( \alpha, \alpha^2 \beta + \beta^2 \) and \( \beta^2 \). If \( \epsilon = 0 \) and \( 2^t \geq 2^p \), the characteristic ring of the bundle is generated by the classes \( \alpha, \alpha^2 \beta + \beta^2 \) and \( \beta^2 \).

If \( \epsilon = 1 \), then write \( 2^t < 2m+1 < 2^{t+1} \), where \( 2^t \) is the largest common term of \( 2m+1 \) and \( 8n+4 \) (8n+4 = \( 2^{t+2} 2^q + 2^{t+4} - 4 = 2^{t+2} 2q' + 2^{t-1} + \cdots + 2^{t+1} + 2^{t-1} \cdots + 4 \), \( j \leq n' \)). By Theorem 1, \( 2m+1 = 2^t (2g+1) + x < 2^t+1 \). It forces \( g = 0, i = 2^t \) and \( 8n+4 = 2^t + 2^{t+1} + y = 2^{t-1} + \cdots + 2^{t+1} + y = 2^{t+2} 2q' + 2^{t-1} + \cdots + 2^{t+1} + 2^{t-1} \cdots + 4 \).

Thus \( y = 2^t + \cdots + 4 \), \( 2m+1 = 2^{t-1} + \cdots + 2^{t+1} + 2^t + \cdots + 4 = 2^{t-1} - 4 \) and \( 2m+1 = 2^{t-2} - 2^{t-1} + 2^t - 1 \). If \( 2^t < 2^p \), then the characteristic ring of \( \nu \) is generated by the classes \( \alpha, \alpha^2 \beta + \beta^2 \) and \( \alpha \beta^2 \). If \( 2^t \geq 2^p \), then \( \beta^2 \nu = \beta^2 \nu = 0 \). This characteristic ring is generated by the classes \( \alpha, \alpha^2 \beta + \beta^2 \) and \( \alpha \beta^2 \).

For \( \epsilon = 0 \) or 1, write \( 2m+1 = 2^t - 1 = 2^t l \). If \( 2^t l - 2 < 2m+1 < 2^{t+1} - 2 \), we have

\[
(\alpha^2 \beta + \beta^2) 2^{t-1} (\beta^2 l) \alpha^{2m+1-(2^{t+1}-2)(CP(2m+1) \times HP(2n+1))} \neq 0,
\]

which shows that the bundle is non-bounding.

Now we suppose \( 2m+1 < 2^{t+1} - 2 \).

If the class \( \alpha \beta^{2^t-2} \) is present (i.e. \( \epsilon = 1 \)), then \( \beta^2 \beta^2 y = 0 \), \( \alpha \beta^{2^t-2} = 0 \), \( \alpha \beta^{2^t-2} = 0 \) and \( \alpha \beta^{2^t-2} = 0 \). Since \( 2^t + 2^t - 2^t - 2^t + 2^t - 2^t < 2^t \), we have \( \beta^{2^t} \)
\[
\alpha^{2^t} \beta^{2^t-2} = 0. \]

Only possible characteristic number involving \( \alpha^{2^t} \beta^{2^t-2} \) which could be nonzero would be of the form

\[
\alpha^{2^t} (\beta (\alpha^2 + \beta) \beta^{2^t-2^t-1}) \cdot (CP(2m+1) \times HP(2n+1)),
\]

and the value of this class is the coefficient of \( \alpha^{2m+1-2^t} \beta^{2^t-1} - 1 \) in \( \beta (\alpha^2 + \beta) \beta^{2^t-1} - 1 \), where \( 2^{t'} + 8y' = 4m+2+8n+4-2^{t+1}-(2^{t+2}+2^{t+1}) = 4m-2 \) and \( y' \leq 2^{t-1} - 1 \).

The coefficient is

\[
\left( \frac{y'}{2^{m+1-2^{t'-t}}-1} \right) \equiv \left( \frac{y'}{2^{2^{t-1}-1}} \right) \equiv \left( \frac{y'}{2^{t-1}-1-y'} \right) \mod 2.
\]
It is nonzero if and only if \( y' = 2^{l-1} - 1 \), and in this case \( 2m + 1 = 2^{l+1} - 1 \).

If \( \varepsilon = 0 \), or \( \varepsilon = 1 \) and \( 2m + 1 \neq 2^{l+1} - 1 \), then the characteristic numbers which could be nonzero would involve only polynomials in \( \alpha, \alpha^2 + \beta^2 \) and \( \beta^{2^{k'}} \), where \( 2^{k'} = \min(2^j, 2^{l'}) \). We will show that every characteristic number involving \( \alpha, \alpha^2 + \beta^2 \) and \( \beta^{2^{k'}} \) is zero.

Suppose that there exist some \( \tilde{x}, \tilde{y} \) and \( \tilde{z} \) such that
\[
\alpha^2(\beta(\alpha^2 + \beta))^k \beta^{2^{k'}} \{C P(2m + 1) \times H P(2n + 1)\} = \left( \frac{\tilde{y}}{2^{m+1-\tilde{x}}} \right) \equiv 1 \pmod{2},
\]
where
\[
\begin{cases}
2\tilde{x} + 8\tilde{y} + 2^{k'+2} \cdot \tilde{z} = 4m + 2 + 8n + 4, \\
2\tilde{y} - \frac{2m+1-\tilde{x}}{2} + 2^{k'} \cdot \tilde{z} = 2n + 1.
\end{cases}
\]
If \( \tilde{x} = 2m + 1 \), we have \( 2\tilde{y} + 2^{k'} \cdot \tilde{z} = 2n + 1 \), which is impossible since \( k' \geq 1 \). So \( \tilde{x} < 2m + 1 \) and \( \tilde{x} \) is odd.

Writing \( 2n + 1 = 2^{k'} - 1 + 2^{k'} l \), we have \( \beta^{2^{k'}(l+1)} = 0 \). Thus \( \tilde{z} \leq l \). Recall that \( 2n + 1 < 2^{k'+1} - 2 \), then \( (\alpha^2 + \beta) 2^{k'} = \beta 2^{k'+1} \). Suppose \( \tilde{y} < 2^{k'} \). We have \( 4\tilde{y} < 2^{k'+2} \). From
\[
4\tilde{y} = 4n + 2 + 2m + 1 - 2^{k'+1} \tilde{z} - \tilde{x},
\]
\[
= 2^{k'+1} - 2 + 2^{k'+1} l + 2m + 1 - 2^{k'+1} \tilde{z} - \tilde{x},
\]
\[
= 2^{k'+1} (l - \tilde{z}) + 2^{k'+1} - 2 + 2m + 1 - \tilde{x},
\]
\[
\geq 2^{k'+1} (l - \tilde{z}) + 2^{k'+1},
\]
we know that \( \tilde{z} = l \) and \( 4\tilde{y} = 2^{k'+1} - 2 + 2m + 1 - \tilde{x} \). Thus \( \tilde{y} = 2^{k'-1} + \frac{2m+1-\tilde{x}-2}{4} \).

\[
\left( \frac{2m+1-\tilde{x}}{2} \right) \equiv \left( \frac{\tilde{y}}{2^{k'-1} - 1} \right) \equiv \left( \frac{\tilde{y}}{2^{k'} - 1} \right) \equiv 1 \pmod{2}
\]
implies \( \tilde{y} = 2^{k'-1} \). So \( 2m + 1 = 2^{k'+1} - 2 + \tilde{x} \geq 2^{k'+1} - 2 \), and this is a contradiction. Thus every characteristic number involving \( \alpha, \alpha^2 + \beta^2 \) and \( \beta^{2^{k'}} \) is zero and \( \nu^r \) bounds.

The proof is completed. \( \square \)

**Proposition 2.** For \( 2m + 1 = 2^n - 1 \) and \( 2n + 1 = 2^{l'} - 1 \), every involution fixing \( CP(2m + 1) \times HP(2n + 1) \) bounds.

**Proof.** If \( 2m + 1 = 2^n - 1 \) and \( 2n + 1 = 2^{l'} - 1 \), then \( w = (1 + \alpha^{2^n})(1 + \beta^{2^{l'}}) = 1 \) is odd. So the bordism class of the normal bundle \( \nu^r \) is totally determined by the class \( w \).

By \( R_* \) we denote the characteristic ring of the map of \( RP(\nu^r) \) into \( RP^{\infty} \), i.e. the subring of \( H^*(RP(\nu^r); Z_2) \) generated by \( c \) and the classes \( w_i(RP(\nu^r)) \), where
\[
w(RP(\nu^r)) = (1 + c)^r + u_1 (1 + c)^{r-1} + \cdots + u_r.
\]
Since \( c \in R_* \), we can solve inductively to obtain \( u_i \in R_* \) for \( 1 \leq i \leq r \). So \( R_* \) contains the characteristic ring of \( \nu^r \) (i.e. the classes \( u_1, u_2, \ldots, u_r \)). For every partition \( \omega \) of \( 4m + 2 + 8n + 4 \), we have \( u_\omega [CP(2m + 1) \times HP(2n + 1)] = u_{\omega', c^{r-1}} \nu^r) = 0 \). So \( \nu^r \) bounds. \( \square \)
Let \( M^n \), \( T \) be a smooth involution on a closed \( n \)-dimensional manifold with the fixed point data \((F, \nu) = \bigsqcup (F^{n-r}, \nu')\). If \( f(x_1, \ldots, x_n) \) is a symmetric polynomial over \( \mathbb{Z}_2 \) in \( n \) variables of degree at most \( n \), then

\[
f(x_1, \ldots, x_n)[M^n] = \sum_r \frac{f(1+y_1, \ldots, 1+y_r, z_1, \ldots, z_{n-r})}{\prod (1+y_i)}[F^{n-r}],
\]

where the expressions are evaluated by replacing the elementary symmetric functions \( \sigma_i(x) \), \( \sigma_i(y) \), and \( \sigma_i(z) \) by the Stiefel-Whitney classes \( w_i(M) \), \( w_i(F') \), and \( w_i(F) \), respectively, and taking the value of the resulting cohomology class on the fundamental homology class of \( M \) or \( F \).

**Lemma 7** (See [5]). Let \( \sigma \) be a non-bounding involution fixing a manifold with the fixed point data \((F, \nu) = \bigsqcup (F^{n-r}, \nu')\). If \( f(x_1, \ldots, x_n) \) is a symmetric polynomial in \( \mathbb{Z}_2 \) variables of degree at most \( n \), then

\[
f(x_1, \ldots, x_n)[M^n] = \sum_r \frac{f(1+y_1, \ldots, 1+y_r, z_1, \ldots, z_{n-r})}{\prod (1+y_i)}[F^{n-r}],
\]

where the expressions are evaluated by replacing the elementary symmetric functions \( \sigma_i(x) \), \( \sigma_i(y) \), and \( \sigma_i(z) \) by the Stiefel-Whitney classes \( w_i(M) \), \( w_i(F') \), and \( w_i(F) \), respectively, and taking the value of the resulting cohomology class on the fundamental homology class of \( M \) or \( F \).

**Lemma 8** (See [5, p. 317]). Let \( \sigma_j(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n) \) be the \( j \)-th elementary symmetric function in \( n \) variables. Then

\[
\sigma_j(1+y_1, \ldots, 1+y_r, z_1, \ldots, z_{n-r}) = \sum \frac{(r-p)!_{p+q}}{(j-p)!_{j-p}} \sigma_p(y_1, \ldots, y_r) \sigma_q(z_1, \ldots, z_{n-r}).
\]

**Proposition 3.** For \( 2m+1 \geq 5 \), every involution fixing \( CP(2m+1) \times HP(2n+1) \) bounds.

**Proof.** If there is a non-bounding involution fixing \( CP(2m+1) \times HP(2n+1) \), then the total Stiefel-Whitney class of the normal bundle \( \nu' \) has the form

\[
u = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^c(1+\alpha^2\beta^{2r-2})^d
\]

\[
= (1+\alpha+\alpha^2+\alpha^3+\alpha^4+\beta+\beta^2+\alpha\beta+\alpha\beta^2)\hat{u},
\]

where \( a \), \( b \) and \( d \) are all odd and \( \hat{u} = (1+\alpha)^{a-1}(1+\beta)^{b-1}(1+\alpha^2+\beta)^{c-1}(1+\alpha^2\beta^{2r-2})^d. \) Since \( 2m+1 \geq 5 \), we have \( 2^r \geq 16 \). So \( u_1 = 0, u_2 = \alpha, u_3 = 0, u_4 = \varepsilon_1\alpha^2, u_5 = 0, u_6 = \varepsilon_1\alpha^3, u_7 = 0 \) and \( u_8 = \alpha^2\beta + \varepsilon_2\alpha^4 + \varepsilon_3\beta^2 \), where \( \varepsilon_k = 0 \) or \( 1 \) (\( 1 \leq k \leq 3 \)).

The following argument is divided into two cases: (1) \( u_8 = \alpha^2\beta + \varepsilon_2\alpha^4 \), (2) \( u_8 = \alpha^2\beta + \varepsilon_4\alpha^4 + \beta^2 \).

(1) \( u_8 = \alpha^2\beta + \varepsilon_4\alpha^4 \)

Just as in Lemma 6, write \( b + d = 2^k \) (odd) with \( 2^k \geq 4 \).

(i) If \( 2^k > 2^r \), the characteristic ring of \( \nu' \) contains the classes \( \alpha, \alpha^2, \alpha^2\beta, \alpha^2\beta^2, \alpha^2\beta^{2r-2} \) and \( \beta^2 \).

For \( 2^r = 2 \), we have \( w_1 = w_2 = w_3 = w_5 = w_7 = 0, w_4 = (2m+2)^2\alpha^2, w_6 = (2m+2)^2\alpha^3 \) and \( w_8 = (2m+2)^2\alpha^4 + \beta^2 \). From Lemma 8, we know

\[
\sigma_2(1+y, z) = \binom{r}{2} + \left( \binom{r-1}{1} \right) \sigma_1(y) + \left( \binom{r}{1} \right) \sigma_1(z) + \sigma_2(y) + \sigma_1(y)\sigma_1(z) + \sigma_2(z)
\]

\[
= \binom{r}{2} + \alpha.
\]
Let $\sigma'_2(x) = \sigma_2(x) + (\varepsilon_1 x)$. Then $\sigma'_2(1+y,z) = \sigma_2(1+y,z) + (\varepsilon_1 y) = \alpha$.

\[
\sigma_s(1+y,z) = \sum_{p+q\leq s} \binom{r-p}{8-p-q} \sigma_p(y)
\]

\[
\sigma_q(z) = \binom{r}{8} + \binom{r-1}{7} \sigma_1(y) + \binom{r}{7} \sigma_1(z) + \binom{r-2}{6} \sigma_2(y) + \binom{r-1}{6} \sigma_1(y) \sigma_1(z)
\]

\[
+ \binom{r}{6} \sigma_2(z) + \binom{r-3}{5} \sigma_3(y) + \binom{r-2}{5} \sigma_2(y) \sigma_1(z) + \binom{r-1}{5} \sigma_1(y) \sigma_2(z)
\]

\[
+ \binom{r}{5} \sigma_3(z) + \binom{r-4}{4} \sigma_4(y) + \binom{r-3}{4} \sigma_3(y) \sigma_1(z) + \binom{r-2}{4} \sigma_2(y) \sigma_2(z)
\]

\[
+ \binom{r-1}{4} \sigma_1(y) \sigma_3(z) + \binom{r}{4} \sigma_4(z) + \binom{r-5}{3} \sigma_5(y) + \binom{r-4}{3} \sigma_4(y) \sigma_1(z)
\]

\[
+ \binom{r-3}{3} \sigma_3(y) \sigma_2(z) + \binom{r-2}{3} \sigma_2(y) \sigma_3(z) + \binom{r-1}{3} \sigma_1(y) \sigma_4(z)
\]

\[
+ \binom{r}{3} \sigma_5(z) + \binom{r-6}{2} \sigma_6(y) + \binom{r-5}{2} \sigma_5(y) \sigma_1(z) + \binom{r-4}{2} \sigma_4(y) \sigma_2(z)
\]

\[
+ \binom{r-3}{2} \sigma_3(y) \sigma_3(z) + \binom{r-2}{2} \sigma_2(y) \sigma_4(z) + \binom{r-1}{2} \sigma_1(y) \sigma_5(z)
\]

\[
+ \binom{r}{2} \sigma_5(z) + \binom{r-7}{1} \sigma_7(y) + \binom{r-6}{1} \sigma_6(y) \sigma_1(z) + \binom{r-5}{1} \sigma_5(y) \sigma_2(z)
\]

\[
+ \binom{r-4}{1} \sigma_4(y) \sigma_3(z) + \binom{r-3}{1} \sigma_3(y) \sigma_4(z) + \binom{r-2}{1} \sigma_2(y) \sigma_5(z)
\]

\[
+ \binom{r-1}{1} \sigma_1(y) \sigma_6(z) + \binom{r}{1} \sigma_7(z) + \sigma_s(y) + \sigma_7(y) \sigma_1(z) + \sigma_6(y) \sigma_2(z)
\]

\[
+ \sigma_5(y) \sigma_3(z) + \sigma_4(y) \sigma_4(z) + \sigma_3(y) \sigma_5(z) + \sigma_2(y) \sigma_6(z) + \sigma_1(y) \sigma_7(z) + \sigma_s(z)
\]

\[
= \binom{r}{8} + \binom{r-2}{6} \alpha + \varepsilon_1 \binom{r-4}{4} \alpha^2 + \binom{r}{4} \binom{2m+2}{2} \alpha^2 + \varepsilon_1 \binom{r-6}{2} \alpha^3
\]

\[
+ \binom{r-2}{2} \binom{2m+2}{2} \alpha^3 + \binom{r}{2} \binom{2m+2}{3} \alpha^3 + \alpha^2 \beta + \varepsilon_2 \alpha^4
\]

\[
+ \varepsilon_1 \binom{2m+2}{3} \alpha^4 + \binom{2m+2}{4} \alpha^4 + \beta^2\]

Let

\[
\sigma'_s(x) = \sigma_s(x) + \binom{r}{8} + \binom{r-2}{6} \sigma'_2(x) + \varepsilon_1 \binom{r-4}{4} \sigma'_2(x)^2 + \binom{r}{4} \binom{2m+2}{2} \sigma'_2(x)^2
\]

\[
+ \varepsilon_1 \binom{r-6}{2} \sigma'_2(x)^3 + \binom{r-2}{2} \binom{2m+2}{2} \sigma'_2(x)^3 + \binom{r}{2} \binom{2m+2}{3} \sigma'_2(x)^3
\]

\[
+ \varepsilon_2 \sigma'_2(x)^4 + \varepsilon_1 \binom{2m+2}{2} \sigma'_2(x)^4 + \binom{2m+2}{3} \sigma'_2(x)^4 + \binom{2m+2}{4} \sigma'_2(x)^4.
\]

Then $\sigma'_s(1+y,z) = \alpha^2 \beta + \beta^2$. Taking $f(x) = (\sigma'_s(x))^{2q+1} (\sigma'_2(x))^{2m+1-2}$ with $\deg f = 8(2q'+1) + 2(2m+1-2) = 4m + 2 + 8n + 4 < \dim M = 4m + 2 + 8n + 4 + r$, by
Lemma 7 we have

\[ 0 = f(x)[M] = \frac{(\alpha^2 \beta + \beta^2)^{2^q - 1 + \alpha - 2m + 1 - 2}}{\prod_{i=1}^{s} (1 + y_i)}[CP(2m + 1) \times HP(2n + 1)] = 1, \]

which is a contradiction.

For \( 2^j \leq 4 \), we have

\[ w_{2^j} = \left( \frac{m + 2}{i'} \right)^{2^j} \sigma(w_{2^j} + 1) = 0 \] \( (0 \leq i' < 2^j + 1), \)

\[ w_{2^j + 2} = \left( \frac{2m + 2}{2^j + 1} \right)^{2^j + 1 + \beta + 2^j}, \]

\[ u_{2^j} = \alpha^{2^j + 1 + \beta + 2^j}, \quad u_{2^j + 1} = 0, \quad u_{2^j + 2} = \gamma_1 \alpha^{2^j + 1 + \beta + 2^j + 1}, \]

\[ u_{2^j + 3} = 0 \] \( (2 \leq i' \leq 2^j \). \]

Using the above method, we get \( \sigma'_2(x) \) and \( \sigma'_3(x) \) such that \( \sigma'_2(1 + y, z) = \alpha \) and \( \sigma'_3(1 + y, z) = \alpha \beta \). In the same way, adding a polynomial in \( \sigma'_2(x) \) and \( \sigma'_3(x) \) to \( \sigma_{10}(x) \) to get \( \sigma'_{10}(x) \) such that \( \sigma'_{10}(1 + y, z) = \alpha \beta \), adding a polynomial in \( \sigma'_2(x) \), \( \sigma'_3(x) \) and \( \sigma_{12}(x) \) to \( \sigma_{16}(x) \) to get \( \sigma'_{16}(x) \) such that \( \sigma'_{16}(1 + y, z) = \alpha \beta \), \( \sigma'_2(x), \sigma'_3(x), \sigma'_{12}(x), \sigma'_{16}(x) \) to \( \sigma_{20'}(x) \) to get \( \sigma'_{20'}(x) \) such that \( \sigma'_{20'}(1 + y, z) = \alpha \beta \) and taking

\[ f(x) = (\sigma'_{20'}(x))^{2^j + 1} (\alpha^{2})^{2m + 1 - 2}, \]

from Lemma 7 we get a contradiction.

(ii) If \( 4 \leq 2^k \leq 2^j \), writing \( 2n + 1 = 2^k - 1 + 2^k l \), from Lemma 6 we know that the characteristic ring of \( \nu' \) contains the classes \( \alpha, \alpha^2, \alpha^2 \beta, \ldots, \alpha^2 \beta^2 - 2 \) and \( \alpha^2 \beta^2 - 1 + \beta^2 \). So \( u_{2^k} = \alpha^2 \beta^2 - 2 \) \( + \epsilon \alpha^2 \beta^2 \), \( u_{2^k + 1} = 0, \) \( u_{2^k + 2} = \gamma_1 \alpha^2 \beta^2 - 1 + \epsilon \alpha^2 \beta^2 + 1, \)

\[ u_{2^k + 3} = 0 \] \( (2 \leq i' \leq 2^k - 1) \) and \( u_{2^k + 4} = \alpha^2 \beta^2 - 1 + \beta^2 + \epsilon \alpha^2 \beta^2 - 1 \).

Using the above method, we get a series of symmetric function \( \sigma'_2(x), \sigma'_3(x), \ldots, \sigma'_{2k + 2}(x) \) and \( \sigma'_{2k + 2}(x) \) such that \( \sigma'_2(1 + y, z) = \alpha, \quad \sigma'_3(1 + y, z) = \alpha \beta, \ldots, \quad \sigma'_{2k + 2}(1 + y, z) = \alpha^2 \beta^2 - 2 \) and \( \sigma'_{2k + 2}(1 + y, z) = \alpha^2 \beta^2 - 1 + \beta^2 \). Taking

\[ f(x) = (\sigma'_{2k + 2}(x))^{2^j + 1} (\alpha^{2})^{2m + 1 - 2}, \]

from Lemma 7 we get a contradiction. So \( u_{2^k} = \alpha^2 \beta + \epsilon \alpha^2 \beta \) does not occur.

(2) \( u_{2^k} = \alpha^2 \beta + \epsilon \alpha^2 \beta \)

From Lemma 6, we need to consider the following cases:

(a) \( \epsilon = 1 \) and \( 2m + 1 = 2^{j+1} - 1 \),

(b) \( 2^{j+1} - d = 2^{i'}(2f + 1) \) with \( 2^k \geq 4 \),

(c) \( 2 \leq 2^k \leq 2^i \) and \( 2m + 1 > 2^{j+1} - 2 \).
In the case (a), \( \varepsilon = 1 \) implies \( 2n + 1 = 2^t - 1 \). By Proposition 2 we know that every involution fixing \( CP(2m + 1) \times HP(2n + 1) \) bounds.

In the case (b), \( w_{2i'} = \binom{2m + 2}{i'} \alpha^{i'} \) and \( w_{2i'+1} = 0 \) (0 \( \leq i' < 2^t+1 \)).

\[
u = 2 \frac{m}{m+1} \alpha^{i'} \]

\( \text{where } \lambda_j \text{ and } \epsilon_j \text{ are } 0 \text{ or } 1 \) (4 \( < i' < 2^t+1 \)).

\[
u = 2 \frac{m}{m+1} \alpha^{i'} + \lambda_j \alpha^{j'} + \epsilon_j \alpha^{j'} \]

Let \( \sigma'_2(x) \) and \( \sigma'_6(x) \) as in (1)-(i). Then \( \sigma'_2(1+y, z) = \alpha \) and \( \sigma'_6(1+y, z) = \alpha^2 + \beta^2 \). We can add a polynomial in \( \sigma'_2(x) \) and \( \sigma'_6(x) \) to get \( \sigma'_{2+2}(x) \) such that \( \sigma'_{2+2}(1+y, \zeta) = \beta^2 \). Writing \( 2n + 1 = 2^t - 1 + 2^t l \) and taking

\[
f(x) = \binom{2m+1}{i'}(\alpha^{i'})^{2t-1} \alpha^{i'} \]

from Lemma 7 we get a contradiction.

In the case (c), we have \( w_{2i'} = \binom{2m + 2}{i'} \alpha^{i'} \) \( w_{2i'+1} = 0 \) (0 \( < i' < 2^t+1 \)) and

\[
\nu = 2 \frac{m}{m+1} \alpha^{i'} + \binom{2m + 2}{2^t+1} \alpha^{2^t+1} \]

For \( 2^t = 2 \) and \( r \) odd,

\[
w(\nu) = (1 + \alpha^{2^t})^{2^t+1}(1 + \beta^{2^t})^{2^t+1}\{(1+c)^r + u(1+c)^{r-1} + \ldots + u_r\}
= \alpha^{2^t} \beta^{4tq}(rc^{r-1} + (r - 1)u_1c^{r-2}
+ \ldots + u_{r-1}) + \text{terms with a smaller dimension.}

Then

\[
w_{2^t+1:2^t+16q} = (1 + \alpha^{2^t})^{2^t+1}(1 + \beta^{2^t})^{2^t+1}\{(1+c)^r + u(1+c)^{r-1} + \ldots + u_r\}
= \alpha^{2^t} \beta^{4tq}(rc^{r-1} + (r - 1)u_1c^{r-2} + \ldots + u_{r-1})
\]

is the top-dimensional class in \( w(\nu) \). Since \( w[0]_2 = u_1c + (w_2 + u_1w_1 + u_2) = \alpha \) and

\[
w[0]_2 = u_1c^7 + u_2c^6 + (u_3 + w_2u_1)c^5 + (u_4 + w_2u_2)c^4 + (u_5 + w_4u_1)c^3
+ (u_6 + w_4u_2)c^2 + (u_7 + w_2u_5 + w_4u_3 + w_6u_1)c + u_8 + w_2u_6
+ w_4u_4 + w_5u_2 + w_8
= \alpha^6 + \varepsilon_1 \alpha^2 \alpha^4 + \varepsilon_2 \alpha^4 + \varepsilon_1 \left( \frac{2m + 2}{2} \right) \alpha^4 + \left( \frac{2m + 2}{3} \right) \alpha^4 + \left( \frac{2m + 2}{4} \right) \alpha^4
+ \left( \varepsilon_1 + \binom{2m + 2}{2} \right) \alpha^3 c^2,
\]

we have

\[
w[0]_2 = \alpha^{2^t-3}w_{2^t+1:2^t+16q} - \alpha^{2^t-1} \beta^{4q}(rc^{r-1} + (r - 1)u_1c^{r-2} + \ldots + u_{r-1})w[0]_8[\nu] = r
\]

which is a nonzero characteristic number. We know that \( RP(\nu) \) bounds, so this is a contradiction.
For $2^p = 2$ and $r = 4h + 2$, we have $\binom{r}{2} \equiv 1 \pmod{2}$, $\binom{r-2}{2} \equiv 0 \pmod{2}$ and

$$w[0]^{2p-3}w_{2p+1-2q+16q'-r-2}(RP(\nu'))w[0]c[RP(\nu')] = \alpha^{2m+1}\beta^{2n+1}(\epsilon^{r-1} + \cdots + u_r-2c)[RP(\nu')]$$

$$= \alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu')]$$

$$= 1 \neq 0,$$

which is a contradiction.

For $2^p = 2$ and $r = 4h$, we have $\binom{r}{2} \equiv 0 \pmod{2}$, $\binom{r-2}{2} \equiv 1 \pmod{2}$ and

$$w[0]^{2p-4}w_{2p+1-2q+16q'+r-2}(RP(\nu'))w[0]c^3[RP(\nu')] = \alpha^{2m+1}\beta^{2n+1}(\alpha \epsilon^{r-1} + \cdots + u_r-2c^3)[RP(\nu')]$$

$$= \alpha^{2m+1}\beta^{2n+1}c^{r-1}[RP(\nu')]$$

$$= 1 \neq 0,$$

which is also a contradiction.

For $2^p \geq 4$, we have $u_1 = u_3 = u_5 = u_7 = 0$, $u_2 = \alpha$, $u_4 = \epsilon_1\alpha^2$, $u_6 = \epsilon_2\alpha^2$, $u_8 = \alpha^2\beta + \epsilon_2\alpha^4 + \beta^2$ and $u_{2r} = \delta'_r\alpha^2\beta + \beta^2 + \lambda_r\alpha^2\beta$ ($4 < i' \leq 2^{p+1}$), where $\delta_r'$ and $\lambda_r'$ are 0 or 1.

Using the method in (1), we get the symmetric functions $\sigma_2(x)$, $\sigma_6(x)$ and $\sigma_2^{p+2}(x)$ such that $\sigma_2(1 + y, z) = \alpha^2\beta + \beta$ and $\sigma_2^{p+2}(1 + y, z) = \beta^{2^{p+2}}$. Taking

$$f(x) = (\sigma_2^{p+2}(x))^{2^{p+1}}(\sigma_6(x))^{2^{p-1}}(\sigma_4(x))^{2m+1-(2^{p+1}-2)},$$

from Lemma 7, we get a contradiction.

This completes the proof. □

**Proposition 4.** Every involution fixing $CP(3) \times HP(2n + 1)$ bounds.

**Proof.** If $\epsilon = 1$, then $2n + 1 = 2^p - 1$. By Proposition 2, every involution bounds. Thus we need only to consider the case $\epsilon = 0$, i.e. $u = (1 + \alpha)\alpha^2(1 + \alpha^2 + \beta)(1 + \beta)^b$.

(1) If $\binom{b+1}{2} \equiv 1 \pmod{2}$, then $u_8 = \alpha^2\beta$, $b' = 1 \equiv 2 \pmod{4}$ and $b' + 1 \equiv 0 \pmod{4}$. Let $b' + 1 = 2^k(2f + 1) \ (k \geq 2)$.

$$u = (1 + \alpha)\alpha^2[(1 + \alpha^2 + \beta)(1 + \beta^{2^{k-1}})]^f(1 + \beta^{2^{k+1}})^f$$

$$= (1 + \alpha)\alpha^2[1 + \beta^{2^k} + \alpha^2(1 + \beta)^{2^{k}} - 1][1 + \beta^{2^{k+1}}]$$

$$= (1 + \alpha)\alpha^2(1 + \alpha^2 + \beta + \cdots + \alpha^2\beta^{2^{k-1}} + \beta^{2^k})(1 + \beta^{2^{k+1}}).$$

If $2^k > 2^p$, the characteristic ring of $\nu'$ is generated by $\alpha$, $\alpha^2\beta$, $\alpha^2\beta^2$, $\cdots$, $\alpha^2\beta^{2^{p-1}}$ and $\beta^{2^p}$. Just as (1)-(i) in the proof of Proposition 3, taking

$$f(x) = (\sigma_2^{p+2}(x))^{2^{p+1}}\sigma_4(x)^{2m+1-2},$$
from Lemma 7 we get a contradiction. If \(2^k \leq 2^{p'}\), the characteristic ring of \(\nu^r\) is generated by \(\alpha, \alpha^2, \beta, \alpha \beta^2, \ldots, \alpha^2 \beta^{2k-2}, \alpha \beta^{2k-1} + \beta^{2k}\) and \(\beta^{2^{p'}-1}\). None of these monomials can give a monomial \(\alpha^3 \beta^{2n+1}\), so \(\nu^r\) bounds.

(2) If \(\left(\frac{p'-1}{2}\right) \equiv 0 \pmod{2}\), we write \(b' = 1 - 2^f(2f + 1)\). The characteristic ring of \(\nu^r\) is generated by \(\alpha, \alpha^2, \beta^2, \beta^{2^k}\), where \(k = \min(t, p')\).

For \(2^{p'} = 2\) and \(r\) odd, we have \(\left(\frac{2n+2}{2n}\right) \equiv 1 \pmod{2}\),

\[
\begin{align*}
\quad w(RP(\nu^r)) &= (1 + \alpha^2)(1 + \beta)^{2n+2}\{c r + u_1 (1 + c)^{r-1} + \ldots + u_r\} \\
\quad &= (1 + \beta)^{2n+2} r c^{r-1} (r - 1) u_1 c^{r-2} + \ldots + u_{r-1} \\
\quad &+ \text{terms with a dimension smaller than } r - 1 \\
\quad &= \beta^{2n} r c^{r-1} (r - 1) u_1 c^{r-2} + \ldots + u_{r-1} \\
\quad &+ \text{terms with a dimension smaller than } 8n + r - 1,
\end{align*}
\]

\[
\quad w_{8n+r-1}(RP(\nu^r)) = \beta^{2n} r c^{r-1} (r - 1) u_1 c^{r-2} + \ldots + u_{r-1},
\]

\[
\quad w[0]_{[2]} = u_1 c + (w_2 + u_1 w_1 + u_2) = \alpha,
\]

\[
\quad w[0]_{[8]} = u_1 c^3 + w_2 c^3 + u_3 w_1 c^3 + u_4 w_2 c^3 + (u_5 + w_3 u_1) c^n + (u_6 + w_4 u_2) c^n + (u_7 + w_5 u_3 + w_6 u_1) c^n + w_8 c^n + w_9 u_6 + w_{10} u_2 + w_{11}.
\]

\[
\begin{align*}
\quad &= \alpha c^6 + \varepsilon_1 \alpha^2 c^4 + \alpha^2 \beta + \varepsilon_1 \left(\frac{2m+2}{2}\right) \alpha^4 + \left(\frac{2m+2}{3}\right) \alpha^4 \\
\quad &+ \left(\frac{2m+2}{4}\right) \alpha^4 + \varepsilon_1 \left(\frac{2m+2}{2}\right)
\end{align*}
\]

Let

\[
\begin{align*}
\quad (w[0]_{[8]})' &= w[0]_{[8]} + w[0]_{[2]} c^6 + \varepsilon_1 w[0]_{[2]} c^4 + \varepsilon_1 \left(\frac{2m+2}{2}\right) w[0]_{[2]} c^2 \\
\quad &= \alpha^2 \beta.
\end{align*}
\]

Then \(w[0]_{[2]}(w[0]_{[8]})' w_{8n+r-1}(RP(\nu^r)) [RP(\nu^r)] \neq 0\), which is a contradiction.

For \(2^{p'} = 2\) and \(r = 4h + 2\), we have \(\left(\frac{r}{2}\right) \equiv 1 \pmod{2}\), \(\left(\frac{-1}{2}\right) \equiv 0 \pmod{2}\),

\[
\begin{align*}
\quad w(RP(\nu^r)) &= \beta^{2n} \left(\frac{r}{2}\right) c^{r-2} + \left(\frac{r-1}{2}\right) u_1 c^{r-3} + \left(\frac{r-2}{2}\right) w_2 c^{r-4} + \ldots + u_{r-2} \\
\quad &+ \text{terms with a dimension smaller than } 8n + r - 2,
\end{align*}
\]

\[
\begin{align*}
\quad w_{8n+r-2}(RP(\nu^r)) &= \beta^{2n} \left(\frac{r}{2}\right) c^{r-2} + \left(\frac{r-1}{2}\right) u_1 c^{r-3} + \left(\frac{r-2}{2}\right) w_2 c^{r-4} + \ldots + u_{r-2},
\end{align*}
\]

\[
\begin{align*}
\quad w[0]_{[2]} &= u_1 c + (w_2 + u_1 w_1 + u_2) = \alpha.
\end{align*}
\]
Involution fixing $CP(2m+1) \times HP(k)$

Just as above, we also get $(w[0]_{8})^{n}$ such that $(w[0]_{8})^{n} = \alpha^{2}\beta$. Then

$$w[0]_{2}(w[0]_{8})^{n}w_{8n+r-2}(RP(\nu'))c[RP(\nu')]$$

$$= \alpha^{3}\beta^{2n+1}(\nu^{-1} + \cdots + u_{r-2}c)[RP(\nu')]$$

$$= \alpha^{3}\beta^{2n+1}\nu^{-1}[RP(\nu')] \neq 0,$$

which is a contradiction.

For $2^{p} = 2$ and $r = 4h$, we have $(\nu_{2}) \equiv 0 (\mod 2)$ and $(\nu_{2}) \equiv 1 (\mod 2)$. Then

$$(w[0]_{8})^{n}w_{8n+r-2}(RP(\nu'))c[RP(\nu')]$$

$$= \alpha^{2}\beta^{2n+1}(\alpha\nu^{-1} + \cdots + u_{r-2}c)\nu^{3}[RP(\nu')]$$

$$= \alpha^{2}\beta^{2n+1}\nu^{-1}[RP(\nu')] \neq 0,$$

which is also a contradiction.

So for $2^{p} = 2$, there is no non-bounding involution fixing $CP(3) \times HP(2n+1)$.

For $2^{p} > 2$, suppose that there exist some $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ such that

$$\alpha^{x}((\alpha^{2}\beta + \beta^{2})^{y} \beta^{2k} \nu)[CP(3) \times HP(2n+1)] \neq 0,$$

where $2x' + 8y' + 2^{k+2} \cdot z' = 6 + 8n + 4$, i.e. $x' + 4y' + 2^{k+1} \cdot z' = 3 + 4n + 2$. Then $x'$ is odd. Since $x' \leq 3$, $x' = 1$ or $3$. If $x' = 3$, then $2y' + 2^{k} \cdot z' = 2n + 1$, which is impossible. If $x' = 1$, then $(\nu_{2}) \equiv 1 (\mod 2)$, i.e. $y'$ is odd. Thus $1+2(y' - 1) + 2^{k} \cdot z' = 2n + 1 = 2y'(2y' + 1) - 1$, which is also impossible. So $\nu'$ bounds.

The proof is completed.

**Proposition 5.** Every involution fixing $CP(1) \times HP(2n+1)$ bounds.

**Proof.** In this case, $\alpha^{2} = 0$. From Theorem 1 and Lemma 5, we know that every involution fixing $CP(1) \times HP(2n+1)$ has the total Stiefel-Whitney class $u = (1 + \alpha)(1 + \beta)^{b+d}$, where $b$ and $d$ are odd. So we cannot obtain any odd power of $\beta$ from $u$ and $w$ and every involution fixing $CP(1) \times HP(2n+1)$ bounds.

Combining Propositions 1, 3, 4 and 5 together, we have Theorem 2.

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**References**


