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I_σ-Convergence

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Abstract. In this paper, the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$ of positive integers and corresponding I_σ-convergence were introduced. Furthermore, inclusion relations between I_σ-convergence and invariant convergence also I_σ-convergence and $[V_\sigma]_p$-convergence were given.

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1. Introduction and background

A sequence $x = (x_k)$ is said to be strongly Cesaro summable to the number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0.$$

A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be a Banach limit if

(a) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(b) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$, and

(c) $\phi(x_{n+1}) = \phi(x_n)$ for all $x \in l_\infty$.

A sequence $x \in l_\infty$ is said to be almost convergent to the value L if all of its Banach limits are equal to L. Lorentz [4] has given the following characterization.

A bounded sequence (x_n) is said to be almost convergent to L if and only if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{n+k} = L$$

uniformly in n. $\hat{c}$ denotes the set of all almost convergent sequences.

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Maddox [5] has defined a strongly almost convergent sequence as follows:
A bounded sequence (x_n) is said to be strongly almost convergent to L if and only if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{n+k} - L| = 0$$

uniformly in n. [ĉ] denotes the set of all strongly almost convergent sequences.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if it satisfies conditions (a) and (b) stated above and

(d) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in l_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m*th* iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$. In the case σ is the translation mapping $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences ĉ.

It can be shown that

$$V_\sigma = \{x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L \text{ uniformly in } n\},$$

where ℓ_∞ denotes the set of all bounded sequences.

The set of all such σ mappings will be denoted by $\mathfrak{M}$. Raimi [11] proved that

$$\bigcup\{V_\sigma : \sigma \in \mathfrak{M}\} = \ell_\infty$$

and

$$\bigcap\{V_\sigma : \sigma \in \mathfrak{M}\} = c,$$

where c denotes the set of all convergent sequences.

The following inclusion relation between $\hat{c}$ and V_σ can be written:

$$\{\hat{c}\} \subset \{V_\sigma : \sigma \in \mathfrak{M}\}.$$

Several authors including Raimi [11], Schaefer [14], Mursaleen [8], Savaş [12] and others have studied invariant convergent sequences.

The concept of strongly σ-convergence was defined by Mursaleen in [7]:

A bounded sequence $x = (x_k)$ is said to be strongly $\sigma-$ convergent to L if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0$$

uniformly in n.
In this case we will write $x_k \to L[V_\sigma]$. By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences. In the case $\sigma(n) = n+1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences [ĉ].

Recently, the concept of strong σ-convergence was generalized by Savaş [12] as below

$$[V_\sigma]_p := \{x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0 \text{ uniformly in } n\},$$

where $0 < p < \infty$.

If p = 1, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - L| \ge \epsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The idea of statistical convergence was introduced by Fast [3] and studied by many authors. There is a natural relationship between statistical convergence and strong Cesaro summability [2].

The concept of a σ-statistically convergent sequence was introduced by Nuray and Savaş in [10] as follows:
A sequence $x = (x_k)$ is σ-statistically convergent to L if for every$\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} |\{k \le m : |x_{\sigma^k(n)} - L| \ge \epsilon\}| = 0$$

uniformly in n.

In this case we write $S_\sigma - \lim x = L$ or $x_k \to L(S_\sigma)$ and define

$$S_\sigma := \{x = (x_k) : S_\sigma - \lim x = L, \text{ for some } L\}.$$

2. I_σ-convergence

Definition 1. *Let $A \subseteq \mathbb{N}$ and*

$$s_m := min_n |A \cap \{\sigma(n), \sigma^2(n), ..., \sigma^m(n)\}|$$
$$S_m := max_n |A \cap \{\sigma(n), \sigma^2(n), ..., \sigma^m(n)\}|.$$

If the following limits exist

$$\underline{V}(A) := \lim_{m\to\infty} \frac{s_m}{m}, \quad \overline{V}(A) := \lim_{m\to\infty} \frac{S_m}{m}$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.

In the case $\sigma(n) = n + 1$, this definition gives a definition of uniform density u in [1].

A non-empty subset of I of $P(\mathbb{N})$ is called an ideal on $\mathbb{N}$ if

(i) $B \in I$ whenever $B \subseteq A$ for some $A \in I$,

(ii) $A \cup B \in I$ whenever $A, B \in I$.

An ideal I is called proper if $\mathbb{N} \notin I$. An ideal I is called admissible if it is proper and contains all finite subsets. For any ideal I there is a filter $F(I)$ corresponding to I, given by $F(I) = \{K \subseteq \mathbb{N} : \mathbb{N} \setminus K \in I\}$.

Let $I \subset P(\mathbb{N})$ be a proper ideal in $\mathbb{N}$. The sequence $x = (x_k)$ is said to be I-convergent to L, if for $\epsilon > 0$ the set

$$A_\epsilon := \{k : |x_k - L| \geq \epsilon\}$$

belongs to I. If $x = (x_k)$ is I-convergent to L, then we write $I - \lim x = L$.

A sequence $x = (x_k)$ is said to be I^*-convergent to the number L if there exists a set $M = \{m_1 < m_2 < ...\} \in F(I)$ such that $\lim_{k\to\infty} x_{m_k} = L$. In this case we write $I^* - \lim x_k = L$ (see [3]).

Denote by I_σ the class of all $A \subset \mathbb{N}$ with $V(A) = 0$.

Definition 2. *A sequence $x = (x_k)$ is said to be I_σ-convergent to the number L if for every $\epsilon > 0$*

$$A_\epsilon := \{k : |x_k - L| \geq \epsilon\}$$

belongs to I_σ; i.e., $V(A_\epsilon) = 0$. In this case we write $I_\sigma - \lim x_k = L$. The set of all I_σ-convergent sequences will be denoted by $\mathfrak{I}_\sigma$.

In the case $\sigma(n) = n + 1$, I_σ-convergence coincides with I_u- convergence which was defined in [1]. We can also write

$$\{\mathfrak{I}_u\} \subset \{\mathfrak{I}_\sigma : \sigma \in \mathfrak{M}\},$$

where $\mathfrak{I}_u$ denotes the set of all I_u-convergent sequences.

We can easily verify that if $I_\sigma - \lim x_n = L_1$ and $I_\sigma - \lim y_n = L_2$, then $I_\sigma - \lim(x_n + y_n) = L_1 + L_2$ and if a is a constant, then $I_\sigma - \lim ax_n = aL_1$.

Theorem 1. *Suppose $x = (x_k)$ is a bounded sequence. If x is I_σ-convergent to L, then x is invariant convergent to L.*

Proof. Let $m, n \in \mathbb{N}$ be arbitrary and $\epsilon > 0$. We estimate

$$t(n,m) = \left|\frac{x_{\sigma(n)} + x_{\sigma^2(n)} + ... + x_{\sigma^m(n)}}{m} - L\right|.$$

We have

$$t(n,m) \leq t^{(1)}(n,m) + t^{(2)}(n,m),$$

where

$$t^{(1)}(n,m) = \frac{1}{m} \sum_{1\leq j\leq m;\quad |x_{\sigma^j(n)} - L| \geq \epsilon} |x_{\sigma^j(n)} - L|$$

and

$$t^{(2)}(n,m) = \frac{1}{m} \sum_{1\leq j\leq m;\quad |x_{\sigma^j(n)} - L| < \epsilon} |x_{\sigma^j(n)} - L|.$$

We have $t^{(2)}(n,m) < \epsilon$, for every $n = 1, 2, ...$ The boundedness of $x = (x_k)$ implies that there exist $K > 0$ such that $|x_{\sigma^j(n)} - L| \leq K, \quad (j = 1, 2, ...; n = 1, 2, ...)$, then this implies that

$$\begin{aligned} t^{(1)}(n,m) &\leq \frac{K}{m}|\{1 \leq j \leq m : |x_{\sigma^j(n)} - L| \geq \epsilon\}| \\ &\leq K\frac{max_n|\{1 \leq j \leq m : |x_{\sigma^j(n)} - L| \geq \epsilon\}|}{m} = K\frac{S_m}{m}, \end{aligned}$$

hence x is invariant convergent to L. □

The converse of the previous theorem does not hold. For example, $x = (x_k)$ is the sequence defined by $x_k = 1$ if k is even and $x_k = 0$ if k is odd. When $\sigma(n) = n + 1$, this sequence is invariant convergent to $\frac{1}{2}$ but it is not I_σ-convergent.

In [2], Connor gave some inclusion relations between strong p-Cesaro convergence and statistical convergence and showed that these are equivalent for bounded sequences. Now we shall give an analogous theorem which states inclusion relations between $[V_\sigma]_p$-convergence and I_σ-convergence and show that these are equivalent for bounded sequences.

Theorem 2.

(a) If $0 < p < \infty$ and $x_k \to L([V_\sigma]_p)$, then $x = (x_n)$ is I_σ-convergent to L.

(b) If $x = (x_n) \in \ell_\infty$ and I_σ-converges to L, then $x_k \to L([V_\sigma]_p)$.

(c) If $x = (x_n) \in \ell_\infty$, then $x = (x_n)$ is I_σ-convergent to L if and only if $x_k \to L([V_\sigma]_p)$ $(0 < p < \infty)$.

Proof. (a) Let $x_k \to ([V_\sigma]_p)$, $0 < p < \infty$. Suppose $\epsilon > 0$. Then for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_1^m |x_{\sigma^j(n)} - L|^p &\geq \sum_{1 \leq j \leq m; |x_{\sigma^j(n)} - L| \geq \epsilon} |x_{\sigma^j(n)} - L|^p \\ &\geq \epsilon^p |\{1 \leq j \leq m : |x_{\sigma^j(n)} - L| \geq \epsilon\}| \\ &\geq \epsilon^p max_n |\{1 \leq j \leq m : |x_{\sigma^j(n)} - L| \geq \epsilon\}| \end{aligned}$$

and

$$\begin{aligned} \frac{1}{m}\sum_1^m |x_{\sigma^j(n)} - L|^p &\geq \epsilon^p \frac{max_n|\{1 \leq j \leq m : |x_{\sigma^j(n)} - L| \geq \epsilon\}|}{m} \\ &= \epsilon^p \frac{S_m}{m} \end{aligned}$$

for every $n = 1, 2, 3, ...$. This implies $\lim_{m\to\infty} \frac{S_m}{m} = 0$ and so $I_\sigma - \lim x_k = L$.

(b) Now suppose that $x \in \ell_\infty$ and I_σ-convergent to L. Let $0 < p < \infty$ and $\epsilon > 0$. By assumption, we have $V(A_\epsilon) = 0$. The boundedness of $x = (x_k)$ implies that there exist $M > 0$ such that $|x_{\sigma^j(n)} - L| \le M, \quad (j = 1, 2, ...; n = 1, 2, ...)$. Observe that for every $n \in \mathbb{N}$ we have that

$$\begin{aligned}\frac{1}{m}\sum_{j=1}^{m}|x_{\sigma^j(n)} - L|^p &= \frac{1}{m}\sum_{1\le j\le m;|x_{\sigma^j(n)}-L|\ge\epsilon}|x_{\sigma^j(n)} - L|^p \\ &+\frac{1}{m}\sum_{1\le j\le m;|x_{\sigma^j(n)}-L|<\epsilon}|x_{\sigma^j(n)} - L|^p \\ &\le M\frac{max_n|\{1\le j\le m : |x_{\sigma^j(n)} - L| \ge \epsilon\}|}{m} + \epsilon^p \\ &\le M\frac{S_m}{m} + \epsilon^p.\end{aligned}$$

Hence, we obtain

$$\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}|x_{\sigma^j(n)} - L|^p = 0$$

uniformly in n.

(c) This is a corollary of (a) and (b). □

In the case $\sigma(n) = n+1$ in the above theorems, we have Theorem 1 and Theorem 2 in [1].

Definition 3. *A sequence $x = (x_k)$ is said to be I_σ^*-convergent to the number L if there exists a set $M = \{m_1 < m_2 < ...\} \in F(I_\sigma)$ such that $\lim_{k\to\infty} x_{m_k} = L$. In this case we write $I_\sigma^* - \lim x_k = L$.*

I_σ^*- convergence is better applicable in some situations.

Theorem 3. *Let I_σ be an admissible ideal. If a sequence $x = (x_k)$ is I_σ^*-convergent to L, then this sequence is I_σ-convergent to L.*

Proof. By assumption, there exists a set $H \in I_\sigma$ such that for $M = \mathbb{N}\setminus H = \{m_1 < m_2 < ... < m_k < ...\}$ we have

$$\lim_{k\to\infty} x_{m_k} = L. \tag{1}$$

Let $\epsilon > 0$. By (1), there exists $k_0 \in \mathbb{N}$ such that $|x_{m_k} - L| < \epsilon$ for each $k > k_0$. Then obviously

$$\{k \in \mathbb{N} : |x_k - l| \ge \epsilon\} \subset H \cup \{m_1 < m_2 < ... < m_{k_0}\}.(2) \tag{2}$$

The set on the right-hand side of (2) belongs to I_σ (since I_σ is admissible). So $x = (x_k)$ is I_σ-convergent to L. □

The converse of Theorem 3 holds if I_σ has property (AP).

Definition 4 (see [3])**.** *An admissible ideal I is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to I there exits a countable family of sets $\{B_1, B_2, ...\}$ such that the symmetric difference $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = (\bigcup_{j=1}^{\infty} B_j) \in I$.*

Theorem 4. *Let I_σ be an admissible ideal and let it have property (AP). If x is I_σ-convergent to L, then x is I_σ^*-convergent to L.*

Proof. Suppose that I_σ satisfies condition (AP). Let $I_\sigma - \lim x_k = L$. Then for $\epsilon > 0$, $\{k : \quad |x_k - L| \geq \epsilon\}$ belongs to I_σ.

Put $A_1 = \{k : \quad |x_k - L| \geq 1\}$ and $A_n = \{k : \quad \frac{1}{n} \leq |x_k - L| < \frac{1}{n-1}\}$ for $n \geq 2$, $n \in \mathbb{N}$. Obviously, $A_i \cap B_j = \emptyset$ for $i \neq j$. By condition (AP) there exits a sequence of $\{B_n\}_{n\in\mathbb{N}}$ such that $A_j \triangle B_j$ are finite sets for $j \in \mathbb{N}$ and $B = (\bigcup_{j=1}^{\infty} B_j) \in I_\sigma$.

It is sufficient to prove that for $M = \mathbb{N} \setminus B$ we have

$$\lim_{k\in M; k\to\infty} x_k = L. \tag{3}$$

Let $\lambda > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \lambda$. Then

$$\{k : \quad |x_k - L| \geq \lambda\} \subset \bigcup_{j=1}^{n+1} A_j.$$

Since $A_j \triangle B_j, j = 1, 2, ..., n+1$ are finite sets, there exists $k_{0\in\mathbb{N}}$ such that

$$(\bigcup_{j=1}^{n+1} B_j) \cap \{k : k > k_0\} = (\bigcup_{j=1}^{n+1} A_j) \cap \{k : k > k_0\} \tag{4}$$

If $k > k_0$ and $k \notin B$, then $k \notin \bigcup_{j=1}^{n+1} B_j$ and by (4), $k \notin \cup_{j=1}^{n+1} A_j$. But then

$$|x_k - L| < \frac{1}{n+1} < \lambda$$

so (3) holds and hence we have $I_\sigma^* - \lim x_k = L$. □

Now we shall state a theorem that gives a relation between S_σ-convergence and I_σ-convergence.

Theorem 5. *A sequence $x = (x_k)$ is S_σ-convergent to L if and only if it is I_σ-convergent to L.*

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