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Pullback diagram of Hilbert C^*-modules

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Abstract. In this paper, we generalize the construction of a pullback diagram in the framework of Hilbert C^*-modules and investigate some conditions under which a diagram of Hilbert C^*-modules is pullback.

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1. Introduction

G. K. Pedersen [8] introduced the notion of a pullback diagram in the category of C^*-algebras and investigated some properties of these diagrams. The pullback diagrams are stable under tensoring with a fixed algebra and stable under crossed products with a fixed group. The relations between the theory of extensions of Hilbert C^*-modules and pullback diagrams of Hilbert C^*-modules were investigated in [3]. In this paper, we generalize the construction of a pullback diagram in the framework of Hilbert C^*-modules and investigate some conditions under which a diagram of Hilbert C^*-modules is pullback.

A pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space X which is an algebraic right $\mathcal{A}$-module, $\lambda(xa) = (\lambda x)a = x(\lambda a)$ and equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : X \times X \to \mathcal{A}$ satisfying the following properties:

(i) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$;

(iii) $\langle x, ya \rangle = \langle x, y \rangle a$;

(iv) $\langle y, x \rangle = \langle x, y \rangle^*$; for all $x, y, z \in X,\ \lambda \in \mathbb{C},\ a \in \mathcal{A}$.

A pre-Hilbert $\mathcal{A}$-module X is called a (right) Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$, where the latter norm denotes that of C^*-algebra $\mathcal{A}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way.
It is a full Hilbert $\mathcal{A}$-module if the ideal $I = span\{\langle x, y \rangle :\ x, y \in X\}$ is dense in $\mathcal{A}$. For example, every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to the inner

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product $\langle x, y\rangle = x^*y$. If X and Y are Hilbert $\mathcal{A}$-modules, the mapping $T : X \to Y$ is called adjointable if there exists a mapping $T^* : Y \to X$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$. The set of all adjointable mappings from X to Y is denoted by $B(X, Y)$. For $x \in X$ and $y \in Y$ we also define the operator $\theta_{y,x} : X \to Y$ by $\theta_{y,x}(z) = y\langle x, z\rangle$ for all $z \in X$. In fact, $\theta_{y,x} \in B(X, Y)$ with $(\theta_{y,x})^* = \theta_{x,y}$, and $B(X)$ is a C^*-algebra with respect to the operator norm. The closure of the span of $\{\theta_{y,x} : x, y \in X\}$ in $B(X)$ is denoted by $K(X)$, and elements of this set will be called "compact" operators. The basic theory of Hilbert C^*-modules can be found in [4, 5, 10]. Throughout the paper X and Y denote Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. We study some conditions under which the diagram of Hilbert C^*-modules are pullback. We follow the terminology and notation of [1, 2, 3].

Definition 1. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras. A mapping $\Phi : X \to Y$ is said to be a φ-morphism of Hilbert C^*-modules if $\langle\Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ for all x, y in X.*

Using polarization identity, one can conclude that Φ is a φ-morphism if and only if $\langle\Phi(x), \Phi(x)\rangle = \varphi(\langle x, x\rangle)$ for each $x \in X$. It is easy to see that each φ-morphism is necessarily a linear operator and a module mapping in the sense that $\Phi(xa) = \Phi(x)\varphi(a)$ for all $x \in X, a \in \mathcal{A}$.

Theorem 1 (see [1, Corollary 2.13])**.** *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a surjective morphism of C^*-algebras and let $\Phi : X \to Y$ be a surjective φ-morphism. Then there exists a morphism of C^*-algebras $\Phi^+ : B(X) \to B(Y)$ satisfying $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$ for $x, y \in X$ and $\Phi^+(K(X)) = K(Y)$.*

If X is a right Hilbert $\mathcal{A}$-module, then X is a left Hilbert $K(X)$-module with respect to the natural left action $T.x = T(x)$ and the inner product $[x, y] = \theta_{x,y}$, (see [9, Lemma 2.30]). By Theorem 1, $[\Phi(x), \Phi(y)] = \theta_{\Phi(x),\Phi(y)} = \Phi^+(\theta_{x,y}) = \Phi^+([x, y])$.

It is well known that $B(\mathcal{A}, X)$ is a Hilbert $B(\mathcal{A})$-module under the $B(\mathcal{A})$-valued inner product $\langle r_1, r_2\rangle = r_1^*r_2$ such that the resulting norm coincides with the operator norm on $B(\mathcal{A}, X)$. Each $x \in X$ induces the mappings $r_x \in B(\mathcal{A}, X)$ and $l_x \in B(X, \mathcal{A})$ given by $r_x(a) = xa$ and $l_x(y) = \langle x, y\rangle$ such that $l_x^* = r_x$. The mapping $x \to l_x$ is an isometric conjugate linear isomorphism of X into $K(X, \mathcal{A})$ and $x \to r_x$ is an isometric linear isomorphism of X to $K(\mathcal{A}, X)$. Furthermore, every $a \in \mathcal{A}$ induces the mapping $T_a \in K(\mathcal{A})$ given by $T_a(b) = ab$ and the mapping $a \to T_a$ is an isomorphism of the C^*-algebra $\mathcal{A}$ into $K(\mathcal{A})$.

The linking algebra $\mathcal{L}(X)$ may be formally defined as the matrix algebra of the form

$$\mathcal{L}(X) = \begin{bmatrix} K(\mathcal{A}) & K(X,\mathcal{A}) \\ K(\mathcal{A},X) & K(X) \end{bmatrix} = \left\{ \begin{bmatrix} T_a & l_x \\ r_y & T \end{bmatrix} \ : \ a \in \mathcal{A}, \ \ x, y \in X, \ \ T \in K(X) \right\}.$$

See [9, Lemma 2.32 and Corollary 3.21]. Observe that $\mathcal{L}(X)$ is the C^*-algebra of all compact operators acting on $\mathcal{A} \oplus X$. We aim to describe morphisms of Hilbert C^*-modules in terms of the corresponding linking algebras. If X and Y are full, then every surjective φ-morphism $\Phi : X \to Y$ induces a morphism of the linking algebras $\rho_{\varphi,\Phi} : \mathcal{L}(X) \to \mathcal{L}(Y)$ given by

$$\rho_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}\right) = \begin{bmatrix} T_{\varphi(a)} & l_{\Phi(y)} \\ r_{\Phi(x)} & \Phi^+(T) \end{bmatrix}.$$

Conversely, if $\rho : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a morphism of linking algebras such that $\rho(K(\mathcal{A})) \subseteq K(\mathcal{B})$ and $\rho(K(X)) \subseteq K(Y)$, then there exist a morphism of C^*-algebras $\varphi : \mathcal{A} \to \mathcal{B}$ and a φ-morphism $\Phi : X \to Y$ such that $\rho = \rho_{\varphi,\Phi}$, cf. [1, Theorem 2.15].

Using the fact that $\theta_{\Phi_2(x_2),\Phi_1(x_1)}\Phi_1 = \Phi_2\theta_{x_2,x_1}$ and by [3, Theorem 3.15] it is easy to prove the following theorem.

Theorem 2. *Let X_1, X_2 be full Hilbert $\mathcal{A}$-modules and Y_1, Y_2 full Hilbert $\mathcal{B}$-modules. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras and let $\Phi_1 : X_1 \to Y_1$, $\Phi_2 : X_2 \to Y_2$ be surjective φ-morphisms of Hilbert C^*-modules. Then the following diagram commutes.*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \Big\downarrow{\scriptstyle \theta_{x_2,x_1}} & & \Big\downarrow{\scriptstyle \theta_{\Phi_2(x_2),\Phi_1(x_1)}} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array}$$

2. Pullback constructions in Hilbert C^*-modules

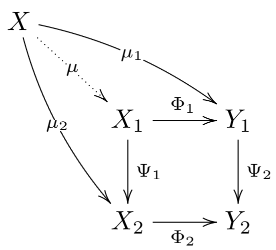
In this section we want to show under what condition the diagram of Hilbert C^*-modules in Theorem 2 is pullback.

Lemma 1 (see [3, Lemma 2.1])**.** *Let $\Psi_2 : Y_1 \to Y_2$ and $\Phi_2 : X_2 \to Y_2$ be morphisms of Hilbert C^*-modules. Let $\varphi_1 : \mathcal{A} \to \mathcal{C}$ and $\varphi_2 : \mathcal{B} \to \mathcal{C}$ denote the corresponding morphisms of underlying C^*-algebras. Denote by $X_2 \oplus_{Y_2} Y_1$ the set $\{(x_2, y_1) \in X_2 \oplus Y_1 : \Phi_2(x_2) = \Psi_2(y_1)\}$. Then $X_2 \oplus_{Y_2} Y_1$ is a Hilbert C^*-module (with operations inherited from a Hilbert $\mathcal{A}\oplus\mathcal{B}$-module $X_2 \oplus Y_1$) over the restricted direct sum $\mathcal{A} \oplus_{\mathcal{C}} \mathcal{B}$.*

Definition 2. *A commutative diagram of Hilbert C^*-modules*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \Big\downarrow{\scriptstyle \Psi_1} & & \Big\downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array}$$

is pullback if $Ker\Phi_1 \bigcap Ker\Psi_1 = \{0\}$ and for every other pair of morphisms $\mu_1 : X \to Y_1$ and $\mu_2 : X \to X_2$ from a full Hilbert C^*-module X that satisfy condition $\Psi_2\mu_1 = \Phi_2\mu_2$, there exists a unique morphism $\mu : X \to X_1$ such that $\mu_1 = \Phi_1\mu$ and $\mu_2 = \Psi_1\mu$.



It follows that X_1 *is isomorphic to* $X_2 \oplus_{Y_2} Y_1$.

Theorem 3. *A commutative diagram of full Hilbert* C^**-modules* X_1 *and* X_2 *and arbitrary Hilbert* C^**-modules* Y_1 *and* Y_2, *in which the corresponding map* φ_1 *to* Φ_1 *is surjective,*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array}$$

is pullback if and only if the following conditions hold:

(i) $Ker\Phi_1 \cap Ker\Psi_1 = \{0\}$,

(ii) $\Psi_2^{-1}(\Phi_2(X_2)) = \Phi_1(X_1)$,

(iii) $\Psi_1(Ker\Phi_1) = Ker\Phi_2$.

Proof. Suppose the diagram above is a pullback. Then there exists a unique isomorphism $\Phi : X_1 \to X_2 \oplus_{Y_2} Y_1$ defined by $\Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1))$.
Conditions (i) and (ii) are clearly satisfied. To prove (iii) let $x_2 \in \Psi_1(Ker\Phi_1)$. Then there is $x_1 \in Ker\Phi_1$ such that $x_2 = \Psi_1(x_1)$. But $\Phi_2(x_2) = \Phi_2(\Psi_1(x_1)) = \Psi_2(\Phi_1(x_1)) = 0$, thus $x_2 \in Ker\Phi_2$. Conversely, let $x_2 \in Ker\Phi_2$, and consider $(x_2, 0)$ in $X_2 \oplus_{Y_2} Y_1$. Since Φ is surjective, then there exists $x_1 \in X_1$ such that $\Phi(x_1) = (x_2, 0)$, i.e. $\Psi_1(x_1) = x_2$ and $\Phi_1(x_1) = 0$. Thus $x_2 \in \Psi_1(Ker\Phi_1)$.

Conversely, suppose that the three conditions above are satisfied and X_1, X_2 are full Hilbert C^*-modules over $\mathcal{A}_1$ and $\mathcal{A}_2$ and Y_1, Y_2 be Hilbert C^*-modules over $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Consider the corresponding diagram of underlying C^*-algebras. Clearly Ψ_1, Ψ_2 are ψ_1, ψ_2 -morphisms and Φ_1, Φ_2 are φ_1, φ_2-morphisms of their corresponding Hilbert C^*-modules.

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\varphi_1} & \mathcal{B}_1 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} \\ \mathcal{A}_2 & \xrightarrow{\varphi_2} & \mathcal{B}_2 \end{array}$$

We shall show that the above three conditions hold for the diagram of underlying C^*-algebras. The diagram of C^*-algebras is commutative, since the diagram of their Hilbert modules is commutative.

(I) Let $a_1 \in Ker\varphi_1 \cap Ker\psi_1$. Then $\varphi_1(a_1) = 0$ and $\psi_1(a_1) = 0$. Let $x_1 \in X_1$ be arbitrary. Then $x_1 a_1 \in X_1$, and $\langle \Phi_1(x_1 a_1), \Phi_1(x_1 a_1) \rangle = \varphi_1(\langle x_1 a_1, x_1 a_1 \rangle) = \varphi_1(a_1^* \langle x_1, x_1 \rangle a_1) = \varphi_1(a_1^*) \varphi_1(\langle x_1, x_1 \rangle) \varphi_1(a_1) = 0$. Hence $\| \langle \Phi_1(x_1 a_1), \Phi_1(x_1 a_1) \rangle \| = \| \Phi_1(x_1 a_1) \|^2 = 0$. Thus $x_1 a_1 \in Ker\Phi_1$. Similarly, $x_1 a_1 \in Ker\Psi_1$. Hence $x_1 a_1 = 0$ for all $x_1 \in X_1$. By [6, Theorem 2.1] we have $a_1 = 0$.

(II) By condition (ii), we have $\Psi_2(\Phi_1(X_1)) \subseteq \Phi_2(X_2)$. Since X_1 and X_2 are full, then $\varphi_2(\mathcal{A}_2) = \varphi_2(\langle X_2, X_2 \rangle) = \langle \Phi_2(X_2), \Phi_2(X_2) \rangle \supseteq \langle \Psi_2(\Phi_1(X_1)), \Psi_2(\Phi_1(X_1)) \rangle = \psi_2 \langle \Phi_1(X_1), \Phi_1(X_1) \rangle = \psi_2 \varphi_1(\langle X_1, X_1 \rangle) = \psi_2 \varphi_1(\mathcal{A}_1)$. Hence $\psi_2^{-1}(\varphi_2(\mathcal{A}_2)) \supseteq \varphi_1(\mathcal{A}_1)$. Since φ_1 is surjective, $\psi_2^{-1}(\varphi_2(\mathcal{A}_2)) \subseteq \mathcal{B}_1 = \varphi_1(\mathcal{A}_1)$.

(III) Since X_2 is a full Hilbert $\mathcal{A}_2$-module, by [1, Proposition 1.3 and 2.3], KerΦ_2 is a full Hilbert Kerφ_2-module. Therefore, $Ker\varphi_2 = \langle Ker\Phi_2, Ker\Phi_2\rangle = \langle \Psi_1(Ker\Phi_1), \Psi_1(Ker\Phi_1)\rangle = \psi_1\langle Ker\Phi_1, Ker\Phi_1\rangle = \psi_1(Ker\varphi_1)$.

By [8, Proposition 3.1], the above diagram of C^*-algebras is pullback. So that $\varphi : \mathcal{A}_1 \to \mathcal{A}_2 \oplus_{\mathcal{B}_2} \mathcal{B}_1$ defined by $\varphi(a_1) = (\psi_1(a_1), \varphi_1(a_1))$ is an isomorphism.

Define $\Phi : X_1 \to X_2 \oplus_{Y_2} Y_1$ by $\Phi(x_1) = (\Psi_1(x_1), \Phi_1(x_1))$. By condition (i), Φ is injective. Let $(x_2, y_1) \in X_2 \oplus_{Y_2} Y_1$, since $\Phi_2(x_2) = \Psi_2(y_1)$, $\quad y_1 \in \Psi_2^{-1}(\Phi_2(X_2)) = \Phi_1(X_1)$. Thus $y_1 = \Phi_1(x_1)$ for some $x_1 \in X_1$. Hence $0 = \Psi_2(y_1 - \Phi_1(x_1)) = \Psi_2(y_1) - \Psi_2\Phi_1(x_1) = \Phi_2(x_2 - \Psi_1(x_1))$. This means that $x_2 - \Psi_1(x_1) \in Ker\Phi_2 = \Psi_1(Ker\Phi_1)$. Therefore there is $x_1' \in Ker\Phi_1$ such that $x_2 - \Psi_1(x_1) = \Psi_1(x_1')$, whence $x_2 = \Psi_1(x_1 + x_1')$. On the other hand, $\Phi(x_1 + x_1') = (\Psi_1(x_1 + x_1'), \Phi_1(x_1 + x_1')) = (x_2, \Phi_1(x_1)) = (x_2, y_1)$. So Φ is surjective. Since

$$\begin{aligned}\langle \Phi(x_1), \Phi(x_1)\rangle &= \langle (\Psi_1(x_1), \Phi_1(x_1)), (\Psi_1(x_1), \Phi_1(x_1))\rangle \\ &= (\langle \Psi_1(x_1), \Psi_1(x_1)\rangle, \langle \Phi_1(x_1), \Phi_1(x_1)\rangle) \\ &= (\psi_1\langle x_1, x_1\rangle, \varphi_1\langle x_1, x_1\rangle) = \varphi\langle x_1, x_1\rangle,\end{aligned}$$

Φ is a φ-morphism. By [3, Proposition 2.3], the diagram of Hilbert C^*-modules is pullback. □

Corollary 1. *Let X_1, X_2, Y_1 and Y_2 be full Hilbert C^*-modules with surjective φ-morphisms Φ_1 and Φ_2 and linking morphisms $\theta_{x_2,x_1}, \theta_{\Phi_2(x_2),\Phi_1(x_1)}$. If θ_{x_2,x_1} is an isometry and $\varphi : \mathcal{A} \to \mathcal{B}$ is injective, then the following left diagram is pullback.*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow{\scriptstyle \theta_{x_2,x_1}} & & \downarrow{\scriptstyle \theta_{\Phi_2(x_2),\Phi_1(x_1)}} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} \\ \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \end{array}$$

Proof. First we show that θ_{x_2,x_1} is a I-morphism. Since θ_{x_2,x_1} is an isometry, then for $z \in X$, $\quad a \in \mathcal{A}$, we have

$$\begin{aligned}\|\langle \theta_{x_2,x_1}(z), \theta_{x_2,x_1}(z)\rangle^{\frac{1}{2}} a\|^2 &= \|a^*\langle \theta_{x_2,x_1}(z), \theta_{x_2,x_1}(z)\rangle a\| = \|\langle \theta_{x_2,x_1}(za), \theta_{x_2,x_1}(za)\rangle\| \\ &= \|\theta_{x_2,x_1}(za)\|^2 = \|za\|^2 = \|I\langle za, za\rangle\| = \|I\langle z, z\rangle^{\frac{1}{2}} a\|^2.\end{aligned}$$

By [4, Lemma 3.4] $\langle \theta_{x_2,x_1}(z), \theta_{x_2,x_1}(z)\rangle = I\langle z, z\rangle$.

Similarly, we can show that $\theta_{\Phi_2(x_2),\Phi_1(x_1)}$ is an isometry. Hence it is an I-morphism.

Let $y_1 \in Y_1$, then there is $x_1' \in X_1$ such that $\Phi_1(x_1') = y_1$. So

$$\begin{aligned}\|\theta_{\Phi_2(x_2),\Phi_1(x_1)}(y_1)\| &= \|\theta_{\Phi_2(x_2),\Phi_1(x_1)}(\Phi_1(x_1'))\| = \|\Phi_2(x_2)\langle \Phi_1(x_1), \Phi_1(x_1')\rangle\| \\ &= \|\Phi_2(x_2)\varphi\langle x_1, x_1'\rangle\| = \|\Phi_2(x_2\langle x_1, x_1'\rangle\| = \|\Phi_2(\theta_{x_2,x_1}(x_1'))\| \\ &= \|\theta_{x_2,x_1}(x_1')\| = \|x_1'\| = \|\Phi_1(x_1')\| = \|y_1\|.\end{aligned}$$

Note that the φ- morphisms Φ_1 and Φ_2 are contractions and φ is injective, then Φ_1 and Φ_2 are isometries.

By Theorem 2, the left diagram above is commutative.

(i) Since φ is injective, Φ_1 is injective, so $Ker\Phi_1 \cap Ker\theta_{x_2,x_1} = \{0\}$.

(ii) Since I is injective, the I-morphism $\theta_{\Phi_2(x_2),\Phi_1(x_1)}$ is injective. So $\theta^{-1}_{\Phi_2(x_2),\Phi_1(x_1)}$ is surjective. By our assumption, the φ-morphisms Φ_1 and Φ_2 are surjective, so $\theta^{-1}_{\Phi_2(x_2),\Phi_1(x_1)}(\Phi_2(X_2)) = \theta^{-1}_{\Phi_2(x_2),\Phi_1(x_1)}(Y_2) = Y_1 = \Phi_1(X_1)$.

(iii) If x'_2 is in $\theta_{x_2,x_1}(Ker\Phi_1)$, then $x'_2 = x_2\langle x_1, x'_1\rangle$ for some $x'_1 \in Ker\Phi_1$. Thus $\Phi_2(x'_2) = \Phi_2(x_2\langle x_1, x'_1\rangle) = \Phi_2(x_2)\varphi\langle x_1, x'_1\rangle) = \Phi_2(x_2)\langle\Phi_1(x_1), \Phi_1(x'_1)\rangle = 0$, i.e. $\theta_{x_2,x_1}(Ker\Phi_1) \subseteq Ker\Phi_2$. Conversely, for $x'_2 \in Ker\Phi_2$, we have $\Phi_2(x'_2) = 0 = \Phi_2(x_2)\langle\Phi_1(x_1), \Phi_1(x'_1)\rangle = \Phi_2(x_2\langle x_1, x'_1\rangle) = \Phi_2(\theta_{x_2,x_1}(x'_1))$, for some $x'_1 \in Ker\Phi_1$. Since φ is injective, Φ_1 and Φ_2 are injective. Thus $x'_2 = \theta_{x_2,x_1}(x'_1)$. Therefore $x'_2 \in \theta_{x_2,x_1}(Ker\Phi_1)$, i.e. $Ker\Phi_2 \subseteq \theta_{x_2,x_1}(Ker\Phi_1)$. □

Corollary 2. *Suppose that*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} \\ X_2 & \xrightarrow{\Phi_2} & Y_2 \end{array}$$

is a pullback diagram of full Hilbert C^*-modules in which all morphisms are surjective. Then the following diagrams of compact operators and linking algebras are pullback.

$$\begin{array}{ccc} K(X_1) & \xrightarrow{\Phi_1^+} & K(Y_1) \\ \downarrow{\scriptstyle \Psi_1^+} & & \downarrow{\scriptstyle \Psi_2^+} \\ K(X_2) & \xrightarrow{\Phi_2^+} & K(Y_2) \end{array} \qquad \begin{array}{ccc} \mathcal{L}(X_1) & \xrightarrow{\rho_{\varphi_1,\Phi_1}} & \mathcal{L}(Y_1) \\ \downarrow{\scriptstyle \rho_{\psi_1,\Psi_1}} & & \downarrow{\scriptstyle \rho_{\psi_2,\Psi_2}} \\ \mathcal{L}(X_2) & \xrightarrow{\rho_{\varphi_2,\Phi_2}} & \mathcal{L}(Y_2) \end{array}$$

Proof. The left diagram is clearly commutative.

Since the diagram of Hilbert modules is pullback, there exists a unique isomorphism $\sigma_1 : X_1 \to X_2 \oplus_{Y_2} Y_1$ defined by $\sigma_1(x_1) = (\Psi_1(x_1), \Phi_1(x_1))$. We show that $\sigma_2 : K(X_1) \to K(X_2) \oplus_{K(Y_2)} K(Y_1)$ defined by $\sigma_2(\theta_{x_1,x'_1}) = (\Psi_1^+(\theta_{x_1,x'_1}), \Phi_1^+(\theta_{x_1,x'_1})) = (\theta_{\Psi_1(x_1),\Psi_1(x'_1)}, \theta_{\Phi_1(x_1),\Phi_1(x'_1)})$ is an isomorphism.
Suppose $\sigma_2(\theta_{x_1,x'_1}) = 0$, i.e. for each $z \in X_1$ we have $\theta_{\Phi_1(x_1),\Phi_1(x'_1)}(\Phi_1(z)) = 0$, then $\Phi_1(x_1)\langle\Phi_1(x'_1), \Phi_1(z)\rangle = \Phi_1(x_1\langle x'_1, z\rangle) = \Phi_1(\theta_{x_1,x'_1}(z)) = 0$, so $\theta_{x_1,x'_1}(z) \in Ker\Phi_1$. Also $\theta_{\Psi_1(x_1),\Psi_1(x'_1)}(\Psi_1(z)) = 0$. We have $\Psi_1(\theta_{x_1,x'_1}(z)) = 0$, so $\theta_{x_1,x'_1}(z) \in Ker\Psi_1$. Therefore $\theta_{x_1,x'_1}(z) \in Ker\Phi_1 \cap Ker\Psi_1 = \{0\}$, this means σ_2 is injective.

To prove the surjectivity, take an arbitrary $(\theta_{x_2,x'_2}, \theta_{y_1,y'_1})$ in $K(X_2) \oplus_{K(Y_2)} K(Y_1)$. Since σ_1 is surjective, for all (x_2, y_1) and (x'_2, y'_1) in $X_2 \oplus_{Y_2} Y_1$ there are $x_1, x'_1 \in X_1$ such that $\sigma_1(x_1) = (\Psi_1(x_1), \Phi_1(x_1)) = (x_2, y_1)$ and $\sigma_1(x'_1) = (\Psi_1(x'_1), \Phi_1(x'_1)) = (x'_2, y'_1)$. Consequently, $\sigma_2(\theta_{x_1,x'_1}) = (\theta_{\Psi_1(x_1),\Psi_1(x'_1)}, \theta_{\Phi_1(x_1),\Phi_1(x'_1)}) = (\theta_{x_2,x'_2}, \theta_{y_1,y'_1})$.

The first diagram and the corresponding diagram of underlying C^*-algebras are pullback, hence there are unique isomorphisms $\gamma_1 : \mathcal{A}_1 \to \mathcal{A}_2 \oplus_{\mathcal{B}_2} \mathcal{B}_1$ and $\gamma_2 : X_1 \to X_2 \oplus_{Y_2} Y_1$ defined by $\gamma_1(a_1) = (\psi_1(a_1), \varphi_1(a_1))$ and $\gamma_2(x_1) = (\Psi_1(x_1), \Phi_1(x_1))$, respectively. We obtain a new diagram using the induced morphisms

$$\begin{array}{ccc} \mathcal{A}_1 \oplus X_1 & \xrightarrow{\varphi_1 \oplus \Phi_1} & \mathcal{B}_1 \oplus Y_1 \\ \downarrow{\scriptstyle \psi_1 \oplus \Psi_1} & & \downarrow{\scriptstyle \psi_2 \oplus \Psi_2} \\ \mathcal{A}_2 \oplus X_2 & \xrightarrow{\varphi_2 \oplus \Phi_2} & \mathcal{B}_2 \oplus Y_2 \end{array}$$

Clearly, the map $\gamma : \mathcal{A}_1 \oplus X_1 \to (\mathcal{A}_2 \oplus_{\mathcal{B}_2} \mathcal{B}_1) \oplus (X_2 \oplus_{Y_2} Y_1)$ defined by $\gamma(a_1, x_1) = (\gamma_1(a_1), \gamma_2(x_1))$ is an isomorphism (we know that the map $(\mathcal{A}_2 \oplus X_2) \oplus_{\mathcal{B}_2 \oplus Y_2} (\mathcal{B}_1 \oplus Y_1) \mapsto (\mathcal{A}_2 \oplus_{\mathcal{B}_2} \mathcal{B}_1) \oplus (X_2 \oplus_{Y_2} Y_1)$ is a natural isomorphism). Then the diagram above is pullback. Therefore, the diagram of compact operators is also pullback, this means that the diagram of linking algebras is a pullback. (Recall that $K(\mathcal{A} \oplus X) = \mathcal{L}(X)$, where X is a Hilbert $\mathcal{A}$-module [1].) □

Remark 1. *Suppose the diagram in the Theorem 2 is a pullback diagram of full Hilbert C^*-modules in which all morphisms are surjective. Then the diagrams of compact operators and linking algebras are pullback.*

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