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Operation approach to β-open sets and applications

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Abstract. In this paper, we introduce the concept of an operation γ on a family of β-open sets denoted by $\beta O(X)$ in a topological space (X, τ). Using the operation γ on $\beta O(X)$, we introduce the concept of β-γ-open sets, and investigate the related topological properties. We also introduce the notion of β-γ-T_i spaces $(i = 0, 1/2, 1, 2)$ and study some topological properties on them. Further, we introduce β-(γ, b)-continuous maps and investigate basic properties. Finally, we investigate a general operation approach to β-closed graphs of mappings.

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1. Introduction

Throughout this paper, (X, τ) and (Y, σ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The closure and interior of $A \subseteq X$ will be denoted by $Cl(A)$ and $Int(A)$, respectively. Kasahara [5] defined the concept of an operation on a topological space and introduced the concept of α-closed graphs of functions. Further, Monsef et al. [1] initiated the study of β-open sets and β-continuity in a topological space and Janković [4] defined the concept of an α-closed set and investigated the functions on α-closed graphs. Ogata [8] called the operation α (respectively, α-closed set) as a γ-operation (respectively γ-closed sets) and introduced the notion of τ_γ which is the collection of all γ-open sets in a topological space (X, τ). Morever, he introduced the concept of γ-T_i (i = 0, 1/2, 1, 2) spaces and characterized γ-T_i spaces by the notion of γ-closed sets or γ-open sets.

The family of all β-open sets of X is denoted by $\beta O(X)$. In this paper, Section 2 is the introduction of the concept of the family $\beta O(X)_\gamma$ of all β-γ-open sets by using the operation γ on $\beta O(X)$ in (X, τ). Further, we introduce the concept of β-γ-closure, $\beta O(X)_\gamma$-closure and study their relationships. In Section 3, we introduce the concept of β-γ-T_i $(i = 0, 1/2, 1, 2)$ spaces and characterize β-γ-T_i spaces by the notion β-γ-closed or β-γ-open sets. In Section 4, we define a new class of maps called β-(γ, b)-continuous maps and study some of their properties. In Section 5, we investigate a general operation approach of β-closed graphs of mappings.

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2. β-γ-open sets

Definition 1. *Let (X,τ) be a topological space. A subset A of space X will be called β-open [1] if $A \subseteq Cl(Int(Cl(A)))$. The complement of a β-open set is said to be β-closed [1]. The intersection of all β-closed sets containing U, the subset of X, is known as a β-closure [3] of U and it is denoted by $\beta Cl(U)$. It is obvious by definition that $U \subseteq \beta Cl(U)$.*

Definition 2. *An operation γ [4] on the topology τ is a mapping $\gamma : \tau \to P(X)$ from τ to the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V.*

Definition 3. *(i) A subset A of a topological space is called a γ-open set [8] of (X,τ) if for each $x \in A$ there exists an open set U such that $x \in U$ and $U^\gamma \subseteq A$. The complement of a γ-open set is said to be γ-closed.*
(ii) The point $x \in X$ is in the γ-closure [4] of a set $A \subseteq X$ if $U^\gamma \cap A \neq \phi$ for each open set U of x. The γ-closure of a set A is denoted by $Cl_\gamma(A)$. Also τ_γ-$Cl(A) = \bigcap\{F : A \subseteq F, X \setminus F \in \tau_\gamma\}$, where τ_γ denotes the set of all γ-open sets in (X,τ).

Definition 4. *An operation γ on $\beta O(X)$ is a mapping $\gamma : \beta O(X) \to P(X)$ from $\beta O(X)$ to the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \beta O(X)$ and V^γ denotes the value of γ at V.*

Definition 5. *Let (X,τ) be a topological space and γ an operation on $\beta O(X)$. Then a subset A of X is said to be β-γ-open if for each $x \in A$, there exists a β-open set U such that $x \in U$ and $U^\gamma \subseteq A$. Also $\beta O(X)_\gamma$ denotes the family of β-γ-open sets in X.*

The following is an example:

Example 1. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and γ be an operation on $\beta O(X)$ such that $A^\gamma = A$, if $b \in A; A^\gamma = Cl(A)$ if $b \notin A$. Then $\beta O(X)_\gamma = \{\emptyset, X, \{a,c\}, \{b,c\}, \{b\}, \{a,b\}\}$.*

Theorem 1. *Let γ be an operation on $\beta O(X)$. Then the following statements hold:*

(i) Every β-γ-open set of (X,τ) is β-open in (X,τ), i.e, $\beta O(X)_\gamma \subseteq \beta O(X)$.

(ii) Every γ-open set of (X,τ) is β-γ-open.

(iii) Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of β-γ-open sets in (X,τ).

Then $\bigcup\{A_\alpha : \alpha \in J\}$ is also a β-γ-open set in (X,τ).

Proof. (i): Let $A \in \beta O(X)_\gamma$. Let $x \in A$. Then there exists a β-open set U such that $x \in U \subseteq U^\gamma \subseteq A$. As U is a β-open set, this implies that $x \in U \subseteq Cl(Int(Cl(U))) \subseteq Cl(Int(Cl(A)))$. Thus we show that $A \subseteq Cl(Int(Cl(A)))$ and hence $A \in \beta O(X)$. Therefore $\beta O(X)_\gamma \subseteq \beta O(X)$.

(ii): Let A be a γ-open set in (X,τ) and $x \in A$. There exists an open set U such that $x \in U \subseteq U^\gamma \subseteq A$. Since every open set is β-open, this implies that A is a β-γ-open set.

(iii): If $x \in \bigcup\{A_\alpha : \alpha \in J\}$, then $x \in A_\alpha$ for some $\alpha \in J$. Since A_α is a β-γ-open set, so there exists a β-open set U such that $U^\gamma \subseteq A_\alpha \subseteq \bigcup\{A_\alpha : \alpha \in J\}$. Therefore, $\bigcup\{A_\alpha : \alpha \in J\}$ is also a β-γ-open set in (X, τ). □

Remark 1. *(i) The converse of (i) and (ii) of the above theorem is not true. Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. *Let* γ *be the operation on* $\beta O(X)$ *such that* $A^\gamma = A$ *if* $b \in A$; $A^\gamma = Cl(A)$ *if* $b \notin A$. *Then* $\{a\}$ *is a* β*-open but not a* β-γ*-open set.*
(ii) Consider Example 1. $\{a, c\}$ *is a* β-γ*-open set but not a* γ*-open set.*
(iii) In general, the intersection of two β-γ*-open sets need not be a* β-γ*-open set. Consider Example 1. The sets* $A = \{a, c\}$ *and* $B = \{b, c\}$ *are* β-γ*-open sets in* (X, τ), *but* $A \cap B = \{c\}$ *is not a* β-γ*-open set in* (X, τ).

Definition 6. *(i) A space* (X, τ) *is said to be a* β-γ*-regular space if for each* $x \in X$ *and for each* β*-open set* V *containing* x, *there exists a* β*-open set* U *containing* x *such that* $U^\gamma \subseteq V$.
(ii) Let γ *be an operation on* $\beta O(X)$. *Then* γ *is said to be* β*-regular if for each* $x \in X$ *and for every pair of* β*-open sets* U *and* V *containing* x, *there exists a* β*-open set* W *such that* $x \in W$ *and* $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Theorem 2. *Let* (X, τ) *be a topological space and* γ *an operation on* $\beta O(X)$. *Then the following statements are equivalent:*

(1) $\beta O(X) = \beta O(X)_\gamma$.

(2) (X, τ) *is a* β-γ*-regular space.*

(3) For every $x \in X$ *and every* β*-open set* U *of* (X, τ) *containing* x *there exists a* β-γ*-open set* W *of* (X, τ) *such that* $x \in W$ *and* $W \subseteq U$.

Proof. (1)⇒(2): Let $x \in X$ and let V be a β-open set containing x. Then by assumption, V is a β-γ-open set. This implies that for each $x \in V$, there exists a β-γ-open set U such that $U^\gamma \subseteq V$. Therefore (X, τ) is a β-γ-regular space.

(2)⇒(3): Let $x \in X$ and let U be a β-open set containing x. Then by (2), there is a β-open set W containing x and $W \subseteq W^\gamma \subseteq U$. Applying (2) to set W shows that W is β-γ-open. Hence W is a β-γ-open set containing x such that $W \subseteq U$.

(3)⇒(1): By (3) and Theorem 1 (iii), it follows that every β-open set is β-γ-open, i.e., $\beta O(X) \subseteq \beta O(X)_\gamma$. Also from Theorem 1 (i), $\beta O(X)_\gamma \subseteq \beta O(X)$. Hence we have the result. □

Theorem 3. *Let* γ *be a* β*-regular operation on* $\beta O(X)$. *Then the following statements hold:*

(i) If A *and* B *are* β-γ*-open sets in* (X, τ), *then* $A \cap B$ *is also a* β-γ*-open set in* (X, τ).

(ii) $\beta O(X)_\gamma$ *forms a topology on* X.

Proof. (i): Let $x \in A \cap B$. Since A and B are β-γ-open sets, there exist β-open sets U and V such that $x \in U, V$ and $U^\gamma \subseteq A$ and $V^\gamma \subseteq B$. By β-regularity of γ, there exists a β-open set W containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma \subseteq A \cap B$. Therefore $A \cap B$ is a β-γ-open set.

(ii): It follows by (i) above and Theorem 1 (iii). □

Definition 7. *Let γ be an operation on $\beta O(X)$. The set A is said to be β-γ-closed if $X \setminus A$ is β-γ-open.*

Definition 8. *Let γ be an operation on $\beta O(X)$. The point $x \in X$ is said to be β-γ-closure of the set A if $U^\gamma \cap A \neq \emptyset$ for each β-open set U containing x. $\beta Cl_\gamma(A)$ denotes the β-γ-closure of a set A.*

Definition 9. *Let γ be an operation on $\beta O(X)$. Then $\beta O(X)_\gamma$-$Cl(A)$ is defined as the intersection of all β-γ-closed sets containing A.*

Theorem 4. *Let (X, τ) be a topological space and A a subset of X. Let γ be an operation on $\beta O(X)$. Then for each point $y \in X, y \in \beta O(X)_\gamma$-$Cl(A)$ if and only if $V \cap A \neq \emptyset$ for every $V \in \beta O(X)_\gamma$ such that $y \in V$.*

Proof. Let $E = \{y \in X |\ (V \cap A) \neq \emptyset$ for every $V \in \beta O(X)_\gamma$ and $y \in V\}$. To prove the theorem, it is enough to show that $E = \beta O(X)_\gamma$-$Cl(A)$. Let $x \notin E$. Then there exists a $V \in \beta O(X)_\gamma$ such that $V \cap A = \emptyset$. This implies that $X \setminus V$ is β-γ-closed and $A \subseteq X \setminus V$. Hence $\beta O(X)_\gamma$-$Cl(A) \subseteq X \setminus V$. It follows that $x \notin \beta O(X)_\gamma$-$Cl(A)$. Thus $\beta O(X)_\gamma$-$Cl(A) \subseteq E$. Conversely, let $x \notin \beta O(X)_\gamma$-$Cl(A)$. Then there exists a β-γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then we have $x \in X \setminus F, X \setminus F \in \beta O(X)_\gamma$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin E$. Hence $E \subseteq \beta O(X)_\gamma$-$Cl(A)$. □

Theorem 5. *Let (X, τ) be a topological space, A and B subsets of X and γ an operation on $\beta O(X)$. Then the following relations holds:*

(i) *The set $\beta O(X)_\gamma$-$Cl(A)$ is β-γ-closed and $A \subseteq \beta O(X)_\gamma$-$Cl(A)$.*

(ii) *A is β-γ-closed if and only if $A = \beta O(X)_\gamma$-$Cl(A)$.*

(iii) *If $A \subseteq B$, then $\beta O(X)_\gamma$-$Cl(A) \subseteq \beta O(X)_\gamma$-$Cl(B)$.*

(iv) *$(\beta O(X)_\gamma$-$Cl(A)) \cup (\beta O(X)_\gamma$-$Cl(B)) \subseteq (\beta O(X)_\gamma$-$Cl(A \cup B))$.*

(v) *If γ is β-regular, then $(\beta O(X)_\gamma$-$Cl(A)) \cup (\beta O(X)_\gamma$-$Cl(B)) = (\beta O(X)_\gamma$-$Cl(A \cup B))$.*

(vi) *$(\beta O(X)_\gamma$-$Cl(A \cap B)) \subseteq (\beta O(X)_\gamma$-$Cl(A)) \cap (\beta O(X)_\gamma$-$Cl(B))$.*

(vii) *$(\beta O(X)_\gamma$-$Cl(\beta O(X)_\gamma$-$Cl(A)) = \beta O(X)_\gamma$-$Cl(A))$.*

Proof. (i): It is obvious from Theorem 1 (iii), Definitions 7 and 9.

(ii): It is clear from (i), Definitions 7, 9.

(iii): It is obvious from Definition 9.

(iv), (vi): These proofs are obvious from (iii).

(v): Let $x \notin (\beta O(X)_\gamma$-$Cl(A)) \cup (\beta O(X)_\gamma$-$Cl(B))$. Then there exists two β-γ-open sets U and V containing x such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. By Theorem 3 (i), it is proved that $U \cap V$ is β-γ-open in (X, τ) such that $(U \cap V) \cap (A \cup B) = \emptyset$. Thus we have $x \notin (\beta O(X)_\gamma\text{-}Cl(A \cup B))$ and hence $\beta O(X)_\gamma\text{-}Cl(A \cup B) \cup (\beta O(X)_\gamma\text{-}Cl(A)) \subseteq (\beta O(X)_\gamma\text{-}Cl(B))$. Using (iv), we have the equality.

(vii): From (i), we have $\beta O(X)_\gamma\text{-}Cl(A) \subseteq \beta O(X)_\gamma\text{-}Cl(\beta O(X)_\gamma\text{-}Cl(A))$. For $\beta O(X)_\gamma\text{-}Cl(\beta O(X)_\gamma\text{-}Cl(A)) \subseteq \beta O(X)_\gamma\text{-}Cl(A)$, let $x \in \beta O(X)_\gamma\text{-}Cl(\beta O(X)_\gamma\text{-}Cl(A))$ and V be any β-γ-open set containing x. We claim that $V \cap A \neq \emptyset$. Indeed, by Theorem 4, $V \cap (\beta O(X)_\gamma\text{-}Cl(A)) \neq \emptyset$ and so there exists a point z such that $z \in V$ and $z \in \beta O(X)_\gamma\text{-}Cl(A)$. Moreover, by Theorem 4, for a point z, it is shown that $V \cap A \neq \emptyset$. Thus, we have that for any point $x \in V, V \cap A \neq \emptyset$ and so $x \in \beta O(X)_\gamma\text{-}Cl(A)$. Hence we conclude that $\beta O(X)_\gamma\text{-}Cl(\beta O(X)_\gamma\text{-}Cl(A)) \subseteq \beta O(X)_\gamma\text{-}Cl(A)$. Hence we have $\beta O(X)_\gamma\text{-}Cl(\beta O(X)_\gamma\text{-}Cl(A)) = \beta O(X)_\gamma\text{-}Cl(A)$. □

Theorem 6. *Let $\gamma : \beta O(X) \to P(X)$ be an operation on $\beta O(X)$ and A and B subsets of X. Then the following relations hold:*

(i) $\beta Cl_\gamma(A)$ is a β-closed set in (X, τ) and $A \subseteq \beta Cl_\gamma(A)$.

(ii) A is β-γ-closed in (X, τ) if and only if $A = \beta Cl_\gamma(A)$ holds.

(iii) If (X, τ) is β-γ-regular, then $\beta Cl_\gamma(A) = \beta Cl(A)$.

(iv) If $A \subseteq B$, then $\beta Cl_\gamma(A) \subseteq \beta Cl_\gamma(B)$.

(v) $\beta Cl_\gamma(A) \cup \beta Cl_\gamma(B) \subseteq \beta Cl_\gamma(A \cup B)$ holds for any subsets A and B of X.

(vi) Let γ be a β-regular operation on $\beta O(X)$, then $\beta Cl_\gamma(A \cup B) = \beta Cl_\gamma(A) \cup \beta Cl_\gamma(B)$ holds for any subsets A and B of X.

(vii) $\beta Cl_\gamma(A \cap B) \subseteq \beta Cl_\gamma(A) \cap \beta Cl_\gamma(B)$ holds.

(viii) If γ is β-open, then $\beta Cl_\gamma(A) = \beta O(X)_\gamma\text{-}Cl(A)$ and $\beta Cl_\gamma(\beta Cl_\gamma(A)) = \beta Cl_\gamma(A)$.

Proof. (i): Let $x \in \beta Cl(\beta Cl_\gamma(A))$. Then $U \cap \beta Cl_\gamma(A) \neq \emptyset$ for every β-open set U containing x. Let $y \in U \cap \beta Cl_\gamma(A)$. Then $y \in U$ and $y \in \beta Cl_\gamma(A)$. Since U is a β-open set containing y, this implies $U^\gamma \cap A \neq \emptyset$. Thus, $x \in \beta Cl_\gamma(A)$. Hence $\beta Cl(\beta Cl_\gamma(A)) \subseteq \beta Cl_\gamma(A)$. This implies that $\beta Cl_\gamma(A)$ is a β-closed set (from Definition 2.1). Also, $A \subseteq \beta Cl_\gamma(A)$ is clear by Definition 8.

(ii) (Necessity): Suppose that $X \setminus A$ is β-γ-open in (X, τ). We claim that $\beta Cl_\gamma(A) \subseteq A$. Let $x \notin A$. There exists a β-open set U containing x such that $U^\gamma \subseteq X \setminus A$, i.e., $U^\gamma \cap A = \emptyset$. Hence using Definition 8, we have that $x \notin \beta Cl_\gamma(A)$ and so $\beta Cl_\gamma(A) \subseteq A$. So by (i), it is proved that $A = \beta Cl_\gamma(A)$.

(ii) (Sufficiency): Suppose that $A = \beta Cl_\gamma(A)$. Let $x \in X \setminus A$. Since $x \notin \beta Cl_\gamma(A)$, there exists a β-open set U containing x such that $U^\gamma \cap A = \emptyset$, i.e., $U^\gamma \subseteq X \setminus A$, namely $X \setminus A$ is β-γ-open in (X, τ) and so A is β-γ-closed.

(iii): By Definition 8, we have $\beta Cl(A) \subseteq \beta Cl_\gamma(A)$. Let $x \notin \beta Cl(A)$. Then, there exists a β-open set U containing x such that $U \cap A = \emptyset$. Using β-γ-regularity of (X, τ), there exist a β-open set V containing x such that $V^\gamma \subseteq U$ and so $V^\gamma \cap A = \emptyset$. Thus we have that $x \notin \beta Cl_\gamma(A)$. Therefore we have that $\beta Cl_\gamma(A) \subseteq \beta Cl(A)$.

(iv): It is obvious by Definition 8.

(v): It is obvious from (iv).

(vi): It is enough to show that $\beta Cl_\gamma(A \cup B) \subseteq \beta Cl_\gamma(A) \cup \beta Cl_\gamma(B)$. Let $x \notin \beta Cl_\gamma(A) \cup \beta Cl_\gamma(B)$. Then, there exist β-open sets U and V such that $x \in U, x \in V, U^\gamma \cap A = \emptyset$ and $V^\gamma \cap B = \emptyset$. Since γ is β-regular, by Definition 6, there exists a β-open set W containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. Thus we have $W^\gamma \cap (A \cup B) \subseteq (U^\gamma \cap V^\gamma) \cap (A \cup B) \subseteq (U^\gamma \cap A) \cup (V^\gamma \cap B) = \emptyset$, i.e., $W^\gamma \cap (A \cup B) = \emptyset$. Hence $x \notin \beta Cl_\gamma(A \cup B)$ and so $\beta Cl_\gamma(A \cup B) \subseteq \beta Cl_\gamma(A) \cup \beta Cl_\gamma(B)$.

(vii): It is obvious by Definition 8.

(viii): By Theorem 5 (i), we have $\beta Cl_\gamma(A) \subseteq \beta O(X)_\gamma\text{-}Cl(A)$. Now we prove that $\beta O(X)_\gamma\text{-}Cl(A) \subseteq \beta Cl_\gamma(A)$. Let $x \notin \beta Cl_\gamma(A)$. Then, there exists a β-open set U containing x such that $U^\gamma \cap A = \emptyset$. Since γ is β-open, there exists a β-γ-open set S such that $x \in S \subseteq U^\gamma$. Therefore $S \cap A = \emptyset$. This implies that $x \notin \beta O(X)_\gamma\text{-}Cl(A)$ and so $\beta O(X)_\gamma\text{-}Cl(A) = \beta Cl_\gamma(A)$. Since $\beta O(X)_\gamma\text{-}Cl(\beta O(X)_\gamma\text{-}Cl(A)) = \beta O(X)_\gamma\text{-}Cl(A)$ (Theorem 5 (vii)). Hence we have that $\beta Cl_\gamma(\beta Cl_\gamma(A)) = \beta Cl_\gamma(A)$. ◻

Remark 2. *We cannot remove the assumption of β-regularity of γ in Theorem 6 (vi). Consider Example 1 and let $\gamma : \beta O(X) \to P(X)$ be an operation defined by $\gamma(A) := Cl(A)$ for any $A \in \beta O(X)$. Now $\beta O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ and $\beta O(X)_\gamma = \{\emptyset, X, \{b,c\}, \{a,c\}\}$. Let $A = \{a\}$ and $B = \{b\}$. Then, $\beta Cl_\gamma(A \cup B) = X$;$\beta Cl_\gamma(A) = \{a\}$;$\beta Cl_\gamma(B) = \{b\}$. The operation γ is not β-regular.*

Theorem 7. *For any subset A of a topological space (X, τ) and any operation $\gamma : \beta O(X) \to P(X)$, the following inclusions hold.*

(i) $\beta Cl(A) \subseteq \beta Cl_\gamma(A) \subseteq \beta O(X)_\gamma\text{-}Cl(A) \subseteq \tau_\gamma\text{-}Cl(A)$.

(ii) $\beta Cl(A) \subseteq Cl(A) \subseteq Cl_\gamma(A) \subseteq \tau_\gamma\text{-}Cl(A)$.

Proof. (i): The implication $\beta Cl(A) \subseteq \beta Cl_\gamma(A)$ is obtained by Definitions 2 and 8. The implication $\beta Cl_\gamma(A) \subseteq \beta O(X)_\gamma\text{-}Cl(A)$ is obtained from Definitions 5, 8 and 9. The implication $\beta O(X)_\gamma\text{-}Cl(A) \subseteq \tau_\gamma\text{-}Cl(A)$ is obtained from Definition 9.

(ii): The implication $\beta Cl(A) \subseteq Cl(A)$ is trivial, the implication $Cl(A) \subseteq Cl_\gamma(A)$ is obtained by Definition 3. The implication $Cl_\gamma(A) \subseteq \tau_\gamma\text{-}Cl(A)$ is obtained from Definitions 3 and 9. ◻

Theorem 8. *Let (X, τ) be a topological space, A a subset of X and γ an operation on $\beta O(X)$. Then the following are equivalent.*

(1) A is β-γ-open.

(2) $\beta Cl_\gamma(X \setminus A) = X \setminus A$.

(3) $\beta O(X)_\gamma\text{-}Cl(X \setminus A) = X \setminus A$.

(4) $X \setminus A$ is β-γ-closed.

Proof. (1)⇔(2): It is obtained by Theorem 6(ii).

(3)⇔(4): It is proved by Theorem 5 (ii).

(4)⇔(1): It is proved by Definitions 9 and 7. ◻

3. β-γ-T_i Spaces

Definition 10. *(i) A space (X,τ) is called β-γ-T_0 if for any two distinct points $x, y \in X$, there exists a β-open set U such that either $x \in U$ and $y \notin U^\gamma$ or $y \in U$ and $x \notin U^\gamma$.*
(ii) A space (X,τ) is called a β-γ-T_0' if for any two distinct points $x, y \in X$, there exists β-γ-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

Definition 11. *(i). A space (X,τ) is called β-γ-T_1 if for any two distinct points $x, y \in X$, there exist two β-open sets U and V containing x and y, respectively, such that $y \notin U^\gamma$ and $x \notin V^\gamma$.*
(ii) A space (X,τ) is called β-γ-T_1' if for any two distinct points $x, y \in X$, there exists two β-γ-open sets U and V containing x, y respectively such that $y \notin U$ and $x \notin V$.

Definition 12. *(i) A space (X,τ) is called β-γ-T_2 if for any two distinct points $x, y \in X$, there exists β-open set U, V such that $x \in U, y \in V$ and $U^\gamma \cap V^\gamma = \emptyset$.*
(ii) A space (X,τ) is called β-γ-T_2' if for any two distinct points $x, y \in X$, there exist β-γ-open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 13. *Let γ be an operation on $\beta O(X)$. Then γ is said to be β-open if for each point $x \in X$ and for every open set U containing x, there exists a β-γ-open set V such that $x \in V$ and $V \subseteq U^\gamma$.*

Theorem 9.

(i) A space (X,τ) is a β-γ-T_0' space if and only if, for every pair $x, y \in X$ with $x \neq y$, $\beta O(X)_\gamma$-$Cl(x) \neq \beta O(X)_\gamma$-$Cl(y)$.

(ii) Let γ be a β-open operation. A space (X,τ) is a β-γ-T_0 space if and only if, for every pair $x, y \in X$ with $x \neq y$, $\beta Cl_\gamma(\{x\}) \neq \beta Cl_\gamma(\{y\})$.

(iii) Let γ be a β-open operation. A space (X,τ) is β-γ-T_0 if and only if it is β-γ-T_0'.

Proof. (i) (Necessity): Let x and y be any two distinct points of a β-γ-T_0' space (X,τ). Then, by definition, we assume that there exists a β-γ-open set U such that $x \in U$ and $y \notin U$. Hence $y \in X \setminus U$. Because $X \setminus U$ is a β-γ-closed set, we have $\beta O(X)_\gamma$-$Cl(\{y\}) \subseteq X \setminus U$ and so $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$.

(i) (Sufficiency): Suppose that for any $x, y \in X, x \neq y$. Thus we have $\beta O(X)_\gamma$-$Cl(\{x\}) \neq \beta O(X)_\gamma$-$Cl(\{y\})$. Thus we assume that there exists $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ such that $z \notin \beta O(X)_\gamma$-$Cl(\{y\})$. We shall prove that $x \notin \beta O(X)_\gamma$-$Cl(\{y\})$. Indeed if $x \in \beta O(X)_\gamma$-$Cl(\{y\})$, then we get $\beta O(X)_\gamma$-$Cl(\{x\}) \subseteq \beta O(X)_\gamma$-$Cl(\{y\})$ (by Definition and Theorem 5 (iii)). This contradiction shows that $X \setminus (\beta O(X)_\gamma$-$Cl(\{y\}))$ is a β-γ-open set containing x but not y. Hence (X,τ) is a β-γ-T_0' space.

(ii) (Necessity): Let x and y be any two distinct points of a β-γ-T_0' space (X,τ). Then by definition, we assume that there exists a β-open set U such that $x \in U$ and $y \notin U^\gamma$. It follows from the assumption that there exists a β-γ-open set S such that $x \in S$ and $S \subseteq U^\gamma$. Hence $y \in X \setminus U^\gamma \subseteq X \setminus S$. Because $X \setminus S$ is a β-γ-closed set, we obtain that $\beta Cl_\gamma(\{y\}) \subseteq X \setminus S$ and so $\beta Cl_\gamma(\{x\}) \neq \beta Cl_\gamma(\{y\})$.

(ii) (Sufficiency): Suppose that $x \neq y$ for any $x, y \in X$. Then we have that $\beta Cl_\gamma(\{x\}) \neq \beta Cl_\gamma(\{y\})$. Thus we assume that there exists $z \in \beta Cl_\gamma(\{x\})$ but $z \notin \beta Cl_\gamma(\{y\})$. If $x \in \beta Cl_\gamma(\{y\})$, then we get $\beta Cl_\gamma(\{x\}) \subseteq \beta Cl_\gamma(\{y\})$ (Theorem 5 (iii)). This implies that $z \in \beta Cl_\gamma(\{y\})$. This contradiction shows that $x \notin \beta Cl_\gamma(\{y\})$. So by Definition 8, there exists a β-open set W such that $x \in W$ and $W^\gamma \cap \{y\} = \emptyset$. Thus, we have that $x \in W$ and $y \notin W^\gamma$. Hence (X, τ) is β-γ-T_0.

(iii): This follows from (i), (ii) and the fact that, for any subset A of (X, τ), $\beta O(X)_\gamma$-$Cl(A) = \beta Cl_\gamma(A)$ holds under the assumption that γ is β-open (Theorem 6 (iii)). □

Definition 14. *A space (X, τ) is said to be β-γ-$T_{1/2}$ if every β-γ-g.closed set of (X, τ) is β-γ-closed.*

Theorem 10. *Let (X, τ) be a topological space and γ be an operation on $\beta O(X)$. Then the following statements are equivalent:*

(1) A is β-γ-g.closed in (X, τ).

(2) $(\beta O(X)_\gamma$-$Cl(x)) \cap A \neq \emptyset$ for every $x \in \beta Cl_\gamma(A)$.

(3) $\beta Cl_\gamma(A) \subseteq \beta O(X)_\gamma$-$Ker(A)$ holds, where $\beta O(X)_\gamma$-$Ker(E) = \cap\{V | E \subseteq V, V \in \beta O(X)_\gamma\}$ for any subset E of (X, τ).

Proof. (1)⇒(2): Let A be a β-γ-g.closed set of (X, τ). Suppose that there exists a $x \in \beta Cl_\gamma(A)$ such that $(\beta O(X)_\gamma$-$Cl(\{x\})) \cap A = \emptyset$. By Theorem 5 (i), $\beta O(X)_\gamma$-$Cl(\{x\})$ is β-γ-closed. Since $A \subseteq X \setminus (\beta O(X)_\gamma$-$Cl(\{x\}))$ and A is β-γ-g.closed, we have that $\beta Cl_\gamma(A) \subseteq X \setminus (\beta O(X)_\gamma$-$Cl(\{x\}))$ and hence $x \notin \beta Cl_\gamma(A)$. This is a contradiction. Therefore, $(\beta O(X)_\gamma$-$Cl(\{x\})) \cap A \neq \emptyset$.

(2)⇒(3): Let $x \in \beta Cl_\gamma(A)$. By (2), there exists a point z such that $z \in \beta O(X)_\gamma$-$Cl(\{x\})$ and $z \in A$. Let $U \in \beta O(X)_\gamma$ be a subset of X such that $A \subseteq U$. Since $z \in U$ and $z \in \beta O(X)_\gamma$-$Cl(\{x\})$, we have that $U \cap \{x\} \neq \emptyset$. Hence we show that $x \in \beta O(X)_\gamma$-$Ker(A)$. Therefore $\beta Cl_\gamma(A) \subseteq (\beta O(X)_\gamma$-$Ker(A))$.

(3)⇒(1): Let U be any β-γ-open set such that $A \subseteq U$. Let x be a point such that $x \in \beta Cl_\gamma(A)$. By (3), $x \in \beta O(X)_\gamma$-$Ker(A)$ holds. So we have $x \in U$ because $A \subseteq U$ and $U \in \beta O(X)_\gamma$. □

Theorem 11. *Let (X, τ) be a topological space and γ an operation on $\beta O(X)$. If a subset A of X is β-γ-g.closed, then $\beta Cl_\gamma(A) \setminus A$ does not contain any non-empty β-γ-closed set.*

Proof. Suppose that there exists a non-empty β-γ-closed set F such that $F \subseteq \beta Cl_\gamma(A) \setminus A$. Then we have $A \subseteq X \setminus F$ and $X \setminus F$ is β-γ-open. It follows from the assumption that $\beta Cl_\gamma(A) \subseteq X \setminus F$ and so $F \subseteq (\beta Cl_\gamma(A) \setminus A) \cap (X \setminus \beta Cl_\gamma(A))$. Therefore, we have $F = \emptyset$. □

Remark 3. *In the above theorem, if γ is a β-open operation, then the converse of the above theorem is true.*

Proof. Let U be a β-γ-open set such that $A \subseteq U$. Since γ is a β-open operation, it follows from Theorem 6 (iii) that $\beta Cl(A)$ is β-γ-closed in (X,τ). Thus by Theorem 6 (iii) and Definition 7, we have $\beta Cl(A) \cap (X \setminus U) = F$ is β-γ-closed in (X,τ). Since $X \setminus U \subseteq X \setminus A, F \subset \beta Cl(A) \setminus A$. Using the assumptions of the converse of Theorem 11 above, $F = \emptyset$ and hence $\beta Cl_\gamma(A) \subseteq U$. □

Theorem 12. *Let (X,γ) be a topological space and γ an operation on $\beta O(X)$. Then for each $x \in X, \{x\}$ is β-γ-closed or $X \setminus \{x\}$ is β-γ-g.closed in (X,τ).*

Proof. Suppose that $\{x\}$ is not β-γ-closed, then $X \setminus \{x\}$ is not β-γ-open. Let U be any β-γ-open set such that $X \setminus \{x\} \subseteq U$. Then $U = X$. Hence, $\beta Cl_\gamma(X \setminus \{x\}) \subseteq U$. Therefore, $X \setminus \{x\}$ is a β-γ-g.closed set. □

Theorem 13. *Let (X,τ) be a topological space and γ an operation on $\beta O(X)$. Then, the following properties are equivalent.*

(1) A space (X,τ) is β-γ-$T_{1/2}$.

(2) For each $x \in X$, $\{x\}$ is β-γ-closed or β-γ-open.

Proof. (1)⇒(2): Suppose $\{x\}$ is not β-γ-closed in (X,τ). Then, $X \setminus \{x\}$ is β-γ-g.closed by Theorem 12. Since (X,τ) is a β-γ-$T_{1/2}$ space, so by definition, $X \setminus \{x\}$ is β-γ-closed and so $\{x\}$ is β-γ-open.

(2)⇒(1): Let F be a β-γ-g.closed set in (X,τ). We shall prove that $\beta Cl_\gamma(F) = F$ (from Theorem 6 (ii)). It is sufficient to show that $\beta Cl_\gamma(F) \subseteq F$. Assume that there exists a point x such that $x \in \beta Cl_\gamma(F) \setminus F$. Then by assumption, $\{x\}$ is β-γ-closed or β-γ-open.

Case 1. $\{x\}$ is a β-γ-closed set: for this case, we have a β-γ-closed set $\{x\}$ such that $\{x\} \subseteq \beta Cl_\gamma(F) \setminus F$. This is a contradiction to Theorem 11.

Case 2. $\{x\}$ is a β-γ-open set: we have $x \in \beta O(X)_\gamma$-$Cl(F)$. Since $\{x\}$ is β-γ-open, it implies that $\{x\} \cap F \neq \emptyset$ by Theorem 4. This is a contradiction. Thus we have $\beta Cl_\gamma(F) = F$ and so by Theorem 6 (ii), F is β-γ-closed. □

Theorem 14. *For a topological space (X,τ), let γ be an operation on $\beta O(X)$.*

(i) Then, the following properties are equivalent.

  (1) (X,τ) is β-γ-T_1.

  (2) For every point $x \in X, \{x\}$ is a β-γ-closed set.

  (3) (X,τ) is β-γ-T_1'.

(ii) Every β-γ-T_i' space is β-γ-T_i, where $i \in \{2,0\}$.

(iii) Every β-γ-T_2 space is β-γ-T_1.

(iv) Every β-γ-T_1 space is β-γ-$T_{1/2}$.

(v) Every β-γ-$T_{1/2}$ space is β-γ-T_0'.

(vi) Every β-γ-T_i' space is β-γ-T_{i-1}', where $i \in \{2,1\}$.

Proof. (i) (1)⇒(2): Let $x \in X$ be a point. For each point $y \in X\setminus\{x\}$, there exists a β-open set V_y such that $y \in V_y$ and $x \notin (V_y)^\gamma$. Then $X\setminus\{x\} = \bigcup\{(V_y)^\gamma | y \in X\setminus\{x\}\}$. It is shown that $X \setminus \{x\}$ is β-γ-open in (X, τ).

(2)⇒(3): Let x and y be two distinct points of X. By (2), $X\setminus\{x\}$ and $X\setminus\{y\}$ are required β-γ-open sets such that $y \in X\setminus\{x\}, x \notin X\setminus\{x\}$ and $x \in X\setminus\{y\}, x \notin X\setminus\{y\}$.

(3)⇒(1): It is shown that if $x \in U$, where $U \in \beta O(X)_\gamma$, then there exists a β-open set V such that $x \in V \subseteq V^\gamma \subseteq U$. Using (3), we have that (X, τ) is β-γ-T_1'.

(ii), (iii), (vi): These proofs are obvious by definition.

(iv): This follows from (i) above and Theorem 13.

(v): This follows from Theorem 13 and Definition 10 (ii). □

Remark 4. *By Theorems 13 and 14 we have the following diagram of implications:*

$$\begin{array}{ccccc} \beta-\gamma-T_2' \Rightarrow & \beta-\gamma-T_1' & \Rightarrow & \beta-\gamma-T_0' & \\ \Downarrow & \Downarrow & & \Downarrow & \\ \beta-\gamma-T_2 \Rightarrow & \beta-\gamma-T_1 & & \beta-\gamma-T_0 & \\ & \searrow & & \swarrow & \\ & & \beta-\gamma-T_{1/2} & & \end{array}$$

Remark 5. *None of the above reverse implications (except β-γ-$T_2' \Rightarrow \beta$-γ-T_2 and β-γ-$T_1' \Rightarrow \beta$-γ-T_1) are true as shown in examples below.*
(i) Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Define an operation γ on $\beta O(X)$ such that $A^\gamma = A$ if $b \in A$; $A^\gamma = Cl(A)$ if $b \notin A$. Then (X, τ) is β-γ-T_0 but not β-γ-$T_{1/2}$.
(ii) Consider Example 1. There (X, τ) is β-γ-$T_{1/2}$ but not β-γ-T_1.
(iii) Let $X = \{a, b, c\}, \tau = P(X)$, the power set on X. Define an operation γ on $\beta O(X)$ such that $A^\gamma = A \cup \{c\}$ if $A = \{a\}$ or $\{b\}$; $A^\gamma = A \cup \{a\}$ if $A = \{c\}$; $A^\gamma = A$ if $A \neq \{a\}, \{b\}, \{c\}$. Then (X, τ) is β-γ-T_1 but not β-γ-T_2. Also, (X, τ) is β-γ-T_1' (Theorem 14 (i)) but not β-γ-T_2'.
(iv) Consider (i) above. Define the operation γ on $\beta O(X)$ such that $A^\gamma = A$ for every set A such that $A \neq \{a\}; A^\gamma = \{a, b\}$ if $A = \{a\}$. Then (X, τ) is β-γ-T_0' but not β-γ-T_1'.
(v) Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Define an operation γ on $\beta O(X)$ such that $\{a\}^\gamma = \{a, c\}$, $\{b\}^\gamma = \{a, b\}, \{a, b\}^\gamma = \{a, b\}, \{b, c\}^\gamma = \{a, b\}, \{a, c\}^\gamma = \{a, b\}, \emptyset^\gamma = \emptyset, X^\gamma = X$. Now the β-γ-open sets are $\{\emptyset, X, \{a, b\}\}$ and γ is not β-open. Then (X, τ) is not β-γ-T_0'. Indeed for every β-γ-open set V_a containing a, we have $b \in V_a$, for every β-γ-open set V_b containing b, we have $a \in V_b$. Hence by Definition 1 (ii), (X, τ) is not β-γ-T_0'. Moreover, (X, τ) is β-γ-T_0.

Remark 6. *In Remark 5, Example (v) shows that β-openness of γ in Theorem 9 (iii) cannot be dropped.*

4. β-(γ, b)-continuous maps

Throughout Sections 4 and 5, let $\gamma : \beta O(X) \to P(X)$ and $b : \beta O(Y) \to P(Y)$ be operations on $\beta O(X)$ and $\beta O(Y)$, respectively.

Definition 15. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be β-(γ, b)-continuous if for each $x \in X$ and each β-open set V containing $f(x)$, there exists a β-open set U such that $x \in U$ and $f(U^\gamma) \subseteq V^b$.*

Theorem 15. *Let $f : (X, \tau) \to (Y, \sigma)$ be a β-(γ, b)-continuous mapping. Then,*

(i) $f(\beta Cl_\gamma(A)) \subseteq \beta Cl_b(f(A))$ holds for every subset A of (X, τ),

(ii) for every β-b-open set B of $(Y, \sigma), f^{-1}(B)$ is β-γ-open, that is for any $B \in \beta O(Y)_b$, $f^{-1}(B) \subseteq \beta O(X)_\gamma$.

Proof. (i): Let $y \in f(\beta Cl_\gamma(A))$ and let V be any β-open set containing y. Then, there exists a point $x \in \beta Cl_\gamma(A)$ and a β-open set U containing x such that $f(x) = y$ and $f(U^\gamma) \subseteq V^b$. We have $U^\gamma \cap A \neq \emptyset$. Therefore, $\emptyset \neq f(U^\gamma \cap A) \subseteq f(U^\gamma) \cap f(A) \subseteq V^b \cap f(A)$ and so $y \in \beta Cl_b(f(A))$.

(ii): Let B be a β-b-closed set. Then using (i) we have that $f(\beta Cl_\gamma(f^{-1}(B)) \subseteq \beta Cl_b(f(f^{-1}(B)) \subseteq \beta Cl_b(B) = B$. Thus, $\beta Cl_\gamma(f^{-1}(B)) \subseteq f^{-1}(B)$ and hence $(f^{-1}(B)) = \beta Cl_\gamma(f^{-1}(B))$. This implies that $f^{-1}(B)$ is β-γ-closed in (X, τ). □

Remark 7. *In Theorem 15, the properties of β-(γ, b)-continuity of f, (i) and (ii) are equivalent to each other if one of the following conditions (a) and (b) is satisfied:*

(a) (Y, σ) is a β-b-regular space,

(b) b is a β-open operation.

Proof. It follows from the proof of Theorem 15 that we know the following implications: "β-(γ, b)-continuity of f" $\Rightarrow$ (i)$\Rightarrow$(ii). Thus, under condition (a), we first show the implication: (ii) $\Rightarrow$ β-(γ, b)-continuity of f. Let $x \in X$ and let V be a β-open set containing $f(x)$. Since (Y, σ) is a β-b-regular space, $V \in \beta O(Y)_b$. Then, by (ii) of Theorem 15, $x \in f^{-1}(V) \in \beta O(X)_\gamma$. So, by the definition of β-b-openness of $f^{-1}(V)$, there exists a β-open set U containing x such that $U^\gamma \subseteq f^{-1}(V)$ and so $f(U^\gamma) \subseteq V \subseteq V^b$. Therefore, f is β-(γ, b)-continuous.

Finally, under condition (b), we prove the implication: (ii) $\Rightarrow$ β-(γ, b)-continuity of f. Let $x \in X$ and let V be a β-open set containing $f(x)$. Since b is β-open, there exists a β-b-open set U containing $f(x)$ such that $U \subseteq V^b$. By (ii) of Theorem 15, $x \in f^{-1}(U) \in \beta O(X)_\gamma$ and so by definition of β-γ-openness of $f^{-1}(U)$, there exists a β-open set W containing x such that $W^\gamma \subseteq f^{-1}(U) \subseteq f^{-1}(V^b)$. Therefore, we have $f(W^\gamma) \subseteq V^b$ and so f is β-(γ, b)-continuous. □

Definition 16. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(i) β-(γ, b)-closed, if for every β-γ-closed set A of $(X, \tau), f(A)$ is β-b-closed in (Y, σ),

(ii) β-(id, b)-closed, if $f(F)$ is β-b-closed in (Y, σ) for every β-closed set F of (X, τ).

Theorem 16. *Suppose f is β-(γ, b)-continuous and f is β-(id, b)-closed. Then, the following properties hold.*

(i) For every β-γ-g.closed set A of (X,τ), the image $f(A)$ is β-b-g.closed.

(ii) For every β-b-g.closed set B of $(Y,\sigma), f^{-1}(B)$ is β-γ- g.closed.

Proof. (i): Let V be a β-b-open set in (Y,σ) such that $f(A)\subseteq V$. Then by Theorem 15 (ii), $f^{-1}(V)$ is β-γ-open. Since A is β-γ-g.closed and $A\subseteq f^{-1}(V)$, $\beta Cl_\gamma(A)\subseteq f^{-1}(V)$ holds and so $f(\beta Cl_\gamma(A))\subseteq V$. Thus, $f(\beta Cl_\gamma(A))$ is β-b-closed as $\beta Cl_\gamma(A)$ is β-closed by Theorem 6 (i) and the assumption that f is β-(id,b)-closed. Therefore $\beta Cl_b(f(A))\subseteq \beta Cl_b(f(\beta Cl\gamma(A))) = f(\beta Cl_\gamma(A))\subseteq V$. Hence, $f(A)$ is β-b-g.closed.

(ii): Let U be a β-γ-open set in (X,τ) such that $f^{-1}(B)\subseteq U$. Let $F = \beta Cl_\gamma(f^{-1}(B))\cap(X\setminus U)$. Then by Theorem 6 (i), F is β-closed in (X,τ). Since f is β-(id,b)-closed, $f(F)$ is β-b-closed in (Y,σ) and $f(F)\subseteq f(\beta Cl_\gamma(f^{-1}(B))\cap(X\setminus U))\subseteq \beta Cl\gamma(B)\setminus B$. By Theorem 11, $f(F)=\emptyset$ and so $F=\emptyset$. Hence $\beta Cl_\gamma(f^{-1}(B))\subseteq U$. Therefore, $f^{-1}(B)$ is β-γ-g.closed in (X,τ). □

Theorem 17. *Suppose that $f:(X,\tau)\to(Y,\sigma)$ is β-(γ,b)-continuous and β-(id,b)-closed. Then the following properties hold.*

(i) If f is injective and (Y,σ) is β-b-$T_{1/2}$, then (X,τ) is β-γ-$T_{1/2}$.

(ii) If f is surjective and (X,τ) is β-γ-$T_{1/2}$, then (Y,σ) is β-b-$T_{1/2}$.

Proof. (i): Let A be a β-γ-g.closed set of (X,τ). Then by Theorem 16 (i), $f(A)$ is β-b-g.closed. Since (X,τ) is β-γ-$T_{1/2}$, $f(A)$ is β-b-closed. By Theorem 16 (ii), $A=f^{-1}(f(A))$ is β-γ-closed. This implies A is β-γ-closed. Hence, (X,τ) is β-γ-$T_{1/2}$ space.

(ii): Let B is a β-b-g.closed set in (Y,σ). By Theorem 16 (ii), $f^{-1}(B)$ is β- γ-g.closed. Since (X,τ) is β-γ-$T_{1/2}$, so $f^{-1}(B)$ is β-γ-closed. Therefore $B=f(f^{-1}(B))$ is β-b-closed in (Y,σ). Hence, (Y,σ) is β-b-$T_{1/2}$ space. □

Definition 17. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then f is said to be β-(γ,b)-homeomorphic, if f is bijective, β-(γ,b)-continuous and f^{-1} is β-(b,γ)-continuous.*

Theorem 18. *Suppose that a mapping $f:(X,\tau)\to(Y,\sigma)$ is β-(γ,b)-homeomorphic. If (X,τ) is β-γ-$T_{1/2}$, then (Y,σ) is β-b-$T_{1/2}$.*

Proof. Let $\{y\}$ be a singleton set of (Y,σ). Then there exists a point $x\in X$ such that $y=f(x)$. By Theorem 13, $\{x\}$ is β-γ-open or β-γ-closed. Therefore by Theorem 15, $\{y\}$ is β-b-closed or β-b-open. Hence, (Y,σ) is β-b-$T_{1/2}$. □

Theorem 19. *Let $f:(X,\tau)\to(Y,\sigma)$ be a β-(γ,b)-continuous injection. If (Y,σ) is β-b-T_2 (resp. β-b-T_1), then (X,τ) is β-γ-T_2 (resp. β-γ-T_1).*

Proof. Suppose that (Y,σ) is β-b-T_2. Let x and y be distinct points of X. Then, there exist two β-open sets V and W of Y such that $f(x)\in V, f(y)\in W$ and $V^b\cap W^b=\emptyset$. Since f is β-(γ,b)-continuous, for V and W there exist two β-open sets U and S such that $x\in U, y\in S$, $f(U^\gamma)\subseteq V^b$ and $f(S^\gamma)\subseteq W^b$. Therefore, we have $U^\gamma\cap S^\gamma=\emptyset$ and hence (X,τ) is β-γ-T_2. Similarly, we can prove the case of β-γ-T_1. □

5. β-closed graphs of mappings

In this section, we further investigate general operator approaches of closed graphs of mappings. Let $(X \times Y, \tau \times \sigma)$ be the product space of topological spaces (X, τ) and (Y, σ) and let $\rho : \beta O(X \times Y) \to P(X \times Y)$ be an operation on $\beta O(X \times Y)$.

Some topological properties on $\beta O(\prod_{i=1}^{n} X_i)$ are investigated in [2] and [7, Lemma 3.1], where $\{X_i | i \in \nabla\}$ is any family of topological spaces with an index set ∇. For subsets $A \subseteq X$ and $B \subseteq Y, A \in \beta O(X)$ and $B \in \beta O(Y)$ if and only if $A \times B \in \beta O(X \times Y)$ hold. It is easily shown that $\beta O(X \times Y) \neq \beta O(X) \times \beta O(Y)$ similar to [6, Remark 8, Example 7]. Some properties of product functions involving b-closure and product spaces are studied in [2].

Definition 18. *Let $(X, \tau), (Y, \sigma)$ be two topological spaces and b an operation on $\beta O(Y)$. We say that the graph $G(f)$ of $f : X \to Y$ is β-b-closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists a β-open set U in X and V in Y contains x and y, respectively, such that $(U \times V^b) \cap G(f) = \emptyset$.*

Example 2. *Let $X = Y = \{x, y, z\}$ and $\tau = \sigma = \{\emptyset, , X, \{x\}, \{x, y\}\}$; then $\beta O(X) = \beta O(Y) = \{\emptyset, X, \{x\}, \{x, y\}, \{x, z\}\}$ hold. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping defined by $f(a) = z$ for every point $a \in X$. Define an operation b on $\beta O(Y)$ such that $A^b = A$ if $y \in A$ and $A^b = Cl(A)$ if $y \notin A$. Then, $G(f) = \{(x, z), (y, z), (z, z)\}$ and $G(f)$ is β-b-closed.*

Definition 19. *An operation $\rho : \beta O(X \times Y) \to P(X \times Y)$ is said to be β-associated with γ and b, if $(U \times V)^\rho = U^\gamma \times V^b$ holds for each set $U \in \beta O(X)$ and $V \in \beta O(Y)$.*

Example 3. *(i) Let $X = Y = \{x, y, z\}$ and $\tau = \sigma = \{\emptyset, X, \{x\}, \{x, y\}\}$. Define an operation γ on $\beta O(X)$ such that $A^\gamma = A$ if $y \in A$ and $A^\gamma = Cl(A)$ if $y \notin A$. Let b be the closure operation on $\beta O(Y)$, i.e., $A^b = Cl(A)$ for every $A \in \beta O(Y)$. Let ρ be the operation on $\beta O(X \times Y)$ defined as $(A \times B)^\rho = A \times B$ if $(x, y) \in A \times B$ and $(A \times B)^\rho = Cl(A \times B)$ if $(x, y) \notin A \times B$. Then, this operation ρ is not β-associated with γ and b. Indeed, for β-open subsets $U = \{x\} \in \beta O(X)$ and $V = \{x, y\} \in \beta O(Y)$, we have $(U \times V)^\rho = U \times V \neq X \times Y$ and $U^\gamma = Cl(U) = X$ and $V^b = Cl(V) = Y$.*
(ii) In general, for subsets $A \subseteq X$ and $B \subseteq Y$, $A \times B \subseteq Cl(A \times B)$ and $Cl(A \times B) = Cl(A) \times Cl(B)$ holds; especially, for subsets $A \in \beta O(X)$ and $B \in \beta O(Y)$, $A \times B \subseteq Cl(Int(Cl(A \times B)))$ and $Cl(Int(Cl(A \times B))) = Cl(Int(Cl(A))) \times Cl(Int(Cl(B)))$; and also, for any subset $U \in \beta O(X \times Y)$, $U \subseteq Cl(Int(Cl(U)))$ holds. Thus, the operations $id : \beta O(X \times Y) \to P(X \times Y)$, $Cl : \beta O(X \times Y) \to P(X \times Y)$ and $Cl \circ Int \circ Cl : \beta O(X \times Y) \to P(X \times Y)$ are well defined by $id(U) := U, Cl(U) := Cl(U)$ and $Cl \circ Int \circ Cl(U) = Cl(Int(Cl(U)))$ for every $U \in \beta O(X \times Y)$, respectively; moreover, they satisfy the condition of Definition 19.

Definition 20. *The operation $\rho : \beta O(X \times Y) \to P(X \times Y)$ is said to be β-regular with respect to γ and b, if for each point $(x, y) \in X \times Y$ and each β-open set W containing (x, y) there exist β-open sets U in (X, τ) and V in (Y, σ) such that $x \in U, y \in V$ and $U^\gamma \times V^b \subseteq W^\rho$.*

Theorem 20. *Let $\rho : \beta O(X \times X) \to P(X \times X)$ be a β-regular operation associated with γ and γ. If $f : (X,\tau) \to (Y,\sigma)$ a β-(γ,b)-continuous and (Y,σ) is β-b-T_2 space, then the set $A = \{(x,y) \in X \times X : f(x) = f(y)\}$ is a β-ρ-closed set of $(X \times X, \tau \times \tau)$.*

Proof. We have to show that $\beta Cl_\rho(A) \subseteq A$. Let $(x,y) \in (X \times X) \setminus A$. Then, there exist two β-open sets U and V in (Y,σ) such that $f(x) \in U, f(y) \in V$ and $U^b \cap V^b = \emptyset$. Moreover, for each U and V there exist β-open sets W and S in (X,τ) such that $x \in W$, $y \in S$ and $f(W^\gamma) \subseteq U^b$ and $f(S^\gamma) \subseteq V^b$. Therefore we have $(W \times S)^\rho \cap A = \emptyset$, because $(x,y) \in W^\gamma \times S^\gamma = (W \times S)^\rho$ and $W \times S \in \beta O(X \times X)$. This shows that $(x,y) \notin \beta Cl_\rho(A)$. □

Definition 21. *Let (X,τ) be a topological space and γ an operation on $\beta O(X)$. A subset K of X is said to be β-γ-compact, if for every β-open cover $\{G_i : i \in \mathbb{N}\}$ of K there exists a finite subfamily $\{G_1, G_2, \ldots, G_n\}$ such that $K \subseteq (G_1)^\gamma \cup (G_2)^\gamma \cup \ldots \cup (G_n)^\gamma$.*

Theorem 21. *Suppose that $\gamma : \beta O(X) \to P(X)$ is β-regular and $\rho : \beta O(X \times Y) \to P(X \times Y)$ is β-regular with respect to γ and b. Let $f : (X,\tau) \to (Y,\sigma)$ be a mapping whose graph $G(f)$ is β-b-closed in $(X \times Y, \tau \times \sigma)$. If a subset B is β-b-compact in (Y,σ), then $f^{-1}(B)$ is β-γ-closed in (X,τ).*

Proof. Let $x \notin f^{-1}(B)$. Then $(x,y) \notin G(f)$ for each $y \in B$. Since $\beta Cl\rho(G(f)) \subseteq G(f)$, there exists a β-open set W in $(X \times Y, \tau \times \sigma)$ such that $(x,y) \in W$ and $W^\rho \cap G(f) = \emptyset$. Since ρ is β-regular with respect to γ and b (cf. Definition 20), for each $y \in B$ we can take two subsets $U(y) \in \beta O(X)$ and $V(y) \in \beta O(Y)$ such that $x \in U(y), y \in V(y)$ and $U(y)^\gamma \times V(y)^b \subseteq W^\rho$. Then we have $f(U(y)^\gamma) \cap V(y)^b = \emptyset$ and so $U(y)^\gamma \cap f^{-1}(V(y)^b) = \emptyset$. Since $\{V(y)|y \in B\}$ is a β-open cover of B, then by β-b-compactness there exists $y_1, y_2, \ldots, y_n \in B$ such that $B \subseteq V(y_1)^b \cup V(y_2)^b \cup \cdots \cup V(y_n)^b$. By β-regularity of γ (cf. Definition 6 (ii)), there exist a β-open set U such that $x \in U$ and $U^\gamma \subseteq U(y_1)^\gamma \cup U(y_2)^\gamma \cup \ldots U(y_n)^\gamma$. Therefore we have $U^\gamma \cap f^{-1}(B) \subseteq \bigcup_{i=1}^n (U^\gamma \cap f^{-1}(V(y_i)^b) \subseteq \bigcup_{i=1}^n (U(y_i)^\gamma \cap f^{-1}(V(y_i)^b) = \emptyset$. This shows that $x \notin \beta Cl_\gamma(f^{-1}(B))$. Therefore, we show $\beta Cl_\gamma(f^{-1}(B)) \subset f^{-1}(B)$ and so $f^{-1}(B)$ is β-γ-closed. □

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