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COMPARATIVE ANALYSIS OF LIMIT BEARING CAPACITY OF A CONTINUOUS BEAM APPLYING THE LIMIT AND SHAKEDOWN ANALYSIS DEPENDING ON THE CHARACTER OF THE LOAD

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Original scientific paper

Applying the theorems of structural limit analysis it is possible to determine the limit load of linear systems exposed to load which increases proportionally until the formation of failure mechanism. In the case when beam systems are exposed to repeated load, the limit theorems do not yield the adequate solutions, thus the adaptation theorems which made safe limit load determination possible were developed simultaneously. In this paper, applying the static and kinematic theorem of limit and shakedown analysis, the failure load of continuous beam with two spans has been determined. Also displayed is the difference between the values of failure forces depending on the character of load and the beam span value in order to assess justification for application of the shakedown method in the analysis of the limit bearing capacity of the beams.

Key words: *continuous beam, failure mechanism, incremental failure force, limit failure force, residual bending moment.*

Komparativna analiza granične nosivosti kontinuiranog nosača primjenom granične analize i metode adaptacije ovisno o karakteru opterećenja

Izvorni znanstveni članak

Primjenom poučaka analize granične nosivosti moguće je odrediti granično opterećenje linearnih sustava izloženih opterećenju koje se proporcionalno povećava dok ne dođe do loma. U slučaju kada su nosivi sustavi izloženi ponovljenom opterećenju, postavke granične analize ne daju odgovarajuća rješenja. Zbog toga su simultano razvijeni adaptacijski poučci koji su omogućili sigurno određivanje graničnog opterećenja. U ovom je radu određeno opterećenje loma kontinuiranog nosača s dva polja, primjenom statičkog i kinematičkog poučka granične analize i metode adaptacije. Pokazana je također razlika između vrijednosti sila loma ovisno o karakteru opterećenja i vrijednosti raspona nosača da bi se procijenila opravdanost primjene metode adaptacije u analizi granične nosivosti nosača.

Ključne riječi: *kontinuirani nosač, mehanizam loma, prirast sile loma, granična sila loma, preostali moment savijanja*

1 Introduction

Many engineering structures or some of their parts are exposed to various types of load, some of which acting independently, so some of these loads are constant while others are not defined in the course of time and they belong to the group of variably repeated loads. In a large number of cases only the domain to which the variably repeated load belongs can be defined.

Limit analysis of structures is an analytic procedure which determines the maximum load parameter of load increment parameter, which can be sustained by an elasto-plastic structure. If the structure is exposed to the action of gradually increasing load, at some point it can surpass a certain critical value, which causes the plastic failure of the structure, after which the structure is not capable of receiving any further increase of load. This critical state is called the limit state of structure, and the load that causes it is called the limit load. Determination of the bearing capacity of a structure, as well as the assessment of the structure failure is very valuable, not only as a simple control of beam bearing capacity, but also as a significant basis and factor in designing of structures.

Even though some ideas appeared in 18th century, the limit analysis is of later date. Its beginning is related to Kazincy [1], who calculated failure load of the beam fixed at both ends, and confirmed the results by experiments. A similar concept was proposed by Kist and Grüning. However, the early work in this field much relied on engineering intuition. Even though the static theorem was first proposed by Kist [2], as an intuitive axiom, it is considered that the basic theorems of limit analysis were first announced by Gvozdev in 1936 and published two years later at a local Russian conference, but they remained unnoticed by the western authors until 1960 when they were translated and published by Haythornthwaite [40]. The limit analysis theorems were independently developed by Hill in 1951, for the stiff perfectly plastic material [21], as well as Drucker and others [22, 23], for elastic perfectly plastic material. In the meantime, a formal proof of these theorems for beams and frames was derived by Horne [24], as well as Greenberg and Prager [25]. Application of limit theorems in designing of civil engineering structures was later applied by many authors among which the following are prominent: Symonds and Neal [26, 27, 41]; Neal [3]; Hodge [4]; Baker and Heyman [5]; Heyman [6]; Horne [7]; Zyczkowski [8]; Mrazik et al. [9]; Save [10].

Limit load of structures determined by the application of the limit analysis is one of the indicators of bearing capacity of the structure exposed to the action of proportional load. When a structure is exposed to the action of variable repeated load, the failure occurs under action of the load which is lower than the load obtained by the application of the limit analysis of structures.

Application of shakedown theory in the assessment of safety of elasto-plastic structures exposed to the action of variable, repeated load is important, and often indispensable. In this context the term "shakedown" introduced by Prager, means that after the onset of initial plastic deformations, the structure acts purely elastic in its further service. The opposite state, which leads to the unsafe structure, is called "non-adaptation" of the structure. The structure in this case undergoes failure due to one or both forms of failure called incremental collapse and alternating collapse. The incremental collapse occurs due to the accumulation of plastic deformations during each load cycle (progressive deformation), causing reduction of structure durability, while the alternating collapse results from the repetition of plastic deformations of the opposite sign, (without accumulation of plastic deformations) causing in this manner a phenomenon of low cycle fatigue.

Shakedown analysis belongs to a class of "simplified" methods where it is not necessary to monitor the entire course of structural response (stress and strain) under the action of repeated load. Also, it represents a significant generalization of the limit analysis theorem.

The concept and method of the calculation of structures applying the shakedown analysis was primarily developed in 1930, even though it has been expanding since 1950. The first papers in this field were presented by Bleich 1932, Melan [28, 29] and Koiter [17, 30]. They proved two basic shakedown theorems: the static shakedown theorem (Melans theorem), i.e. the lower limit of shakedown load and dynamic shakedown theorem (Koiter theorem), i.e. the upper limit of shakedown load which represent a basis of shakedown theory of elasto-plastic structures. These two theorems have been successfully applied in solving of a large number of civil engineering problems (Maier [18], Corradi and Zavelani [19]; Köning and Maier [20]; Kaliszky [11]; König [12]; Polizzoto [31, 32]; Gro-Wedge [33]; Ponter i Karter [34]; Pham[35]).

In the recent years, the shakedown analysis of elasto-plastic structures is becoming increasingly applied in the analysis of engineering problems due to the increased demands of modern technologies. It is thus successfully applied in many engineering problems, such as designing of nuclear reactors, railways, civil engineering designing and safety assessment of some building structures.

The goal of this paper is to present the application of the static and kinematic theorem of limit analysis when the beam is exposed to the load action which is proportionally increasing, as well as the application of shakedown analysis when the beam is exposed to the action of variably repeated load. Also displayed is the difference between the values of failure forces depending on the character of load and the beam span value of continuous beams with two spans in order to assess justification for the application of the shakedown method in the analysis of the limit bearing capacity of the beams.

2
Basic postulates of limit and shakedown analysis

In the area of elastic behavior of beams, the stresses and strains are proportionally dependant. Due to the increase of load, there is a gradual build-up of stress, until the value of the stress in the most loaded fiber reaches the value of yield stress. Further increase of load causes the plastification of the entire cross section, and thus the formation of plastic hinge [13].

It is known that in statically determinate structures, the complete plastification of one cross-section of a beam (formation of a plastic hinge on the location of maximum bending moment) and the transition of the beam into the failure mechanism means the loss of load bearing capacity. In statically indeterminate beams, formation of one plastic joint does not lead to the formation of failure mechanism, and the bearing capacity of one n times statically indeterminate beam is fully exhausted when in the beam an n+1 plastic joint is formed.

In order to determine the limit bearing capacity of a beam applying the limit analysis, previously it should be proved that the limit state relevant for it will occur after formation of the failure mechanism, i.e., the occurrence of any other limit state should be excluded. It is necessary to exclude the occurrence of fatigue under the action of variable load, then the possibility of onset of local instability prior to reaching the complete plastification and exclude the occurrence of any effect which would bring about the failure of the beam before a sufficient number of plastic hinges have been formed and before they transform to failure mechanism [5].

It can be stated that a beam is in the state of limit equilibrium when the bearing capacity of the beam has been fully exhausted, and when the beam behaves fully plastic in a sufficient number of cross sections [14]. On this basis it can be concluded that at the moment of the formation of a sufficient number of plastic hinges, the deformations are progressive, and the beams transit into the failure mechanism. The moment immediately preceding the formation of failure mechanism represents the moment of limit equilibrium of the beam.

If the beam is unloaded prior to the formation of failure mechanism, certain residual strain occurs, which causes the occurrence of retained bending moments. By applying the limit analysis it is not possible to include the retained bending moments in the calculations, in the case of repeated loading of the beam. This is possible by applying the shakedown analysis. In the shakedown analysis all the assumptions of the limit analysis are also valid, whereby this method makes possible the analysis of the behavior of the beam exposed to repeated load.

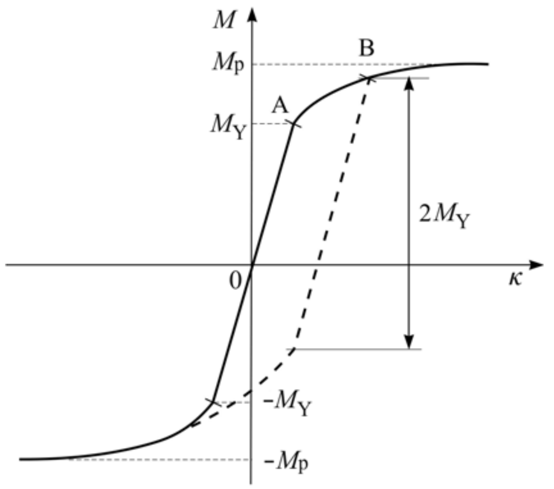

Figure 1 Relation moment–curve in shakedown theory

If the beam is made of a material in which the initial stress state is equal to zero (non-tempered material), the load applied to it will lead to the stress which in certain cross sections can exceed the limit of elasticity. Then the value of the bending moment lies between the elastic stress moment M_Y and the moment of full plastification of the cross section M_p (plastic moment). If the beam is in the elasto-plastic area, in the case of unloading, the connection moment-curve is linear until the value of the bending moment of the cross section is the range $2M_Y$.

In the diagram of dependence of moment-curve (Fig. 1) it can be observed that the value of the elastic moment of unloading is $2M_Y$. Such diagram of dependence of moment and curve is valid in the shakedown theory. The values of yield moment M_Y and moment of plasticity M_p, are equal both at tension and at pressure. Except this, $2M_Y$ remains the range of yield where purely elastic behavior will occur, regardless of the previous history of loading. [4]

3 Limit and shakedown theorems

The basic theorems of limit analysis can be applied to all the types of static systems, irrespective of whether they are statistically determinate or statically indeterminate. The basic theorems of limit analysis consist of:

- static theorem or the theorem of the lower border of limit load and
- kinematic theorem or the theorem of the upper border of limit load.

Static theorem is based on the static equilibrium of the observed system. A large number of distributions of bending moments meeting the equilibrium conditions as a result of the given external load can be assumed for one statically indeterminate system. Greenberg and Prager [25] named such distribution statically admissible. If one such system meets the yield condition, that is, if the bending moment has not exceeded the appropriate value it is claimed that it is also safe. A necessary requirement is that there must be at least one safe distribution of moments in the structure, which is also statically admissible. According to the static theorem, this is a sufficient condition for providing the bearing capacity.

The static theorem can be expressed in the following way: *if there exists any distribution of bending moment throughout structure which is simultaneously safe and statically admissible under the load λF, then the value λ must be less or equal to the factor of failure load λ_C, ($\lambda_C > \lambda$). The actual limit load ($\lambda_C F \leq F_p$) can be equal or higher than the given one.*

On the basis of this theorem, it can be concluded that if under the given load λF there is no distribution of bending moments which is simultaneously safe and statically permissible, that this λ is higher than the factor of failure load λ_C. Also, it can be concluded that one static system can really bear the limit load without failure, considering that λ_C is the maximum factor of load where the static equilibrium cannot be achieved without formation of plastic hinges.

The kinematic theorem relates to the possible failure mechanism. The failure mechanism comprises a kinematically unstable system which a beam becomes after the plastic hinges are formed in the cross sections where there are conditions for this [5]. In the case the failure mechanism is known, the factor of failure load λ_C, i.e. the limit load ($\lambda_C F$), is determined by equalizing the work of external forces with the work absorbed in plastic hinges. In the case when the failure mechanism which corresponds to the limit load is not known in advance, the equation of the work can be written for each assumed failure mechanism, whereby the values (λF) are obtained, that are corresponding to the assumed failure mechanisms.

The kinematic theorem can be expressed in the following way: *for the given static system, subjected to the set of loads λF, the value of which corresponds to any assumed failure mechanism must be higher or equal to the factor of failure load λ_C, that is, $\lambda_C \geq \lambda$.*

Combining the static and kinematic theorem one can form also the theorem of uniqueness which can be expressed in the following way: *if for the given static system and the load there is at least one safe and statically admissible distribution of bending moment at which the plastic moment occurs at sufficient number of cross sections, the corresponding factor of failure load should be λ_C in order to form the failure mechanism.*

Shakedown theorems have a role to set the main conditions under which the plastic yield in the structure finally ceases, regardless of how frequently and in what sequence the load was applied [6]. As well as in the limit analysis, in the shakedown analysis there are static and kinematic theorems, on whose basis it is possible to determine the safe limit load depending on the type of variable repeated load.

The bending moment of the observed cross section *j* can be presented as:

$$M_j = m_j + (M_e)_j, \tag{1}$$

where:
M_j – is the actual bending moment of the cross section,
$(M_e)_j$ – is the elastic bending moment of the cross section,
m_j – is the residual bending moment of the cross section.

Any distribution of residual bending moments, defined in this way must be statically possible in case when the structure is unloaded, because the moment M_j and $(M_e)_j$ must be in equilibrium with the external load [13]. Thus it can be said that the structure has adapted under the action of variable repeated load, if at some point the condition (1) has been satisfied, and all the following loads cause only elastic change of bending moments. Then it is possible to determine the value of safe limit load, which depending on the character of repeated load can be:

– incremental limit load,
– alternating limit load.

On the basis of conditions (1) the static shakedown theorem can be expressed in the following form: *if there exists any distribution of residual bending moment m_j throughout structure, which is statically admissible in the case with zero external loading and which also satisfies at every cross section j, it is necessary to meet one of the conditions:*

$$m_j + \lambda M_j^{\max} \leq \left(M_p\right)_j, \tag{2}$$

$$m_j + \lambda M_j^{\min} \geq -\left(M_p\right)_j, \tag{3}$$

$$\lambda\left(M_j^{\max} - M_j^{\min}\right) \leq 2\left(M_e\right)_j, \tag{4}$$

the value λ will be less than or equal to the shakedown load factor λ_S.

Each girder strives to adapt to the action of variable repeated load in a best possible way. Thus, if λ exceeds the value λ_S, the unlimited plastic yield occurs, and in this case no distribution of residual moments is possible, which is a necessary condition for determination of safe limit load. Similarly, under the action of proportional load, the structure will fail when the load factor λ reaches the value λ_C, above which the structure is not safe, and simultaneously there is a statically possible distribution of bending moments. Depending on the calculated load factor λ it is possible to determine the safe limit load which depends on the type of variable repeated load, on the basis of meeting some of the requirements of the equations (2) and (3), as incremental conditions of plasticity and equation (4), as alternating plasticity conditions.

Application of static shakedown theorem is possible only if distribution of residual bending moments is already known [3]. Application of the static theorem is only justified in the structures with the lower degree of static indeterminacy.

As the application of kinematic shakedown theorem is based on the assumed failure mechanisms, whose form is identical to the form of failure mechanism in the limit analysis of structures, it can be said that this procedure is simpler to apply. The deficiency of this procedure is that the residual bending moments are not included in the calculation.

Assuming that the observed failure mechanism is known, rotations of formed plastic hinges *can be noticed* in a certain number of characteristic cross sections.[5] If the rotation in any cross section is positive (θ^+), then it can be said that the total bending moment in this cross section aspires to reach the value $+M_P$, and if the rotation of the formed plastic joint is negative (θ^-), the bending moment aspires to reach the value $-M_P$. On the basis of the introduced assumptions, the equations (2) and (3) can be written in the form:

$$m_j + \lambda M_j^{\max} = \left(M_{\mathrm{p}}\right)_j \quad \text{for } \theta_j^+, \tag{5}$$

$$m_j + \lambda M_j^{\min} = -\left(M_{\mathrm{p}}\right)_j \quad \text{for } \theta_j^-. \tag{6}$$

If the equations (5) and (6) are multiplied by the corresponding rotation of the formed plastic joint in the cross section *j*, then, they have the form:

$$m_j\theta_j + \lambda M_j^{\max}\theta_j^+ = \left(M_{\mathrm{p}}\right)_j \left|\theta_j\right|, \tag{7}$$

$$m_j\theta_j - \lambda M_j^{\max}\theta_j^- = \left(M_{\mathrm{p}}\right)_j \left|\theta_j\right|. \tag{8}$$

Adding up of equations (7) and (8), of all the plastic hinges which have been formed on the observed failure mechanism, give the following:

$$\sum_j m_j\theta_j + \lambda\left[\sum_j M_j^{\max}\theta_j^+ + \sum_j M_j^{\max}\theta_j^-\right] = \sum_j \left(M_{\mathrm{p}}\right)_j \left|\theta_j\right|. \tag{9}$$

As the distribution of residual bending moments is in equilibrium when the structure is unloaded, and the θ is rotation of the cross section where the plastic joint has been formed, the equation of the principle of virtual work can be written in the following form

$$\sum_j m_j\theta_j = 0, \text{ thus (9) becomes}$$

$$\lambda\left[\sum_j M_j^{\max}\theta_j^+ + \sum_j M_j^{\max}\theta_j^-\right] = \sum_j \left(M_{\mathrm{p}}\right)_j \left|\theta_j\right|, \tag{10}$$

which represents the basic equation of incremental failure.

On the basis of equation (10) it is possible to express the kinematic shakedown theorem in the following way: *the value of parameter λ corresponding to any assumed failure mechanism of alternating plasticity λ_a or of incremental collapse λ_I must be either greater than or equal to the shakedown load factor λ_S.*

The kinematic shakedown theorem in this form was first defined by Koiter [17, 30], though it can be said that he had done that on the basis of the work of P. S. Symods and B. G. Neal [36], which was published at the First National Congress of Applied Mechanics in Chicago in 1951. They started from the assumption that the work of all the residual moments on the possible failure mechanism is equal to zero. In this paper the incremental failure force will be calculated applying the Symonds and Neal method.

4 Analysis of limit bearing capacity of continuous beam depending on the character of the load

Applying the adequate method, and depending on the character of the load, an analysis of the limit load of continuous beam displayed in Fig. 2 was conducted. The span of the beams affects the distribution of internal forces, and therefore on the relevant condition of failure, that is, the value of the failure force. On the example of the continuous beam, a procedure of the failure force calculation has been conducted, depending on the change of beam span, which is defined by the coefficients α and β.

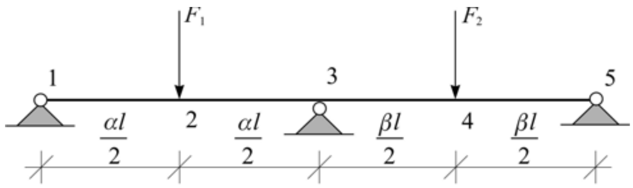

Figure 2 Continuous two–span beam loaded by concentrated forces in the middle of the span

4.1 Failure limit state

Applying the principle of virtual work for the observed beam (Fig. 2), in the paper [16], the equation (11) was derived. In order to obtain the limit load in one-parameter form, it is assumed $F_1 = F_2 = F$:

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & 2 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} \Delta M_2 \\ \Delta M_3 \\ \Delta M_4 \end{bmatrix} + \frac{12EI}{l} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta\theta_4 \end{bmatrix} = \begin{bmatrix} \dfrac{F\alpha l}{2} \\ \dfrac{F\beta l}{2} \\ 0 \end{bmatrix} \tag{11}$$

where ΔM denotes change of actual bending moment and $\Delta\theta$ change of hinge rotation.

When the beam is exposed to the load which proportionally increases, applying the equation (11) it is possible to determine the value of the load which corresponds to the formation of each plastic hinge, until the failure mechanism is formed, whereby also are determined the rotations of the cross sections where formation of plastic hinges occurred.

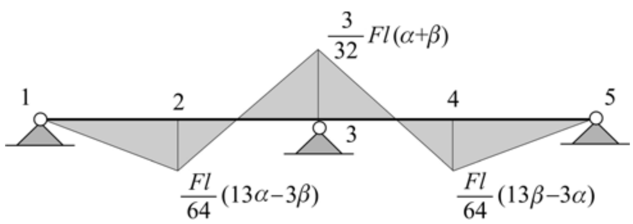

Figure 3 Elastic distribution of bending moment in the function of load

Substituting $\theta_2 = \theta_3 = \theta_4 = 0$ with expression (11) the distribution of bending moments when the beam is in the elastic area is obtained (Fig. 3). By equalizing the highest value of the bending moment (cross section 3) and the plastic moment (M_p) the value of the load causing formation of the first plastic hinge is obtained:

$$M_3 = \frac{3}{32} Fl\left(\alpha+\beta\right) = M_\mathrm{p} \quad \Rightarrow F^{(1)} = \frac{32 M_\mathrm{p}}{3l\left(\alpha+\beta\right)}. \qquad (12)$$

After formation of the plastic joint in the cross section 3, the beam becomes statically determined, and the distribution of bending moments is given in Fig. 4.

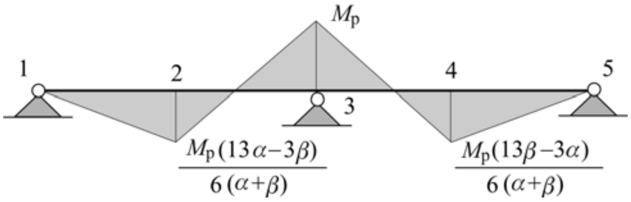

Figure 4 Distribution of the bending moment after formation of the first plastic hinge

If the load is increased, the bending moment of the cross section 3 remains M_p, and the rotation of the cross section where the first plastic joint formed is:

$$\Delta\theta_3 = -\frac{\Delta F_1 l^2 \left(\alpha+\beta\right)}{32EI}. \qquad (13)$$

The load leading to formation of the next plastic joint (cross section 2), when ($\alpha \geq \beta$) is:

$$F_\mathrm{cr} = F^{(2)} = F^{(1)} + \Delta F_1 = \frac{32 M_\mathrm{p}}{3l\left(\alpha+\beta\right)} + \frac{M_\mathrm{p}\left(18\beta - 14\alpha\right)}{3\alpha l\left(\alpha+\beta\right)}, \qquad (14)$$

while the load leading to formation of the plastic joint of the cross section 4, when ($\alpha \leq \beta$) is:

$$F_\mathrm{cr} = F^{(2)} = F^{(1)} + \Delta F_1 = \frac{32 M_\mathrm{p}}{3l\left(\alpha+\beta\right)} + \frac{M_\mathrm{p}\left(18\alpha - 14\beta\right)}{3\beta l\left(\alpha+\beta\right)}. \qquad (15)$$

On the basis of the expressions (14) and (15) it is possible to present the change of the limit force of failure in one-parameter form, depending on the change of span length (Fig. 5). Observing the diagram, it can be concluded that with the increase of the span lengths, the force leading to formation of the relevant failure mechanism decreases. Therefore, if $\alpha \geq \beta$ a partial failure mechanism of the first span forms, while when $\beta \geq \alpha$ a partial failure mechanism of the second span of the beam is generated, and in the case when $\alpha = \beta$ the failure mechanism forms in both spans simultaneously.

The limit failure load can be also determined by applying the kinematic theorem when it is necessary to equalize on the assumed failure mechanism the virtual work of all the external forces with the forces absorbed in the cross sections in which the plastic hinges are assumed. For each of the possible failure mechanisms, one limit force of failure is obtained. The lowest of them is at the same time also the force causing formation of the relevant failure mechanism.

For the observed beam (Fig. 2) three failure mechanisms can be formed, two independent (Fig. 6a and Fig. 6b) and a combined one (Fig. 6c).

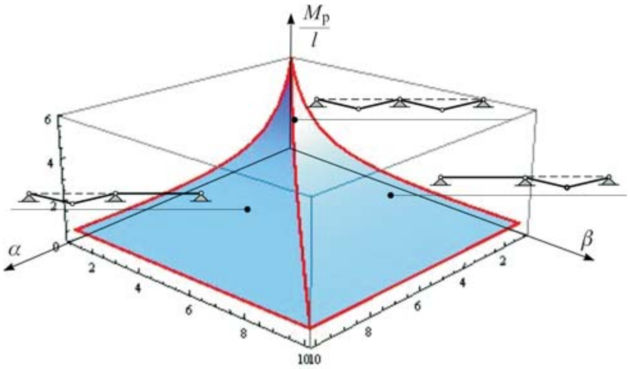

Figure 5 Change of the limit failure force depending on α and β

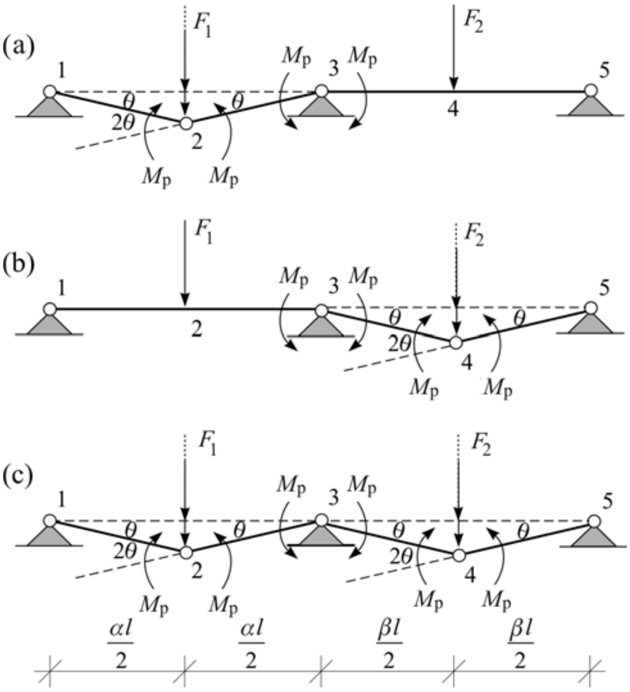

Figure 6. (a) Failure mechanism of the first span, (b) Failure mechanism of the second span, (c) Combined failure mechanism

For each of the possible failure mechanisms, the limit failure forces on the basis of the expressions (16), (17) and (18) are obtained:

$$M_\mathrm{p}\left(2\theta\right) + M_\mathrm{p}\theta = F_1 \frac{\alpha l}{2}\theta \quad \Rightarrow \quad F_1 = \frac{6M_\mathrm{p}}{\alpha l}, \qquad (16)$$

$$M_\mathrm{p}\theta + M_\mathrm{p}\left(2\theta\right) = F_2 \frac{\beta l}{2}\theta \quad \Rightarrow \quad F_2 = \frac{6M_\mathrm{p}}{\beta l}, \qquad (17)$$

$$M_\mathrm{p}(2\theta) + M_\mathrm{p}(\theta) + M_\mathrm{p}(\theta) + M_\mathrm{p}(2\theta) = F_1 \frac{\alpha l}{2}\theta + F_2 \frac{\beta l}{2}\theta \qquad (18)$$
$$\Rightarrow \quad 12 M_\mathrm{p} = l\left(F_1\alpha + F_2\beta\right)$$

When the spans have equal lengths ($\alpha=\beta=1$), and when the beam is simultaneously acted upon by two independent load systems F_1 and F_2, which are in arbitrary relationship, the limit bearing capacity analysis and defining of the area where the beam is safe from the onset of failure can be performed on the basis of the interaction diagram. The

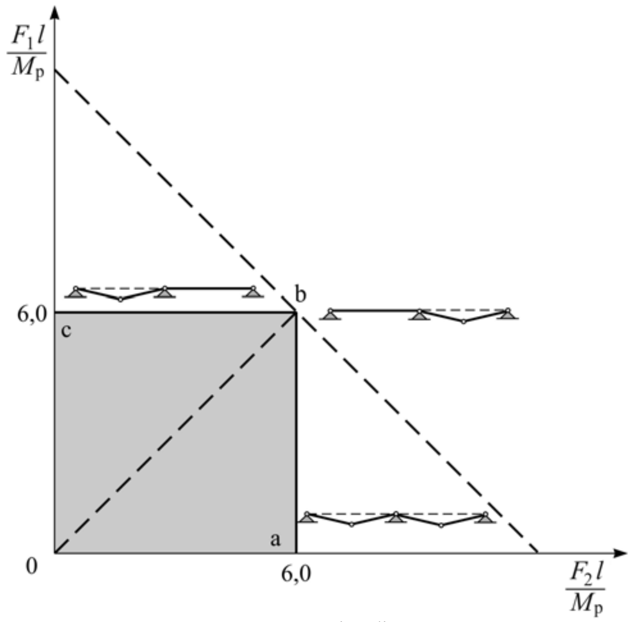

Figure 7 Interaction diagram

mutual relationship of failure mechanisms and mutual load relationship are best observed in the interaction diagram.

From the interaction diagram (Fig. 7) one may conclude that the failure mechanism in the second span forms when the relation of the load is ($F_1/F_2 \leq 1$), while when ($F_1/F_2 \geq 1$), the failure mechanism forms in the first span of the beam. For any relation of loads lying inside the area 0abc0 no failure mechanism will occur and thus no beam failure either. If the relation of the load is such so as to be defined by some of the segments, the failure mechanism defined by this segment is formed.

4.2 Incremental failure load

When the beam (Fig. 2) is exposed to the action of variable repeated load, the area in which the load acts lies within the following range: $0 \leq F_1 \leq F_1$, $0 \leq F_2 \leq F_2$.

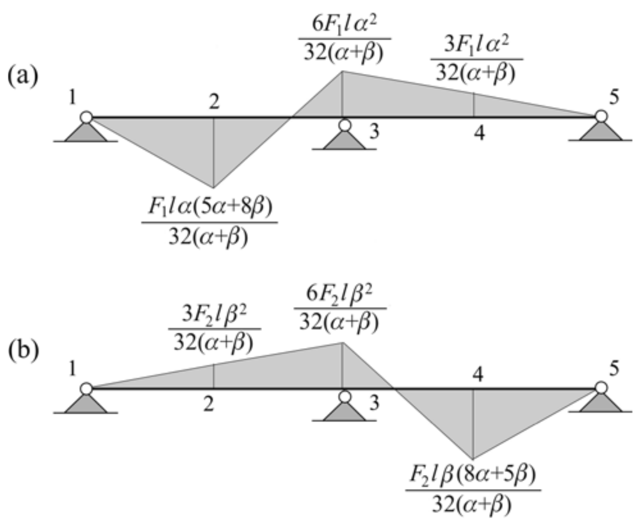

Figure 8 Elastic bending moment of continuous two–span beam

In Fig. 8a is presented the diagram of bending moments when the beam is acted upon only by force F_1, and in Fig. 8b when the beam is acted upon only by force F_2.

The analysis of limit bearing capacity of the beam exposed to the action of variable repeated load will be performed applying the shakedown theorems. For the application of static shakedown theorem it is necessary to know the possible distribution of residual bending moment (Fig. 9).

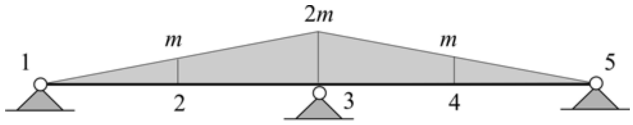

Figure 9. Possible distribution of residual bending moment

Applying the static shakedown theorem on the basis of equations (2) and (3), as well as incremental conditions of plasticity and equation (4) as alternating condition, the values of the forces causing the beam failure are obtained.

The relevant limit failure load depends on the relation of the coefficients α and β. So, if $\alpha \geq \beta$, the failure mechanism is formed in the first span, and the value of incremental failure force is obtained on the basis of the expression:

$$8F_1\,\alpha(\alpha+\beta)+3F_2\beta^2 = \frac{48M_p(\alpha+\beta)}{l}, \tag{19.1}$$

while the residual bending moment is:

$$m = \frac{F_1\,\alpha l(8\beta-\alpha)-6F_2\beta^2 l}{96(\alpha+\beta)}, \tag{19.2}$$

while in the case when $\beta \geq \alpha$, the failure mechanism in the second span is formed. The incremental failure force is obtained on the basis of the expression:

$$8F_2\beta(\alpha+\beta)+3F_1\,\alpha^2 = \frac{48M_p(\alpha+\beta)}{l}, \tag{20.1}$$

while the residual bending moment is:

$$m = \frac{F_2\beta l(8\alpha-\beta)-6F_1\,\alpha^2 l}{96(\alpha+\beta)}. \tag{20.2}$$

On the basis of the alternating plasticity conditions (4) for the cross section 2 is obtained:

$$F_1\,l\alpha(5\alpha+8\beta)+3F_2\,l\beta^2 = 64M_e(\alpha+\beta), \tag{21}$$

for the cross section 3:

$$3F_1\,l\alpha^2+3F_2\,l\beta^2 = 32M_e(\alpha+\beta), \tag{22}$$

and for the cross section 4:

$$F_2\,l\beta(8\alpha+5\beta)+3F_1\,l\alpha^2 = 64M_e(\alpha+\beta). \tag{23}$$

As $M_e = M_p/\alpha_{form}$, it is concluded that the value of alternating failure force depends on the coefficient of cross section form. Here the rectangular cross section is adopted whose form coefficient is $\alpha_{form} = 1{,}50$.

Interaction diagram (Fig. 10) is constructed for the case when the beam spans are equal ($\alpha = \beta = 1$), on the basis of expressions (19.1), (20.1), (21), (22) and (23). From the diagrams, it is observed that the safe area 0abc is defined on the basis of incremental failure condition.

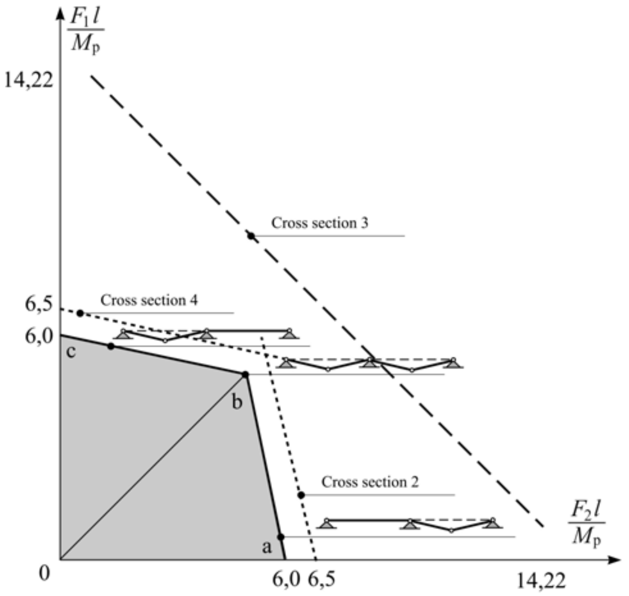

Figure 10 Interaction diagram

On the basis of the expression (19.1) and (20.1), incremental failure load is obtained in one-parameter form:

$$F_{\text{inc}} = \frac{48M_{\text{p}}(\alpha+\beta)}{l(8\alpha^2+8\alpha\beta+3\beta^2)}, F_{\text{inc}} = \frac{48M_{\text{p}}(\alpha+\beta)}{l(3\alpha^2+8\alpha\beta+8\beta^2)}, \quad (24)$$

while on the basis of expression (24.2) and (25.2) the values of residual bending moments are obtained:

$$m = -\frac{M_{\text{p}}(\alpha^2-8\alpha\beta+6\beta^2)}{2(8\alpha^2+8\alpha\beta+3\beta^2)}, m = -\frac{M_{\text{p}}(\beta^2-8\alpha\beta+6\alpha^2)}{2(8\beta^2+8\alpha\beta+3\alpha^2)}. \quad (25)$$

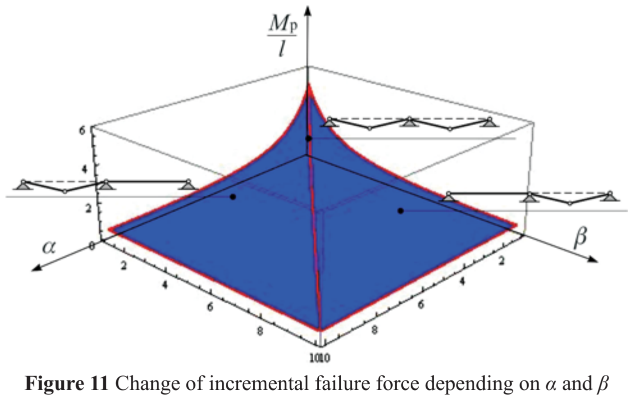

Figure 11 Change of incremental failure force depending on α and β

On the basis of the expressions (24) and (25) the diagrams were constructed (Fig. 11 and Fig. 12) on which the change of incremental force of failure and residual bending moment is displayed for: $1 \leq \alpha \leq 10$ and $1 \leq \beta \leq 10$. In the diagrams it is possible to observe the change of the incremental failure force and residual bending moments depending on the length of beam span, as well as of the relevant beam failure mechanism. In the diagram in Fig. 12 it can be observed that the value of the residual bending moment when $\alpha = \beta$ is constant, and amounts to

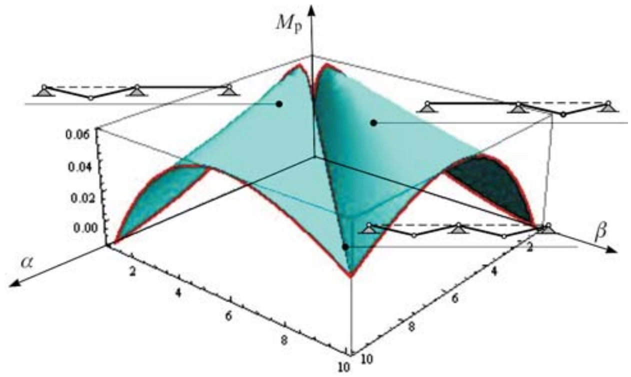

Figure 12 Change of residual bending moment depending on α and β

m=0,0263M_{p}. Depending on the change of the beam span, the maximum value of the residual bending moment is m=0,0595M_{p}, and it occurs when α=1,804β whereby the failure mechanism in the first span is formed, i.e. for β=1,804α when the failure mechanism of the second span is formed.

On the basis of equations (21), (22) and (23) the values of alternating failure forces in one-parameter form are obtained for the cross sections 2, 3 and 4 respectively:

$$F_{\text{alt}} = \frac{64M_{\text{e}}}{l(5\alpha+3\beta)}, \quad (26)$$

$$F_{\text{alt}} = \frac{32M_{\text{e}}(\alpha+\beta)}{3l(\alpha^2+\beta^2)}, \quad (27)$$

$$F_{\text{alt}} = \frac{64M_{\text{e}}}{l(3\alpha+5\beta)}. \quad (28)$$

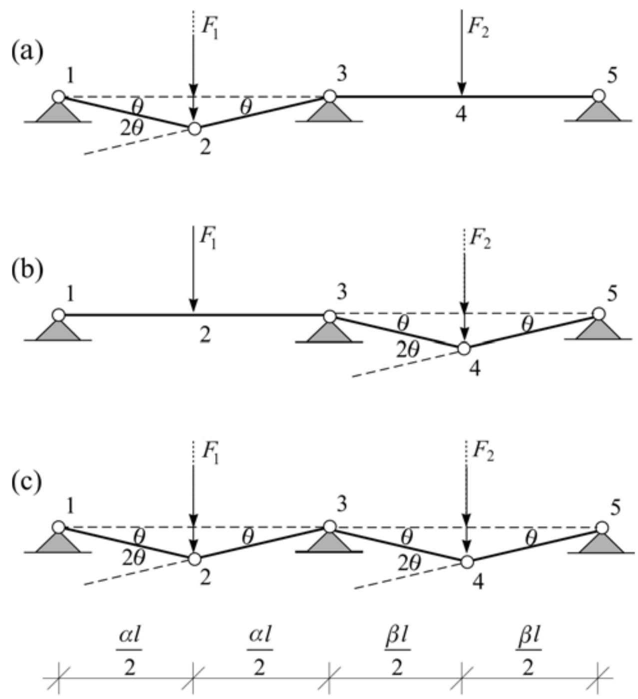

Figure 13 Possible failure mechanism of continuous two–span beam

Failure load was also determined applying kinematic shakedown theorem, that is, Symonds and Neal method, in the cases when the load has been defined by the range: $0 \leq F_1 \leq F_1$, $0 \leq F_2 \leq F_2$. On the basis of the condition that the residual bending moments on possible failure mechanisms

(Fig. 13) are in equilibrium, the following equations can be written:

$$m_2\left(2\theta\right)+m_3\left(-\theta\right)=0, \quad (29)$$

$$m_3\left(-\theta\right)+m_4\left(2\theta\right)=0, \quad (30)$$

$$m_2\left(2\theta\right)+m_3\left(-2\theta\right)+m_4\left(2\theta\right)=0. \quad (31)$$

Solving equations (29), (30) and (31) the following expressions are obtained:

$$8F_1\,\alpha l\left(\alpha+\beta\right)+3F_2\beta^2 l=48M_\mathrm{p}\left(\alpha+\beta\right), \quad (32)$$

$$3F_1\,\alpha^2 l+8F_1\beta l\left(\beta+\alpha\right)=48M_\mathrm{p}\left(\alpha+\beta\right), \quad (33)$$

$$F_1\,\alpha l\left(11\alpha+8\beta\right)+F_1\beta l\left(11\beta+8\alpha\right)=96M_\mathrm{p}\left(\alpha+\beta\right), \quad (34)$$

on whose basis, for the case of one-parameter load the expressions for incremental failure force are obtained:

$$F_\mathrm{inc}=\frac{48M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(8\alpha^2+8\alpha\beta+3\beta^2\right)}, \quad (35)$$

$$F_\mathrm{inc}=\frac{48M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(3\alpha^2+8\alpha\beta+8\beta^2\right)}, \quad (36)$$

$$F_\mathrm{inc}=\frac{96M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(11\alpha^2+16\alpha\beta+11\beta^2\right)}. \quad (37)$$

The expressions obtained applying static (19.1), (20.1) and kinematic shakedown theorem (33), (34) are equal, so the conclusion is drawn that the obtained solution is also singular, which in turn satisfies the theorem of uniqueness. For different values of coefficients α and β the different values of limit and incremental failure loads are obtained, with the relevant failure mechanisms being different, too.

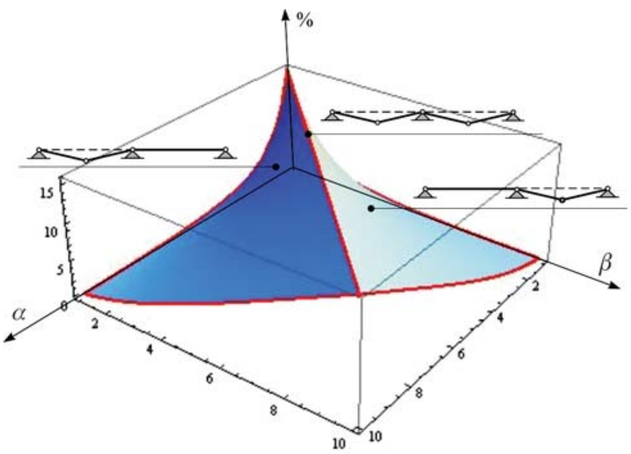

Figure 14 Change of limit and incremental failure force in percents depending on the change of α and β

The difference between the limit and incremental force of failure in percents, depending on the change of and is presented on the diagram in Fig. 14. In the case when the spans of the beams are equal ($\alpha=\beta$), this difference is the largest and amounts to 15,78 %, while for α=1,804β, when the value of the residual moment is maximum, this difference is 6,90 %.

4.3 Alternating failure load

In further analysis of the limit bearing capacity of beams it is assumed that the force in the first span of the alternating character is ($-F_1\le F_1\le F_1$), while the force acting in the second span is in the range $0\le F_2\le F_2$. Applying the static shakedown theorem and failure conditions (2) and (3), when $\alpha\ge\beta$, the value of the failure force is defined on the basis of the expression:

$$8F_1\,\alpha l\left(\alpha+\beta\right)+3F_2\beta^2 l=48M_\mathrm{p}\left(\alpha+\beta\right), \quad (38)$$

and the value of the residual bending moment is:

$$m=\frac{F_1\,\alpha l\left(8\beta-\alpha\right)-6F_2\beta^2 l}{96\left(\alpha+\beta\right)}. \quad (38.1)$$

That is, in the case when $\alpha\le\beta$, the following is obtained:

$$3F_1\,\alpha^2 l+4F_2\beta l\left(\alpha+\beta\right)=24M_\mathrm{p}\left(\alpha+\beta\right), \quad (39)$$

$$m=\frac{F_2\beta l\left(8\alpha-\beta\right)-3F_1\,\alpha^2 l}{96\left(\alpha+\beta\right)}. \quad (39.1)$$

On the basis of the expression (38) and (39) the values of incremental failure force are obtained:

$$F_\mathrm{inc}=\frac{48M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(8\alpha^2+8\alpha\beta+3\beta^2\right)},\ F_\mathrm{inc}=\frac{24M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(3\alpha^2+4\alpha\beta+4\beta^2\right)}, \quad (40)$$

and, on the basis of the expression (38.1) and (39.1) the values of the residual bending moments:

$$m=-\frac{M_\mathrm{p}\left(\alpha^2-8\alpha\beta+6\beta^2\right)}{2\left(8\alpha^2+8\alpha\beta+3\beta^2\right)},\ m=-\frac{M_\mathrm{p}\left(3\alpha^2-8\alpha\beta+\beta^2\right)}{4\left(3\alpha^2+4\alpha\beta+4\beta^2\right)}. \quad (41)$$

For the cross sections 2, 3 and 4 the following expressions are obtained on the basis of alternating condition of plasticity (4) of the static shakedown theorem:

$$F_1\,\alpha l\left(10\alpha+16\beta\right)+3F_2\beta^2 l=64M_\mathrm{e}\left(\alpha+\beta\right), \quad (42)$$

$$3l\left(2\alpha^2 P_1+\beta^2 P_2\right)=32M_\mathrm{e}\left(\alpha+\beta\right), \quad (43)$$

$$6F_1\,\alpha^2 l+F_2\beta l\left(8\alpha+5\beta\right)=64M_\mathrm{e}\left(\alpha+\beta\right), \quad (44)$$

on whose basis the values of alternating forces for these characteristic cross sections are obtained, as follows:

$$F_\mathrm{alt}=\frac{64M_\mathrm{e}\left(\alpha+\beta\right)}{l\left(10\alpha^2+16\alpha\beta+3\beta^2\right)}, \quad (42.1)$$

$$F_\mathrm{alt}=\frac{32M_\mathrm{e}\left(\alpha+\beta\right)}{3l\left(2\alpha^2+\beta^2\right)}, \quad (43.1)$$

$$F_\mathrm{alt}=\frac{64M_\mathrm{e}\left(\alpha+\beta\right)}{l\left(6\alpha^2+8\alpha\beta+5\beta^2\right)}. \quad (44.1)$$

Applying the kinematic theorem, when in the first span of the beam a force of alternating character is acting, for possible failure mechanisms (Fig. 13), the following expressions are formed:

$$8F_1\,\alpha l\left(\alpha+\beta\right)+3F_2\beta^2 l=48M_\mathrm{p}\left(\alpha+\beta\right), \tag{45}$$

$$3\alpha^2 lF_1+4F_2\beta l\left(\alpha+\beta\right)=24M_\mathrm{p}\left(\alpha+\beta\right), \tag{46}$$

$$F_1\,\alpha l\left(14\alpha+8\beta\right)+F_2\beta l\left(8\alpha+11\beta\right)=96M_\mathrm{p}\left(\alpha+\beta\right) \tag{47}$$

on whose basis the values of incremental failure forces in one-parameter form are obtained:

$$F_\mathrm{inc}=\frac{48M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(8\alpha^2+8\alpha\beta+3\beta^2\right)}, \tag{45.1}$$

$$F_\mathrm{inc}=\frac{24M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(3\alpha^2+4\alpha\beta+4\beta^2\right)}, \tag{46.1}$$

$$F_\mathrm{inc}=\frac{96M_\mathrm{p}\left(\alpha+\beta\right)}{l\left(14\alpha^2+16\alpha\beta+11\beta^2\right)}. \tag{47.1}$$

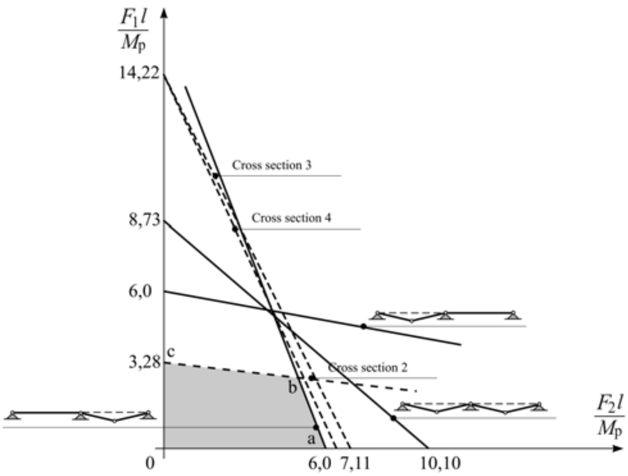

Figure 15 Interaction diagram when the force is action in the middle of the first field of alternating character

On the basis of the expressions (42), (43), (44) as well as of the expressions (45), (46), (47) the interaction diagram was constructed (Fig. 15) on which it can be observed that inside the area 0abc0 the beam is safe against the onset of failure. This area is defined by the alternating failure condition of the cross section 2 and incremental failure condition corresponding to the formation of the second field failure mechanism.

On the basis of the expressions (42.1) and (46.1) a diagram was constructed (Fig. 16) in the case when the failure load is defined in one-parameter form. On the diagram is presented the change of relevant condition of failure depending on the span of the beam. Thus, when $(2{,}137/\alpha)-(5{,}555/\beta)\geq 0$, the failure force is defined on the basis of alternating failure condition of the cross sections 2, and when $(2{,}137/\alpha)-(5{,}555/\beta)\leq 0$, the incremental failure condition is relevant, and the failure mechanism forms in the second span of the beam.

From the diagram presented in Fig. 17 it can be observed that the difference between the alternating and limit forces of failure ranges between 32,74 % and 50,95 % when $\beta=1$, and $\alpha\geq 1$. The largest difference between the

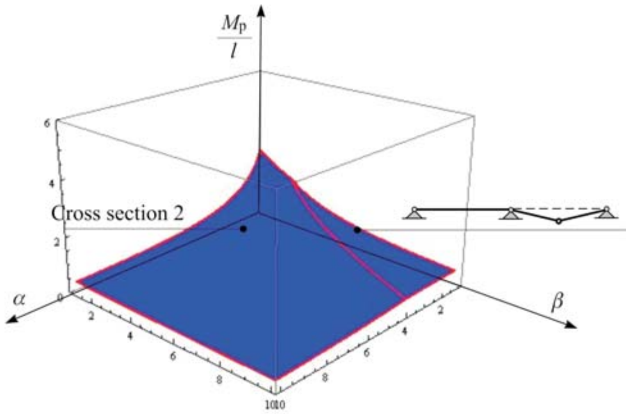

Figure 16 Change of failure force depending on and when the beam in the first field is loaded by the force of alternating character

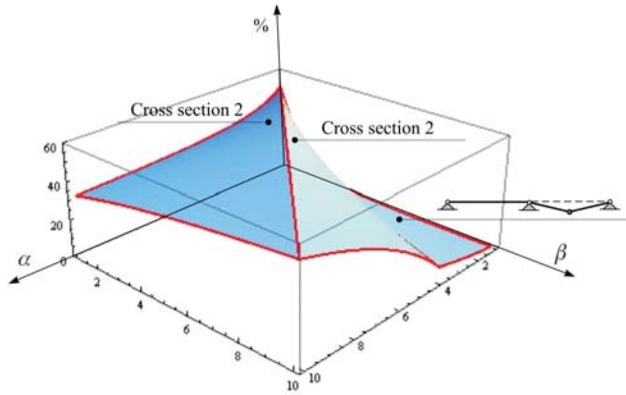

Figure 17 Change of limit and alternating failure forces depending on α and β when the beam is exposed to the action of alternating failure force in the first span

forces of failure, when the spans are of equal lengths $\alpha=\beta$, is 50,95 %. In case when $\beta\geq\alpha$, the difference between the forces sharply decreases, so that it would be the smallest for $\beta\geq 2{,}59\alpha$, when the failure force is defined on the basis of the incremental failure condition.

In case when both forces of alternating character are ($-F_1\leq F_1\leq F_1$, $-F_2\leq F_2\leq F_2$), the failure force is defined on the basis of the alternating condition of failure of the cross section 2, and for $\alpha>\beta$ it is:

$$F_\mathrm{alt}=\frac{64M_\mathrm{e}\left(\alpha+\beta\right)}{l\left(10\alpha^2+16\alpha\beta+3\beta^2\right)}, \tag{48}$$

and for the cross section 4, for $\alpha<\beta$, the failure force is:

$$F_\mathrm{alt}=\frac{64M_\mathrm{e}\left(\alpha+\beta\right)}{l\left(3\alpha^2+16\alpha\beta+10\beta^2\right)}, \tag{49}$$

which is elaborated in the paper in detail [16].

On the diagram (Fig. 19) it can be observed that the greatest difference between the limit and alternating force is 50,59 % when the spans are of equal lengths. Depending on what span is larger, the relevant condition of failure changes, and the difference between the failure forces decreases up to 32,74 %.

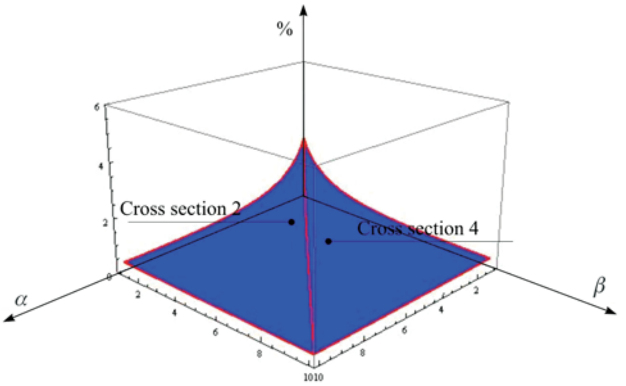

Figure 18 Change of alternating failure force depending on the change of α and β

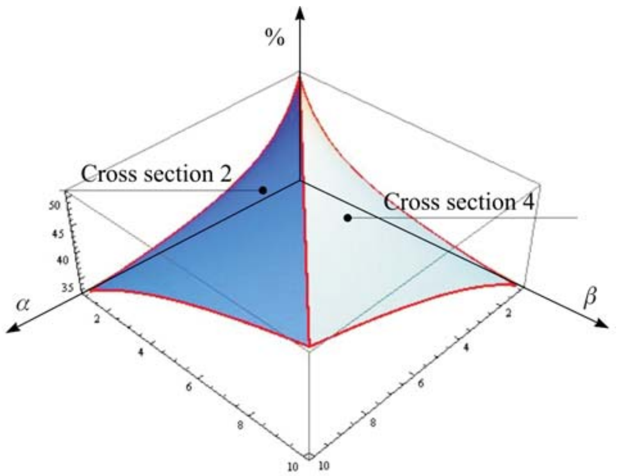

Figure 19 Change of limit and alternating failure force depending on α and β

5
Conclusion

The paper firstly presents the application of static and kinematic theorems of structural limit analysis in determination of the limit load of beams. In both cases the finite, limit state of beam was observed. Application of these theorems was presented on the example of the continuous beam with two spans, loaded by two-parameter and one-parameter load. When the limit load is determined as two-parameter one, the dependence of the load on possible failure mechanism is presented on the interaction diagram, while in the case when the limit load is defined as one-parameter one, its change depending on the span length is presented.

Analysis of the behavior of beams exposed to variable repeated load whose intensity lies in the previously defined range is presented on the example of the continuous beam with two spans which is loaded in the middle of the span with concentrated forces. The limit load was determined applying static and kinematic shakedown theorem, as one-parameter and two-parameter load. When that load is of the same direction, it is possible to determine only the incremental failure load.

On the basis of Fig. 14, where changes of limit and incremental failure force depending on the beam span value are presented it is concluded that the application of shakedown method is justified for certain relations of coefficients α and β, while in some cases the limit load can be determined through the application of limit analysis considering that the difference between these forces is small. However, when one of the forces has alternating character, the difference between the failure forces (Fig. 17) for some spans is up to 50,95 %, so the application of shakedown analysis is obligatory, which is also valid in the case when both forces are of alternating character (Fig. 19).

6
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