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The matching of two sets of factors by decomposition of variance approach

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The usual approach, proposed by several authors, in matching of two factor solutions obtained on two different samples measured by the same set of variables is the determination of the matrix

$$\mathbf{C} = diag(\mathbf{A}_1^{\mathrm{T}}\mathbf{A}_1)^{-1/2}\mathbf{A}_1^{\mathrm{T}}\mathbf{A}_2 diag(\mathbf{A}_2^{\mathrm{T}}\mathbf{A}_2)^{-1/2}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are either matrices of correlations between variables and orthogonal or oblique factors or parallel projections on oblique factors, as reported by Fruchter (1966). When the comparison of oblique factor solutions is carrying out, the difference in values obtained by two definitions of **A** matrices can be substantial, so it is recommended to calculate the congruences of both pairs, the structure and pattern matrices.
In this paper, a new approach is proposed where a single sintetic measure represents the matching of both pairs of A matrices. The matching measure is defined as cosine of the angle between vectors of proportions of variance explained by the factors.

The Matching of Two Sets of Factors by Decomposition of Variance Approach

The generalization of the results of factor analyses over different studies bringing different sources of variation is the most important problem in the field. The number of studies showing the agreement of the results is an indicia of the generalizability of the derived factorial concepts. When matching two factor solutions obtained on two different samples measured by the same set of variables, the problem of measuring the level of agreement is often solved by the determination of the matrix

$$\mathbf{C} = diag(\mathbf{A}_1^{\mathrm{T}}\mathbf{A}_1)^{-1/2}\mathbf{A}_1^{\mathrm{T}}\mathbf{A}_2 diag(\mathbf{A}_2^{\mathrm{T}}\mathbf{A}_2)^{-1/2}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are either matrices of correlations between variables and orthogonal or oblique factors or parallel projections on oblique factors. Fruchter (1966) pointed out that several authors proposed this or similar solution. He mentioned Wrigley and Neuhaus (1955) to be the first to publish the idea, although the procedure has very often been attributed to Tucker (1958). This way of matching of two sets is simpler for orthogonal factors and results in one set of coefficients of congruence. Still, when a more frequent comparison of oblique factor solutions id carried out, the difference in values obtained by inclusion the structure or pattern matrices can be substantial, therefore it is recommended to calculate the congruences of both pairs of matrices.

In this paper, the proposed matching measure is defined as the cosine of the angle between vectors of proportions of variance explained by the factors. This approach provides a single synthetic measure representing the matching of the pairs of structure and pattern matrices.

ALGORITHM

Let $U = \{u_i;\ i = 1,..., n_1\}$ and $W = \{w_i;\ i = 1,..., n_2\}$ be two sets of entities described under the set of variables $V = \{v_j;\ j = 1,..., m\}$. Let Cartesian product of set and sets V and W, respectively, result in matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively, so that

$$\mathbf{Z}_1^{\mathrm{T}}\mathbf{1}_{n1}(\mathbf{1}_{n1}^{\mathrm{T}}\mathbf{1}_{n1})^{-1} = 0 \qquad diag\mathbf{Z}_1^{\mathrm{T}}\mathbf{Z}_1 1/n = \mathbf{I}$$

and

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$$\mathbf{Z}_2^T\mathbf{1}_{n2}(\mathbf{1}_{n2}^T\mathbf{1}_{n2})^{-1} = 0 \qquad diag\mathbf{Z}_2^T\mathbf{Z}_2 1/n = \mathbf{I}$$

Correlation coefficients of variables from $\mathbf{V}$ in two samples are in matrices

$$\mathbf{R}_1 = \mathbf{Z}_1^T\mathbf{Z}_1 1/n \quad \text{and} \quad \mathbf{R}_2 = \mathbf{Z}_2^T\mathbf{Z}_2 1/n$$

By decomposition

$$\mathbf{R}_1 = \sum_{p=1}^{m} \mathbf{x}_{1p}\mathbf{x}_{1p}^T\lambda_{1p} \left| \begin{array}{l} \mathbf{x}_{1p}^T\mathbf{x}_{1p^*} = 0;_{p \neq p^*} \\ \mathbf{x}_{1p}^T\mathbf{x}_{1p} = 1 \end{array} \right.$$

and

$$\mathbf{R}_2 = \sum_{q=1}^{m} \mathbf{x}_{1q}\mathbf{x}_{1q}^T\lambda_{1q} \left| \begin{array}{l} \mathbf{x}_{1q}^T\mathbf{x}_{1q^*} = 0;_{q \neq q^*} \\ \mathbf{x}_{1q}^T\mathbf{x}_{1q} = 1 \end{array} \right.$$

respectively, let eigenvalues Λ_1 and Λ_1 and eigenvectors $\mathbf{X}_1$ and $\mathbf{X}_2$ of correlation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively, be defined.

Let matrices of principle axes $\mathbf{H}_1$ and $\mathbf{H}_2$, be defined as

$$\mathbf{H}_1 = \mathbf{X}_1\Lambda_1^{1/2} \quad \text{and} \quad \mathbf{H}_2 = \mathbf{X}_2\Lambda_2^{1/2}$$

respectively.

Let transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ be used to transform the initial orthogonal solutions into oblique factor solutions

$\mathbf{A}_1 = \mathbf{H}_1\mathbf{T}_1$ and $\mathbf{A}_2 = \mathbf{H}_2\mathbf{T}_2$, where matrices

$$\mathbf{B}_1 = \mathbf{A}_1(\mathbf{A}_1^T\mathbf{A}_1)^{-1} \quad \text{and} \quad \mathbf{B}_2 = \mathbf{A}_2(\mathbf{A}_2^T\mathbf{A}_2)^{-1}$$

contain the regression weights for the calculation of the factor scores of entities

$$\Phi_1^* = \mathbf{Z}_1\mathbf{B}_1 \quad \text{and} \quad \Phi_2^* = \mathbf{Z}_2\mathbf{B}_2$$

Let matrices of covariances

$$\mathbf{M}_1^* = \Phi_1^{*T}\Phi_1^* 1/n \quad \text{and} \quad \mathbf{M}_2^* = \Phi_2^{*T}\Phi_2^* 1/n$$

and diagonal matrices of variances

$$\mathbf{D}_1^{*2} = diag(\mathbf{M}_1^*) \quad \text{and} \quad \mathbf{D}_2^{*2} = diag(\mathbf{M}_2^*)$$

be determined.

Let factor values

$$\Phi_1 = \mathbf{Z}_1\mathbf{B}_1\mathbf{D}_1^{-1} \quad \text{and} \quad \Phi_2 = \mathbf{Z}_2\mathbf{B}_2\mathbf{D}_2^{-1}$$

be standardized, and define matrices of factor correlations, with property to be different from identity matrix

$$\mathbf{M}_1 = \Phi_1^T\Phi_1 1/n \quad \text{and} \quad \mathbf{M}_2 = \Phi_2^T\Phi_2 1/n$$

Structure matrices are then

$$\mathbf{F}_1 = \mathbf{Z}_1^T\Phi_1 1/n \quad \text{and} \quad \mathbf{F}_2 = \mathbf{Z}_2^T\Phi_2 1/n$$

The reproduction of matrices of variable correlations $\mathbf{R}_1$ and $\mathbf{R}_2$ is therefore

$$\mathbf{R}_1^* = \mathbf{F}_1\mathbf{A}_1^T \quad \text{and} \quad \mathbf{R}_2^* = \mathbf{F}_2\mathbf{A}_2^T$$

where diagonal values of $\mathbf{R}_1^*$ and $\mathbf{R}_2^*$ are the proportions of variance explained by the factors in Φ_1 and Φ_2.

By Hadamard's product, let matrices of variance decomposition

$$\mathbf{V}_1 = \mathbf{F}_1 \otimes \mathbf{A}_1 \quad \text{and} \quad \mathbf{V}_2 = \mathbf{F}_2 \otimes \mathbf{A}_2$$

be defined.

Sums of values in rows of $\mathbf{V}_1$ and $\mathbf{V}_2$ matrices are equal to diagonal values of $\mathbf{R}_1^*$ and $\mathbf{R}_2^*$ matrices.

The values in the decomposition of variance matrices unite information from matrices of pattern $\mathbf{A}_1$ and $\mathbf{A}_2$, and matrices of structure $\mathbf{F}_1$ and $\mathbf{F}_2$. Therefore, cosine of the angle between pairs of vectors from $\mathbf{V}_1$ and $\mathbf{V}_2$ is proposed as a synthetic matching measure of two factors, and

$$\mathbf{K} = diag(\mathbf{V}_1^T\mathbf{V}_1)^{-1/2}\mathbf{V}_1^T\mathbf{V}_2 diag(\mathbf{V}_2^T\mathbf{V}_2)^{-1/2}$$

consists of coefficients of congruence defined over decomposition of variance values.

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