Original scientific paper
Accepted 18. 12. 2011.

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Equidistant Surfaces in $\mathbf{H}^2\times\mathbf{R}$ Space

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ABSTRACT

After having investigated the equidistant surfaces ("perpendicular bisectors" of two points) in $\mathbf{S}^2\times\mathbf{R}$ space (see [6]) we consider the analogous problem in $\mathbf{H}^2\times\mathbf{R}$ space from among the eight Thurston geometries. In [10] the third author has determined the geodesic curves, geodesic balls of $\mathbf{H}^2\times\mathbf{R}$ space and has computed their volume, has defined the notion of the geodesic ball packing and its density. Moreover, he has developed a procedure to determine the density of the geodesic ball packing for generalized Coxeter space groups of $\mathbf{H}^2\times\mathbf{R}$ and he has applied this algorithm to them.

In this paper we introduce the notion of the equidistant surface to two points in $\mathbf{H}^2\times\mathbf{R}$ geometry, determine its equation and we shall visualize it in some cases. The pictures have been made by the Wolfram Mathematica software.

Key words: non-Euclidean geometries, geodesic curve, geodesic sphere, equidistant surface in $\mathbf{H}^2\times\mathbf{R}$ geometry

MSC 2010: 53A35, 51M10, 51M20, 52C17, 52C22

Ekvidistantne plohe u prostoru $\mathbf{H}^2\times\mathbf{R}$

SAŽETAK

Nakon istraživanja ekvidistanthih ploha ("okomitih simetrala" dviju točaka) u prostoru $\mathbf{S}^2\times\mathbf{R}$ (vidi [6]), razmatramo analogni problem u prostoru $\mathbf{H}^2\times\mathbf{R}$ iz osam Thurstonovih geometrija. U radu [10] treći je autor odredio geodetske krivulje i kugle prostora $\mathbf{H}^2\times\mathbf{R}$ te definirao pojam popunjavanja geodetskim kuglama i njegovu gustoću. Pored toga, razvio je metodu određivanja gustoće popunjavanja geodetskim kuglama za generalizirane Coxeterove grupe prostora $\mathbf{H}^2\times\mathbf{R}$ i primijenio taj algoritam na njih. U ovom radu uvodimo pojam ekvidistantne plohe dviju točaka u geometriji $\mathbf{H}^2\times\mathbf{R}$, određujemo njihovu jednadžbu i vizualiziramo neke slučajeve. Slike su napravljene u Wolframovom programu Mathematica.

Ključne riječi: neeuklidske geometrije, geodetska krivulja, geodetska sfera, ekvidistantna ploha u $\mathbf{H}^2\times\mathbf{R}$ geometriji

1 Basic notions of $\mathbf{H}^2\times\mathbf{R}$ geometry

The $\mathbf{H}^2\times\mathbf{R}$ geometry is one one of the eight simply connected 3-dimensional maximal homogeneous Riemannian geometries. This Seifert fibre space is derived by the direct product of the hyperbolic plane $\mathbf{H}^2$ and the real line $\mathbf{R}$. The points are described by (P,p) where $P\in\mathbf{H}^2$ and $p\in\mathbf{R}$.

In [2] E. Molnár has shown, that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4,\boldsymbol{V}_4,\mathbf{R})$. In our work we shall use this projective model of $\mathbf{H}^2\times\mathbf{R}$ and the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0)$,$E_1^\infty(\mathbf{e}_1)$,$E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\}\subset\mathbf{V}^4$ with the unit point $E(\mathbf{e}=\mathbf{e}_0+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3))$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y}=c\mathbf{x}$ with $0<c\in\mathbf{R}$ (or $c\in\mathbb{R}\setminus\{0\}$) defines a point $(\mathbf{x})=(\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(\boldsymbol{e}^i)\}\subset\boldsymbol{V}_4$ describes the simplex planes, especially the plane at infinity $(\boldsymbol{e}^0)=E_1^\infty E_2^\infty E_3^\infty$, and generally, $\boldsymbol{v}=\boldsymbol{u}\frac{1}{c}$ defines a plane $(\boldsymbol{u})=(\boldsymbol{v})$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$, respectively). Thus $0=\mathbf{x}\boldsymbol{u}=\mathbf{y}\boldsymbol{v}$ defines the incidence of point $(\mathbf{x})=(\mathbf{y})$ and plane $(\boldsymbol{u})=(\boldsymbol{v})$, as $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$ also denotes it. Thus $\mathbf{H}^2\times\mathbf{R}$ can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well.

The point set of $\mathbf{H}^2\times\mathbf{R}$ in the projective space $\mathcal{P}^3$, are the following open cone solid (see Fig. 1-2):

$$\mathbf{H}^2\times\mathbf{R}:=$$
$$\left\{X(\mathbf{x}=x^i\mathbf{e}_i)\in\mathcal{P}^3:-(x^1)^2+(x^2)^2+(x^3)^2<0<x^0,\ x^1\right\}.$$

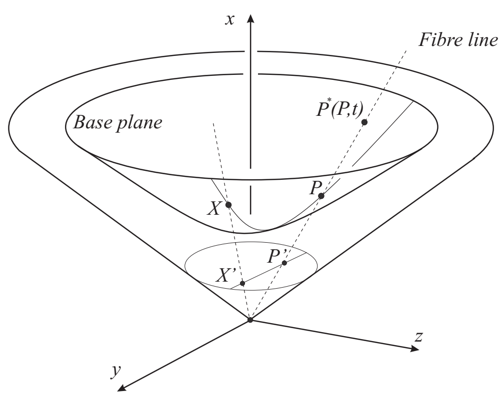

Figure 1: *Projective model of* $\mathbf{H}^2\times\mathbf{R}$ *geometry*

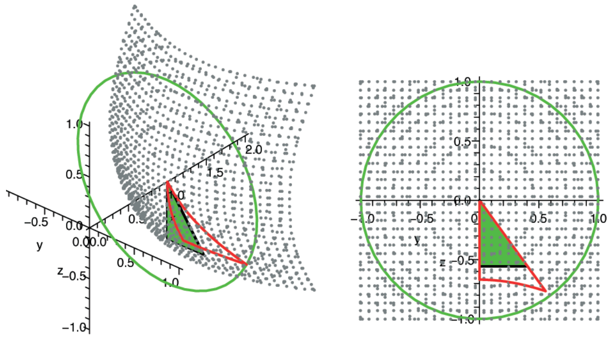

Figure 2: *The connection between Cayley-Klein model of the hyperbolic plane and the "base plane" of the model of* $\mathbf{H}^2\times\mathbf{R}$ *geometry.*

In this context E. Molnár [2] has derived the infinitesimal arc-length square at any point of $\mathbf{H}^2\times\mathbf{R}$ as follows

$$\begin{aligned}(ds)^2 = &\frac{1}{-x^2+y^2+z^2}\cdot\Big[(x^2+y^2+z^2)(dx)^2+\\ &2dxdy(-2xy)+2dxdz(-2xz)+\\ &(x^2+y^2-z^2)(dy)^2+\\ &2dydz(2yz)(x^2-y^2+z^2)(dz)^2\Big].\end{aligned} \tag{1}$$

By introducing the new (t,r,α) coordinates in (2), our formula becomes simplier in (3): $-\pi<\alpha\leq\pi$ and $r\geq 0$ with $t\in\mathbf{R}$ the fibre coordinate. The proper points can be described by the following equations:

$$\begin{aligned}&x^0=1,\ \ x^1=e^t\cosh r,\\ &x^2=e^t\sinh r\cos\alpha,\ \ x^3=e^t\sinh r\sin\alpha.\end{aligned} \tag{2}$$

We apply the usual Cartesian coordinates for the visualization and further computations, i.e. $x=x^1/x^0, y=x^2/x^0, z=x^3/x^0$. So the infinitesimal arc length square with coordinates (t,r,α) at any proper point of $\mathbf{H}^2\times\mathbf{R}$ - and the symmetric metric tensor g_{ij} obtained from it - are the following:

$$(ds)^2=(dt)^2+(dr)^2+\sinh^2 r(d\alpha)^2, \tag{3}$$

$$g_{ij}:=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\sinh^2 r\end{pmatrix}. \tag{4}$$

By the usual method of the differential geometry we have obtained the equation system of the geodesic curves [5]:

$$\begin{aligned}&x(\tau)=e^{\tau\sin v}\cosh(\tau\cos v),\\ &y(\tau)=e^{\tau\sin v}\sinh(\tau\cos v)\cos u,\\ &z(\tau)=e^{\tau\sin v}\sinh(\tau\cos v)\sin u,\\ &\qquad -\pi<u\leq\pi,\ \ -\frac{\pi}{2}\leq v\leq\frac{\pi}{2}.\end{aligned} \tag{5}$$

Remark 1.1 *The starting point of our geodesics can be chosen at* $(1,1,0,0)$ *by the homogeneity of* $\mathbf{H}^2\times\mathbf{R}$.

Definition 1.2 *The distance* $d(P_1,P_2)$ *between the points* P_1 *and* P_2 *is defined by the arc length* $s=\tau$ *in (5) of the geodesic curve from* P_1 *to* P_2.

Definition 1.3 *The geodesic sphere of radius* ρ *(denoted by* $S_{P_1}(\rho)$*) with center at the point* P_1 *is defined as the set of all points* P_2 *in the space with the condition* $d(P_1,P_2)=\rho$. *We also require that the geodesic sphere is a simply connected surface without selfintersection in* $\mathbf{H}^2\times\mathbf{R}$ *space (see Fig. 3).*

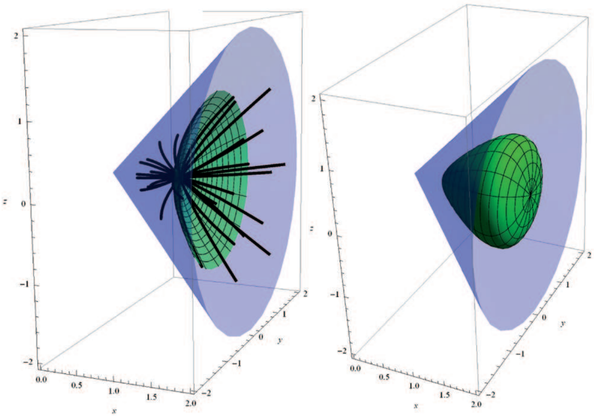

Figure 3: *Geodesics with varying parameters and the "base-hyperboloid" in the cone and a geodesic sphere with radius* $\frac{2}{3}$ *centered at* $(1,1,0,0)$.

1.1 Equidistant surfaces in $\mathbf{H}^2\times\mathbf{R}$ geometry

One of our further goals is to visualize and examine the Dirichlet-Voronoi cells of $\mathbf{H}^2\times\mathbf{R}$ where the faces of the DV-cells are equidistant surfaces. The definition below comes naturally.

Definition 1.4 *The equidistant surface* $\mathcal{S}_{P_1P_2}$ *of two arbitrary points* $P_1,P_2\in\mathbf{H}^2\times\mathbf{R}$ *consists of all points* $P'\in\mathbf{H}^2\times\mathbf{R}$, *for which* $d(P_1,P')=d(P',P_2)$. *Moreover, we require that this surface is a simply connected piece without selfintersection in* $\mathbf{H}^2\times\mathbf{R}$ *space.*

It can be assumed by the homogeneity of $\mathbf{H}^2\times\mathbf{R}$ that the starting point of a given geodesic curve segment is $P_1(1,1,0,0)$. The other endpoint will be given by its homogeneous coordinates $P_2(1,a,b,c)$. We consider the geodesic curve segment $\mathcal{G}_{P_1P_2}$ and determine its parameters (τ,u,v) expressed by a,b,c. We obtain by equation system (5) the following identity :

$$\sqrt{a^2-b^2-c^2}=e^{\tau\sin v} \tag{6}$$

If we substitute this into (5), the equation system can be solved for (τ,u,v).

$$\tau=\frac{\log\sqrt{a^2-b^2-c^2}}{\sin v},\quad \text{if } v\neq 0. \tag{7}$$

$$v=\arctan\left(\frac{\log\sqrt{a^2-b^2-c^2}}{\operatorname{arccosh}\left(\frac{a}{\sqrt{a^2-b^2-c^2}}\right)}\right), \tag{8}$$

if $P_2(a,b,c)$ does not lie on the axis $[x]$ i.e. $(b,c)\neq(0,0)$.

$$\tan u=\frac{z(\tau)}{y(\tau)}=\frac{c}{b}\Rightarrow u=\arctan\left(\frac{c}{b}\right). \tag{9}$$

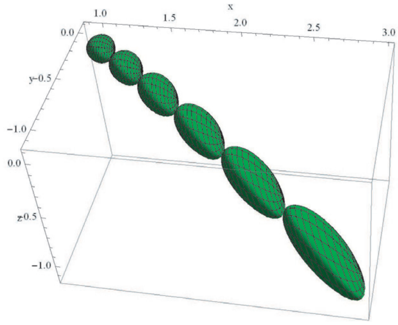

Figure 4: *Touching geodesic spheres of radius $\frac{1}{10}$ centered on the geodesic curve with starting point $(1,1,0,0)$ and parameters $u=\frac{\pi}{4}$, $v=\frac{\pi}{3}\neq 0$.*

Remark 1.5 *If $P_2\in[x]$, then $v=\frac{\Pi}{2}$ and $u=0$, and the geodesic curve is an Euclidean line segment between P_1 and P_2. If $v=0$, then $\tau=\operatorname{arccosh} a$ and the two points are on the same hyperboloid surface. These special cases will be discussed in section 3 in terms of the equidistant surfaces belonging to them.*

It is clear that $X\in\mathcal{S}_{P_1P_2}$ iff $d(P_1,X)=d(X,P_2)\Rightarrow d(P_1,X)=d(X^{\mathcal{F}},P_2^{\mathcal{F}})$, where $\mathcal{F}$ is a composition of isometries which maps X onto $(1,1,0,0)$, and then by (7) the length of the geodesic (e.g. the distance between the two points) is comparable to $d(P_1,X)$. This method leads to the implicit equation of the equidistant surface of two proper points $P_1(1,a,b,c)$ and $P_2(1,d,e,f)$ in $\mathbf{H}^2\times\mathbf{R}$:

$$\begin{gathered}\mathcal{S}_{P_1P_2}(x,y,z)\Rightarrow\\ 4\operatorname{arccosh}^2\left(\frac{ax-by-cz}{\sqrt{a^2-b^2-c^2}\sqrt{x^2-y^2-z^2}}\right)+\\ \log^2\left(\frac{a^2-b^2-c^2}{x^2-y^2-z^2}\right)=\\ =4\operatorname{arccosh}^2\left(\frac{dx-ey-fz}{\sqrt{d^2-e^2-f^2}\sqrt{x^2-y^2-z^2}}\right)+\\ \log^2\left(\frac{d^2-e^2-f^2}{x^2-y^2-z^2}\right).\end{gathered} \tag{10}$$

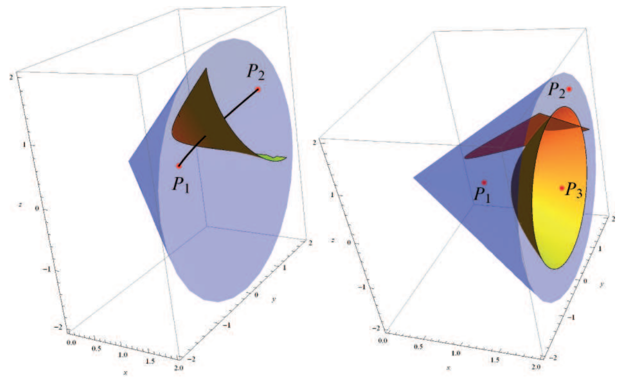

Figure 5: *Equidistant surfaces with $P_1(1,1,0,0)$ and $P_2(1,2,1,1)$, and the two special cases.*

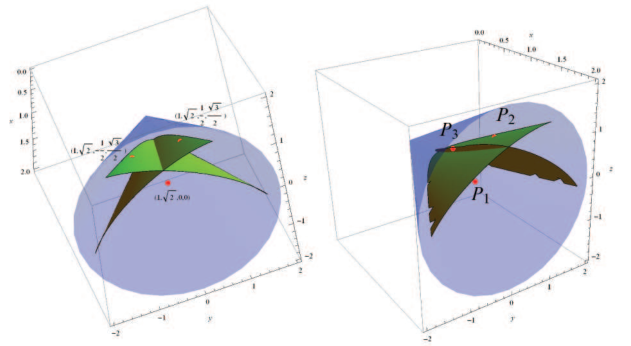

Figure 6: *Equidistant surfaces to points $P_1(1,\sqrt{2},0,0)$, $P_2(1,\sqrt{2},\frac{1}{2},\frac{\sqrt{3}}{2})$ and $P_1(1,\sqrt{2},0,0)$, $P_3(1,\sqrt{2},-\frac{1}{2},\frac{\sqrt{3}}{2})$.*

1.2 Some observations

We introduce the next denotations to simplify the equation (10): $\mathbf{a}=\overrightarrow{OP_1}$, $\mathbf{b}=\overrightarrow{OP_2}$ and $\mathbf{x}=\overrightarrow{OX}$. We define the scalar product for all vectors $\mathbf{u}(u_1,u_2,u_3)$ and $\mathbf{v}(v_1,v_2,v_3)$ by the following equation:

$$\langle\mathbf{u},\mathbf{v}\rangle=-u_1v_1+u_2v_2+u_3v_3,$$

moreover, we introduce the denotation $|\mathbf{v}|=\sqrt{-\langle\mathbf{v},\mathbf{v}\rangle}$ similarly to the $\mathbf{S}^2\times\mathbf{R}$ space (see [6]).

With these denotations, the equation of the surface becomes shorter and gives important informations about equidistant surfaces:

$$\operatorname{arccosh}^2\left(\frac{-\langle\mathbf{a},\mathbf{x}\rangle}{|\mathbf{a}||\mathbf{x}|}\right)+\log^2\left(\frac{|\mathbf{a}|}{|\mathbf{x}|}\right)=$$
$$\operatorname{arccosh}^2\left(\frac{-\langle\mathbf{x},\mathbf{b}\rangle}{|\mathbf{x}||\mathbf{b}|}\right)+\log^2\left(\frac{|\mathbf{b}|}{|\mathbf{x}|}\right).$$

The last step is to notice that $\operatorname{arccosh}\left(\frac{-\langle \mathbf{a},\mathbf{x}\rangle}{|\mathbf{a}||\mathbf{x}|}\right)$ is the hyperbolic disance between points **a** and **x** in the projective model of the hyperbolic plane. So let $\varepsilon = d_h(\mathbf{a},\mathbf{x})$ and $\delta = d_h(\mathbf{x},\mathbf{b})$. The final form of the equation is the following:

$$\varepsilon^2+\log^2(|\mathbf{a}||\mathbf{x}|^{-1}) = \delta^2+\log^2(|\mathbf{b}||\mathbf{x}|^{-1}) \tag{11}$$

Remark 1.6 *This formula also describes the equidistant surface of* $\mathbf{S}^2\times\mathbf{R}$ *with the usual Euclidean scalar product, vector length and angle formula (see [6]).*

It is now easy to examine some special cases: when $|\mathbf{a}| = |\mathbf{b}|$, the equidistant surface consists of those points of an Euclidean plane in our model, which are inner points of the cone (e.g. proper point of $\mathbf{H}^2\times\mathbf{R}$). Another special case appears when **a** and **b** are on the same *fibre*. In this case ($\delta = \varepsilon$) the equidistant surface is the "positive side" of a hyperboloid of two sheets.

Our projective method gives us a way of investigation the $\mathbf{H}^2\times\mathbf{R}$ space, which suits to study and solve similar problems (see [10]). In this paper we have examined only some problems, but analogous questions in $\mathbf{H}^2\times\mathbf{R}$ geometry or, in general, in other homogeneous Thurston geometries are timely (see [11], [8], [9]).

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Acknowledgement: We thank Prof. Emil Molnár for helpful comments to this paper.