Equidistant Surfaces in $\mathbb{H}^2 \times \mathbb{R}$ Space

ABSTRACT

After having investigated the equidistant surfaces (“perpendicular bisectors” of two points) in $S^2 \times \mathbb{R}$ space (see [6]) we consider the analogous problem in $\mathbb{H}^2 \times \mathbb{R}$ space from among the eight Thurston geometries. In [10] the third author has determined the geodesic curves, geodesic balls of $\mathbb{H}^2 \times \mathbb{R}$ space and has computed their volume, has defined the notion of the geodesic ball packing and its density. Moreover, he has developed a procedure to determine the density of the geodesic ball packing for generalized Coxeter space groups of $\mathbb{H}^2 \times \mathbb{R}$ and he has applied this algorithm to them.

In this paper we introduce the notion of the equidistant surface to two points in $\mathbb{H}^2 \times \mathbb{R}$ geometry, determine its property and we shall visualize it in some cases. The pictures have been made by the Wolfram Mathematica software.

Key words: non-Euclidean geometries, geodesic curve, geodesic sphere, equidistant surface in $\mathbb{H}^2 \times \mathbb{R}$ geometry

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1 Basic notions of $\mathbb{H}^2 \times \mathbb{R}$ geometry

The $\mathbb{H}^2 \times \mathbb{R}$ geometry is one one of the eight simply connected 3-dimensional maximal homogeneous Riemannian geometries. This Seifert fibre space is derived by the direct product of the hyperbolic plane $\mathbb{H}^2$ and the real line $\mathbb{R}$. The points are described by $(P, p)$ where $P \in \mathbb{H}^2$ and $p \in \mathbb{R}$.

In [2] E. Molnár has shown, that the homogeneous 3-spaces have a unified interpretation in the projective space $\mathbb{P}^3(V^4, V_4, \mathbb{R})$. In our work we shall use this projective model of $\mathbb{H}^2 \times \mathbb{R}$ and the Cartesian homogeneous coordinate simplex $E_0(e_0)|E_1^+(e_1)|E_2^+(e_2)|E_3^+(e_3)$, $\{e_i\} \subset V^4$ with the unit point $E(e = e_0 + e_1 + e_2 + e_3)$ which is distinguished by an origin $E_0$ and by the ideal points of coordinate axes, respectively. Moreover, $y = cx$ with $0 < c \in \mathbb{R}$ (or $c \in \mathbb{R} \setminus \{0\}$) defines a point $(x) = (y)$ of the projective 3-space $\mathbb{P}^3$ (or that of the projective space $\mathbb{P}^3$ where opposite rays $(x)$ and $(-x)$ are identified). The dual system $\{e_i\} \subset V_4$ describes the simplex planes, especially the plane at infinity $(e_0) = E_1^+|E_2^+|E_3^+$, and generally, $v = u_2^1$ defines a plane $(u) = (v)$ of $\mathbb{P}^3$ (or that of $\mathbb{P}^3$, respectively). Thus $0 = xu = yv$ defines the incidence of point $(x) = (y)$ and plane $(u) = (v)$, as $(x)||y(y)$ also denotes it. Thus $\mathbb{H}^2 \times \mathbb{R}$ can be visualized in the affine 3-space $\mathbb{A}^3$ (so in $E^3$) as well.

The point set of $\mathbb{H}^2 \times \mathbb{R}$ in the projective space $\mathbb{P}^3$, are the following open cone solid (see Fig. 1-2):

$$\mathbb{H}^2 \times \mathbb{R} := \{X(x = x' e_i) \in \mathbb{P}^3 : -(x^1)^2 + (x^2)^2 + (x^3)^2 < 0 < x^0, x^1\}.$$
\[
\begin{align*}
\text{(ds)}^2 &= \frac{1}{-x^2 + y^2 + z^2} \left( (x^2 + y^2 + z^2) (dx)^2 + 2dxdy(-2xy) + 2dxdz(-2xz) + (x^2 + y^2 - z^2) (dy)^2 + 2dydz(2yz)(x^2 - y^2 + z^2) (dz)^2 \right), \\
&= (dt)^2 + (dr)^2 + \sinh^2 r (d\alpha)^2, \\
g_{ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sinh^2 r \end{pmatrix},
\end{align*}
\]

By the usual method of the differential geometry we have obtained the equation system of the geodesic curves [5]:

\[
\begin{align*}
x(\tau) &= e^{\tau \sin \nu} \cosh (\tau \cos \nu), \\
y(\tau) &= e^{\tau \sin \nu} \sinh (\tau \cos \nu) \cos u, \\
z(\tau) &= e^{\tau \sin \nu} \sinh (\tau \cos \nu) \sin u, \\
-\pi < u \leq \pi, &\quad -\pi / 2 \leq v \leq \pi / 2.
\end{align*}
\]

Remark 1.1 The starting point of our geodesics can be chosen at \((1,1,0,0)\) by the homogeneity of \(H^2 \times \mathbb{R}\).

Definition 1.2 The distance \(d(P_1, P_2)\) between the points \(P_1\) and \(P_2\) is defined by the arc length \(s = \tau\) in (5) of the geodesic curve from \(P_1\) to \(P_2\).

Definition 1.3 The geodesic sphere of radius \(\rho\) (denoted by \(S_{\rho}(p)\)) with center at the point \(P_1\) is defined as the set of all points \(P_2\) in the space with the condition \(d(P_1, P_2) = \rho\). We also require that the geodesic sphere is a simply connected surface without selfintersection in \(H^2 \times \mathbb{R}\) space (see Fig. 3).

1.1 Equidistant surfaces in \(H^2 \times \mathbb{R}\) geometry

One of our further goals is to visualize and examine the Dirichlet-Voronoi cells of \(H^2 \times \mathbb{R}\) where the faces of the DV-cells are equidistant surfaces. The definition below comes naturally.

Definition 1.4 The equidistant surface \(S_{P_1, P_2}\) of two arbitrary points \(P_1, P_2 \in H^2 \times \mathbb{R}\) consists of all points \(P' \in H^2 \times \mathbb{R}\), for which \(d(P_1, P') = d(P', P_2)\). Moreover, we require that this surface is a simply connected piece without selfintersection in \(H^2 \times \mathbb{R}\) space.

It can be assumed by the homogeneity of \(H^2 \times \mathbb{R}\) that the starting point of a given geodesic curve segment is \(P_1(1,1,0,0)\). The other endpoint will be given by its homogeneous coordinates \(P_2(1,a,b,c)\). We consider the geodesic curve segment \(\gamma_{P_1P_2}\) and determine its parameters \((\tau, u, v)\) expressed by \(a,b,c\). We obtain by equation system (5) the following identity:

\[
\sqrt{a^2 - b^2 - c^2} = e^{\tau \sin \nu}
\]
If we substitute this into (5), the equation system can be solved for \( \tau, u, v \).

\[
\tau = \frac{\log \sqrt{a^2 - b^2 - c^2}}{\sin \nu}, \quad \text{if } \nu \neq 0.
\]

\[
v = \arctan \left( \frac{\log \sqrt{a^2 - b^2 - c^2}}{\arccosh \left( \frac{u}{\sqrt{a^2 - b^2 - c^2}} \right)} \right),
\]

if \( P_2(a, b, c) \) does not lie on the axis \( |x| \) i.e. \( (b, c) \neq (0, 0) \).

\[
\tan u = \frac{z(\tau)}{y(\tau)} = \frac{c}{b} \Rightarrow u = \arctan \left( \frac{c}{b} \right).
\]

**Remark 1.5** If \( P_2 \in |x| \), then \( v = \frac{u}{\tau} \) and \( u = 0 \), and the geodesic curve is an Euclidean line segment between \( P_1 \) and \( P_2 \). If \( v = 0 \), then \( \tau = \arccosh a \) and the two points are on the same hyperboloid surface. These special cases will be discussed in section 3 in terms of the equidistant surfaces belonging to them.

It is clear that \( X \in S_{P_1} \) if \( d(P_1, X) = d(X, P_2) \Rightarrow d(P_1, X) = d(X^\mathcal{F}, P_2^\mathcal{F}) \), where \( \mathcal{F} \) is a composition of isometries which maps \( X \) onto \((1, 1, 0, 0)\), and then by (7) the length of the geodesic (e.g. the distance between the two points) is comparable to \( d(P_1, X) \). This method leads to the implicit equation of the equidistant surface of two proper points \( P_1(1, a, b, c) \) and \( P_2(1, d, e, f) \) in \( \mathbb{H}^2 \times \mathbb{R} \):

\[
S_{P_1}(x, y, z) \Rightarrow 4\text{arccosh}^2 \left( \frac{ax - by - cz}{\sqrt{a^2 - b^2 - c^2} \sqrt{x^2 - y^2 - z^2}} \right) + \log^2 \left( \frac{x^2 - b^2 - c^2}{x^2 - y^2 - z^2} \right) = 4\text{arccosh}^2 \left( \frac{dx - ey - fz}{\sqrt{d^2 - e^2 - f^2} \sqrt{x^2 - y^2 - z^2}} \right) + \log^2 \left( \frac{d^2 - e^2 - f^2}{x^2 - y^2 - z^2} \right).
\]

1.2 Some observations

We introduce the next denotations to simplify the equation (10): \( a = \overline{OP_1}, b = \overline{OP_2} \) and \( x = \overline{OX} \). We define the scalar product for all vectors \( u(u_1, u_2, u_3) \) and \( v(v_1, v_2, v_3) \) by the following equation:

\[
(u, v) = -u_1v_1 + u_2v_2 + u_3v_3,
\]

moreover, we introduce the denotation \( |v| = \sqrt{(v, v)} \) similarly to the \( S^2 \times \mathbb{R} \) space (see [6]).

With these denotations, the equation of the surface becomes shorter and gives important informations about equidistant surfaces:

\[
\text{arccosh}^2 \left( \frac{\langle a, x \rangle}{|a||x|} \right) + \log^2 \left( \frac{|a|}{|x|} \right) = \text{arccosh}^2 \left( \frac{\langle x, b \rangle}{|x||b|} \right) + \log^2 \left( \frac{|b|}{|x|} \right).
\]
The last step is to notice that \( \arccosh \left( \frac{-\langle a, x \rangle}{||a|| \cdot ||x||} \right) \) is the hyperbolic distance between points \( a \) and \( x \) in the projective model of the hyperbolic plane. So let \( \varepsilon = d_h(a, x) \) and \( \delta = d_h(x, b) \). The final form of the equation is the following:

\[
\varepsilon^2 + \log^2 \left( \frac{||a||}{||x||} - 1 \right) = \delta^2 + \log^2 \left( \frac{||b||}{||x||} - 1 \right)
\]

Remark 1.6 This formula also describes the equidistant surface of \( S^2 \times \mathbb{R} \) with the usual Euclidean scalar product, vector length and angle formula (see [6]).

It is now easy to examine some special cases: when \( |a| = |b| \), the equidistant surface consists of those points of an Euclidean plane in our model, which are inner points of the cone (e.g. proper point of \( H^2 \times \mathbb{R} \)). Another special case appears when \( a \) and \( b \) are on the same fibre. In this case (\( \delta = \varepsilon \)) the equidistant surface is the “positive side” of a hyperboloid of two sheets.

Our projective method gives us a way of investigation the \( H^2 \times \mathbb{R} \) space, which suits to study and solve similar problems (see [10]). In this paper we have examined only some problems, but analogous questions in \( H^2 \times \mathbb{R} \) geometry or, in general, in other homogeneous Thurston geometries are timely (see [11], [8], [9]).

References


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