INVOLUTES AND EVOLUTES IN n-DIMENSIONAL SIMPLY ISOTROPIC SPACE $I_n^{(1)}$

Blaženka Divjak  
University of Zagreb, Faculty of Organization and Informatics, Varaždin, Croatia  
E-mail: bdivjak@foi.hr

Željka Milin Šipuš  
University of Zagreb, Department of Mathematics, Croatia  
E-mail: milin@math.hr

In this paper, the notions of the isotropic involutes (of order k) and the isotropic evolutes in n-dimensional simply isotropic space $I_n^{(1)}$ are defined. We determine the formula of involutes of a given admissible curve in $I_n^{(1)}$ and the curvature and the torsion of involutes and evolutes in $I_n^{(1)}$. The system of differential equations which determines the evolute of a given admissible curve in $I_n^{(1)}$ is found. The explicit formula of the evolutes of admissible curve in $I_n^{(1)}$ is given. The definitions of involutes and evolutes, which are used in this article, are motivated by the analogous definitions for Euclidean case from [2].

Keywords: admissible curve, involutes, evolutes, n-dimensional simply isotropic space.

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1. CURVES IN $I_n^{(1)}$

Let $I$ be an interval, $I \subseteq \mathbb{R}$ and $f : I \rightarrow I_n^{(1)}$ vector function given in affine coordinates as

$$\overrightarrow{OX}(t) = (x_1(t), \ldots, x_n(t)) : = \mathbf{x}(t), \ t \in I.$$  

The set of points $c \in I_n^{(1)}$ is called a $C^r$-curve if there is an open interval $I \subseteq \mathbb{R}$ and $C^r$-function $(r \geq 1)$ $f : I \rightarrow I_n^{(1)}$ with $f(I) = c$.

A $C^r$-curve is a regular $C^r$-curve provided

$$\dot{\mathbf{x}}(t) = (\dot{x}_1(t), \ldots, \dot{x}_n(t)) \neq 0, t \in I,$$

and if $f$ is an injective transformation a curve is called a simple $C^r$-curve.

A regular $C^r$-curve $(r \geq n-1)$ is nondegenerate if the set of vectors

$$\{\dot{x}(t), \ldots, x^{(n-1)}(t)\}$$

is linearly independent for all $t \in I$. 


A curve \( c \subset \mathbb{I}^{(1)}_n \) is said to be an admissible \( C^r \)-curve (\( r \geq n-1 \)) when \( c \) is a simple, nondegenerate \( C^r \)-curve (\( r \geq n-1 \)) without the isotropic osculating hyperplanes.

Let \( c \), which is defined on a closed interval \([a,b]\), be an admissible curve in \( \mathbb{I}^{(1)}_n \). Then
\[
s := \int_a^b \left( \dot{x}_1^2 + \ldots + \dot{x}_{n-1}^2 \right)^{1/2} \, dt
\]
is called the isotropic arc length of the curve \( c \) from \( x(a) \) to \( x(b) \). (From now on, \( s \) always denotes a parameter of the arc length.)

For the admissible curve \( c \) (\( x=x(s) \)) we can define \( n \)-frame \( \{t_1, \ldots, t_n\} \) in any point, as has been done in [1] or [4] for example. Then, there are functions \( \kappa_1(s), \ldots, \kappa_{n-1}(s) \) so that the Frenet formulae
\[
(t'_i) = \kappa_i(t_2)
\]
\[
(t'_i) = \kappa_i(t_{i+1}) - \kappa_{i-1}(t_{i-1}) \quad i = 2, \ldots, n-1,
\]
\[
t'_n = 0
\]
hold. The functions \( \kappa_1(s), \ldots, \kappa_{n-1}(s) \) are called the isotropic curvatures of the curve \( c \).

The definitions of involutes and evolutes, which are used in this article, are motivated by the analogous definitions for the Euclidean case from [2].

2. INVOLUTES

2.1. Involutves in \( \mathbb{I}^{(1)}_n \)

Definition 1. Let \( c \), given by \( x = x(s) \), \( x : I \to \mathbb{I}^{(1)}_n \), \( I \subseteq \mathbb{R} \) be an admissible \( C^r \)-curve (\( r \geq n \)) parameterized by the parameter of the arc length. The orthogonal trajectories of the first tangents of the curve \( c \) are called the involutes of the curve \( c \).

Theorem 1. A one-parameter family of involutes of an admissible curve \( c \) is represented by the formula
\[
\bar{x}(s) = x(s) + t_i(s)(k - s),
\]
where \( k \) is an arbitrary constant and \( s \) is the arc length of the curve \( c \).

Proof. The involute of the curve \( c \) (\( x=x(s) \)) is characterized by
\[
\bar{x}(s) = x(s) + u(s)t_i(s)
\]
where \( u(s) \) is a function of \( s \) on \( I \). Then the differentiation of the relation (2.2) and the Frenet formulae (1.1) give the following equation.
(2.3) \[ \ddot{x}'(s) = (1 + u'(s))t_i(s) + \kappa_i(s)u(s)t_j(s). \]

In accordance with \( \ddot{x}'t_i = 0 \), we have \( 1 + u'(s) = 0 \) and furthermore,

(2.4) \[ u(s) = k-s, \quad k = \text{const}. \]

Inserting the relation (2.4) into (2.2) we obtain the expression (2.1) as desired.

**Corollary 1.** Two different involutes of an admissible curve \( c \) are equidistant.

In addition, we wish to generalize the notion of an involute.

**Definition 2.** Let \( c (x = x(s)) \) be an admissible curve. Curves, which are orthogonal to the system of \( k \)-dimensional osculating hyperplanes of \( c \), are called the *involutes of order \( k \)* of the curve \( c \).

The involutes of order \( k \) are given by

(2.5) \[ \ddot{x} = x(s) + u_1(s)t_1(s) + \ldots + u_k(s)t_k(s), \quad k \leq n-1. \]

In order to determine the functions \( u_1, \ldots, u_k \) from (2.5) we differentiate (2.5) and by using the Frenet formulae (1.1) we have

(2.6) \[
\begin{align*}
\ddot{x}' &= (1 + u_1' - u_2\kappa_2) t_1 + \sum_{i=2}^{k-1} (u_i' + \kappa_{i-1}u_{i-1} - \kappa_i u_{i+1}) t_i + \\
&\quad + (u_k' + \kappa_{k-1}u_{k-1}) t_k + \kappa_k u_k t_{k+1}.
\end{align*}
\]

Since we have \( t_iy' = 0 \) for \( i = 1, \ldots, k; \quad k \leq n-1 \) we obtain

(2.7) \[
\begin{align*}
1 + u_1' - \kappa_2 u_2 &= 0 \\
u_i' + \kappa_{i-1}u_{i-1} - \kappa_i u_{i+1} &= 0, \quad 1 = 2, \ldots, k - \\
u_k' + \kappa_{k-1}u_{k-1} &= 0.
\end{align*}
\]

from (2.6) after a scalar multiplication by \( t_1, t_2, \ldots, t_k \).

The system of differential equations (2.7) is the same as in the Euclidean case and admits a uniquely determined set of solutions \( u_1, \ldots, u_k \), having already prescribed the initial values at the point \( s = a \) of the curve \( c \).

According to the above, the involute, which is defined in Definition 1, is actually the involute of order 1 and then the relations (2.7) are reduced to (2.4).

If \( c \subset I_{n}^{(m)} (m < n) \) is an admissible curve the involutes of \( c \) could be defined in the same way as is done above. Obviously, Theorem 1, Corollary 1 and Corollary 2 are true in a case when \( k \leq n-m-1 \).

**2.2. Involutes in \( I_{3}^{(1)} \)**

**Corollary 2.** Let \( c \), given by \( x = x(s) \), be an admissible curve in \( I_{3}^{(1)} \) where \( s \) is the parameter of the arc length and \( \{t(s), n(s), b(s)\} \) the 3-frame of the given curve. Then the involute \( \tilde{c} (\tilde{x} = \tilde{x}(s)) \) of curve \( c \) has the following form
\[ \bar{x}(s) = x(s) + (k - s)t(s). \]

The proof is analogous to that of Theorem 1.

**Corollary 3.** If \( \kappa(s) \) and \( \tau(s) \) are the curvature and the torsion of an admissible curve \( c \), then the curvature \( \bar{\kappa} \) and the torsion \( \bar{\tau} \) of the involute \( \bar{c} \) of the curve \( c \) are given by

\[ \bar{\kappa}(s) = \frac{\text{sgn} \ \kappa}{|s-k|}, \quad \bar{\tau}(s) = \frac{(\tau')}{\kappa(k-s)}. \]

**Proof.** The parameter \( s \) is not the parameter of the arc length of \( \bar{c} \), so, as is shown in [4], we have

\[ \bar{\kappa}(s) = \frac{\text{Det}(\bar{x}, \bar{x}, \bar{x})}{|\bar{x}|^3}, \quad \bar{\tau}(s) = \frac{\text{Det}(\bar{x}, \bar{x}, \bar{x})}{\text{Det}(\bar{x}, \bar{x}, \bar{x})}. \]

On the other hand, the differentiation of equation (2.8) implies that

\[ \bar{x}(s) = (k-s)\kappa n, \]
\[ \bar{x}(s) = -(k-s)\kappa^2 t + [(k-s)\kappa' - \kappa]n + (k-s)\kappa \tau b, \]
\[ \bar{x}(s) = [2\kappa^3 - 3(k-s)\kappa \kappa']t + [-\kappa'(k-s)^3 - 2\kappa' + (k-s)\kappa'']n + \]
\[ [2(k-s)\kappa \tau + (k-s)\kappa \tau' - 2\kappa \tau]b, \]
\[ \bar{x}(s) = (k-s)\kappa n, \]
\[ \bar{x}(s) = -(k-s)\kappa^2 t + [(k-s)\kappa' - \kappa]n. \]

Now, it is easy to see that

\[ |\bar{x}| = |(k-s)\kappa|, \]
\[ \text{Det}(\bar{x}, \bar{x}) = (k-s)^3 \kappa^3, \]
\[ \text{Det}(\bar{x}, \bar{x}, \bar{x}) = (k-s)^3(\kappa \tau' - \kappa \tau) \kappa^3. \]

And now from the above relations and (2.10) we deduce (2.9).

**Example 1.** The involutes of the helix

\[ x(s) = \left( a \cos \frac{s}{a}, -a \sin \frac{s}{a}, \frac{p}{a} s \right) \]

are the plane curves

\[ x(s) = \left( a \cos \frac{s}{a} + (s-k) \sin \frac{s}{a}, -a \sin \frac{s}{a} + (k-s) \cos \frac{s}{a}, \frac{p}{a} k \right) \]

\( k = \text{const.} \)

We could ask ourselves if there are any other admissible curves in \( U_1 \) which have plane involutes. Because of (2.9) we may conclude that \( \bar{\tau} = 0 \) if and only if
\[ \left( \frac{\tau}{K} \right)' = 0. \] Thus, \[ \frac{\tau}{K} = \text{const.} \] So, only those admissible curves in \( I^{(i)}_1 \) which have plane involutes are the helices.

### 3. EVOLUTES

#### 3.1. Evolutes in \( I^{(i)}_n \)

**Definition 3.** We say that a curve \( c^*(x^*(s)) \) is an evolute of an admissible \( C^n \)-curve \( c(x=x(s)), c \subset I^{(i)}_n \) if \( c \) is the involute of \( c^* \). The parameter \( s \) is the parameter of the arc length of \( c \).

The question that must be asked is: when does an evolute of a given curve exist and what does this evolute look like? The following theorem, which has the same form as in the Euclidean case (see [2]), answers the first part of this question.

**Theorem 2.** Let \( c: I \rightarrow I^{(i)}_n \) be an admissible curve and \( s \) the parameter of the arc length. The evolute of \( c \) exists if and only if there is a nonisotropic unit field \( a(s) \) and a real function \( p(s) \) such that

\[ \tau + a'p = 0. \]  

Let \( c^*(x^*(s)) \) be the involute of \( c (x=x(s)) \). Then, there is a unit field \( a(s) \) and a function \( p(s) \) so that

\[ x^*(s) = x(s) + p(s)a(s) \]  

and

\[ x'^*(s) = \lambda(s)a(s). \]  

By differentiating \( x^*(s) \) we get

\[ (\lambda - p')a = \tau + a'p. \]  

Multiplying the relation \( 3.4 \) by \( a \) we obtain

\[ \lambda - p' = 0 \]  

and then, the relation \( 3.4 \) becomes \( 3.1 \).

\[ \Rightarrow \] Now we suppose that \( 3.1 \) holds. Define \( x^*(s) \) by

\[ x^*(s) = x(s) + a(s)p(s) \]  

and by differentiating that by \( s \) we get

\[ x'^*(s) = \tau_1 + a'p + ap'. \]  

Comparing \( 3.6 \) and \( 3.1 \) we conclude that

\[ x'^* = ap' \]

which means that vectors \( x^*-x \) and \( x'^* \) are linearly dependent. In addition, we have
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\[ t_1x^{*} = t_1ap' = a'pap' = 0, \]

since \( a \) is a unit field. Therefore \( x' \) is orthogonal to \( x^{*} \) which implies \( c \) is the involute of \( c^{*} \).

Now, we shall try to find the expression for the evolute \( c^{*} \) of a given admissible curve \( c (x=x(s)), c \subseteq I^{(n)} \) which is referred to as the parameter of the arc length \( s \).

Obviously,

\[ (3.7) \quad c^{*} = x^{*}(s) = x(s) + p(s)a(s), \quad p(s) \neq 0, \]

where \( a(s) \) is a unit field orthogonal to \( c \) and therefore, collinear with the first tangent of \( c^{*} (x^{*} = \lambda a) \). So, we have

\[ (3.8) \quad a = \sum_{i=2}^{n} a_i t_i \]

and since \( |a| = 1 \) it follows that

\[ (3.9) \quad \sum_{i=2}^{n-1} a_i^2 = 1. \]

By differentiating (3.7) by \( s \), we get

\[ \lambda a = t_1 + p' a + p \sum_{i=2}^{n}[a_i t_1 + a_i(\kappa_i t_{i+1} - \kappa_{i-1} t_{i-1})]. \quad (\kappa_n = 0) \]

and then,

\[ (\lambda - p') a = (1 - \kappa_1 a_2 p)t_1 + p[(a_2' - \kappa_2 a_3)t_2 + \sum_{i=3}^{n-2}(a_i' + a_{i-1}\kappa_{i-1} - a_{i+1}\kappa_i)t_i + (a_{n-1}' + a_{n-2}\kappa_{n-2})t_{n-1} + (a_{n}' + \kappa_{n-1}a_{n-1})t_n]. \]

So now we have

\[ (3.10) \quad \lambda - p' = 0 \]

and

\[ (1 - \kappa_1 a_2 p)t_1 + p[(a_2' - \kappa_2 a_3)t_2 + \sum_{i=3}^{n-2}(a_i' + a_{i-1}\kappa_{i-1} - a_{i+1}\kappa_i)t_i + (a_{n-1}' + a_{n-2}\kappa_{n-2})t_{n-1} + (a_{n}' + \kappa_{n-1}a_{n-1})t_n] = 0. \]

At the end, we will have the following system

\[ \begin{align*}
1 - \kappa_1 a_2 p &= 0 \\
\kappa_2 a_3 &= 0 \\
\kappa_3 a_4 &= 0 \\
\kappa_i a_{i+1} - a_{i-1} a_i &= 0 & i = 3, ..., n-2 \\
\kappa_1 a_{n-1} &= 0 \\
\kappa_2 a_n &= 0 \\
\sum_{i=2}^{n-1} a_i^2 &= 1
\end{align*} \]

which gives us the evolute of \( c \) (up to a constant).
When we consider an admissible curve $c$ from $I_n^{(a)}$ the analogous system to system (3.10) does not determine the evolutes of $c$ completely because the constants $a_{n-m+1}, \ldots, a_n$ aren't actually in that system.

3.2. Evolutes in $I_3^{(1)}$

If we put $n=3$, the system (3.10) becomes

$$
\begin{align*}
a_2 &= \pm 1 \\
1 \mp p\kappa &= 0 \\
a_3 \pm \tau &= 0,
\end{align*}
$$

and then, we have

$$
(3.12) \quad p(s) = \pm \frac{1}{\kappa(s)} (= \pm \rho(s)), \quad a_3(s) = k - \int_0^s \tau(\sigma)d\sigma.
$$

Inserting this into (3.7) we get the following corollary:

**Corollary 4.** The equation of evolute $c^*$ of an admissible curve $c$ $(x=x(s))$ in $I_3^{(1)}$, where $s$ is the parameter of the arc length on $c$, has the following form:

$$
(3.13) \quad c^*\ldots x^*(s) = x(s) + \rho(s) \left[ n(s) + (k - \int_0^s \tau(\sigma)d\sigma)b \right].
$$

The projection of (3.13) on the basic plane $x_3=0$ is

$$
\tilde{x}^* = \tilde{x}(s) + \rho(s)\tilde{n}(s)
$$

and this is a formula of an evolute in the Euclidean case.

**Corollary 5.** The curvature $\kappa^*$ and the torsion $\tau^*$ of the evolute $c^*$ of a curve $c \subset I_3^{(1)}$ depend on the curvature $\kappa$ and torsion $\tau$ of $c$ in the following way:

$$
(3.14) \quad \kappa^*(s) = \frac{\kappa^3(s)}{|\kappa'(s)|}, \quad \tau^*(s) = -\frac{\kappa^3}{\kappa'} (k - \int_0^s \tau(\sigma)d\sigma).
$$

**Proof.** If $c^*$ is given by (3.13) we have

$$
\begin{align*}
\dot{x}^* &= \rho'n + \rho'(k - \int_0^s \tau(\sigma)d\sigma)b, \\
\tilde{x}^* &= -\rho'\kappa t + \rho''n + \rho''(k - \int_0^s \tau(\sigma)d\sigma)b, \\
\ddot{x}^* &= -(2\rho''\kappa + \rho'\kappa')t + (\rho'' - \rho'\kappa^2)n + \rho''(k - \int_0^s \tau(\sigma)d\sigma)b, \\
|\dot{x}^*| &= |\rho'|, \\
Det(\dot{x}^*, \ddot{x}^*) &= \rho^2\kappa, \\
Det(\dot{x}^*, \ddot{x}^*, \dddot{x}^*) &= (\rho')^2 k^3(1 - \int_0^s \tau(\sigma)d\sigma).
\end{align*}
$$

Formulae (2.10) complete the proof.
Example 2. The evolute $c^*$ of the helix, given by (2.11), is an isotropic straight line
\[ c^* \ldots x(s) = (0, 0, ak). \]
In the projection on $x_3 = 0$ it shows that the evolute of a circle is the point.

Corollary 6. The evolute of a given curve is a plain curve if and only if $c$ is a plane curve.

Proof. Namely, $\tau^* = 0$ if and only if $\kappa = 0$ or $\int r(\sigma) d\sigma = k$. The condition $\kappa = 0$ contradicts the fact that $c$ is admissible. The second condition can be written as $r = 0$ which means that $c$ lies in a nonisotropic plane.

Corollary 7. If a curve $c$ has a constant torsion $\tau_0 \neq 0$, then the torsion of its evolute has the form
\[ \tau^* = -\frac{k^3}{\kappa'}(k - \tau_0 s). \]

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Blaženka Divjak
Željka Milin Šipuš

EVLVENTE I EVOLUTE U n-DIMENZIONALNOM JEDNOSTRUKO
IZOTROPNOM PROSTORU

Sažetak
Članak se sastoji od tri dijela. Prvi, uvodni dio, definira pojam dopustive krivulje u n-
dimenzionalnom jednostruko izotropnom prostoru i navodi Frenetove formule kao specijalni
slučaj situacije u n-dimenzionalnom m-struko izotropnom prostoru $I_n^{(m)}$ opisane u [3]. U
drugom dijelu dana je formula evolventi dopustive krivulje, kao i sustav diferencijalnih
jednačbi koji određuje evolvente k-tog reda u $I_n^{(1)}$. Nadalje, izvedena je fleksija i torzija
evolventi dopustive krivulje u trodimenzionalnom jednostrukom izotropnom prostoru $I_{3}^{(1)}$ u ovisnosti o fleksiji i torziji dane krivulje, a dan je primjer evolvente cilindrične spirale. Treći dio bavi se evolutama dopustive krivulje u $I_{n}^{(1)}$. Nađen je sustav diferencijalnih jednadžbi koje određuju evolutu dane dopustive krivulje u $I_{n}^{(1)}$, dana je eksplcitna formula evolute dopustive krivulje u $I_{3}^{(1)}$, kao i fleksija i torzija takve evolute u ovisnosti o fleksiji i torziji dane krivulje. Razmotrene su i neke posljedice izvedenih formula, te pitanje evolventi i evoluta dopustivih krivulja u općem slučaju $I_{n}^{(m)}$. Upotrijebljene definicije evolvente i evolute motivirane su analognim definicijama za euklidski slučaj koje su izrečene u [2].

Ključne riječi: dopustiva krivulja, evolvente, evolute, n-dimenzionalni jednostruko izotropni prostor.