H-CLOSED EXTENSIONS AND ABSOLUTE OF INVERSE LIMIT SPACE

The main purpose of this paper is the application of the Katetov extension \( kX \) to an inverse system and its limit. By the method of the extension theory the theorems concerning continuity of the Katetov functor, H-closedness and nearly-compactness of an inverse limit space are given.

H-closed extension, inverse system

1. KATETOV EXTENSION OF A LIMIT SPACE

If \( X \) is a topological space, then the closure and the interior of a subset \( A \subseteq X \) is denoted by \( C_A^X \) and \( \text{Int}_A^X \) or by \( C_A \) and \( \text{Int}_A \).

A Hausdorff space \( X \) is \( H \)-closed if for every open cover \( U \) of \( X \) there exists a finite subfamily \( \{ U_1, \ldots, U_k \} \) of \( U \) such that

\[
X = C_{\bigcup_{1 \leq j \leq k} U_j} \cap \left( \bigcup_{1 \leq j \leq k} C_{U_j} \right) \quad (17)\]

A continuous mapping \( f: X \to Y \) is said to be proper \([17]\)

if for each \( y \in Y \) and each \( V \ni y \) open in \( Y \) there exists a \( V' \ni y \) which is open in \( Y \) and such that \( \text{Int}_Y \left( C_V \right) \subseteq C_{f^{-1}(V')} \).

An inclusion \( A \subseteq Y \) is proper if for each \( y \in Y \) and each \( V \ni y \) open in \( Y \) there exists a \( V' \ni y \) open in \( Y \) and such that \( \text{Int}_A \left( A \cap C_Y \right) \subseteq C_{A \cap V'} \).

1.1. LEMMA. \([17]\). Let \( f: X \to Y \) be a continuous mapping. Then:

1. \( f \) is proper, if \( Y \) is regular;
2. \( f \) is proper if \( X \) is \( H \)-closed and if \( Y \) is a Hausdorff space;
3. a closed subspace \( A \) of \( H \)-closed \( X \) is \( H \)-closed iff the inclusion \( A \subseteq X \) is proper;
4. each open and dense embedding is proper.

Let \( F \) be a family of all open free ultrafilters on a Hausdorff space \( X \). The Katetov extension \( kX \) of \( X \) \([17]\) is the set \( X \cup F \) with topology consisting of all open subsets of \( X \) and all sets of the form \( \{ x \} \cup U \), where \( x \in F \) and \( U \in x \).

1.2. LEMMA. \((16)\), \((6)\). Let \( X \) be a Hausdorff space. Then:

1. \( kX \) is \( H \)-closed;
2. \( X \) is open and dense (i.e. proper) embedded in \( kX \);
3. \( kX \) is discrete in the topology induced by the topology on \( kX \);
4. a mapping \( f: X \to Y \) into \( H \)-closed space \( Y \) has a unique continuous extension \( k f: kX \to Y \) if and only if \( f \) is proper;
(v) If $U$ and $V$ are disjoint open subsets of $X$ then $C_{kX} U \cap C_{kX} V \subseteq X$.

We say that an extension $Y$ of $X$ is majorizable if there exists an extension $Z$ of $X$ and a map $F: Z \rightarrow Y$ which is an extension of the identity $i: X \rightarrow X$.

An extension will be called r.o.-free if for each regularly open subset $U$ of $X$ the boundary $B_{dX} U$ in $X$ is the same as the boundary $B_{dY} V$ of $V$ in $Y$, where $V$ is an arbitrary open subset of $Y$ such that $U = V \cap X$.

1.3. LEMMA. ([17:1.3.1].) If an H-closed extension $X \subseteq Y$ is such that:

a) $X$ is open in $Y$,

b) The remainder $Y - X$ is discrete in the topology induced from $Y$,

c) $X \subseteq Y$ is r.o. - free

then $X \subseteq Y$ is non-majorizable.

1.4. LEMMA. An extension $X \subseteq Y$ which satisfies a) and b) of Lemma 1.3. is r.o.-free iff the following condition (K) is satisfied:

(K) If $U$, $V$ is a pair of disjoint open subsets of $X$ then $C_{kU'} U \cap C_{kV'} V \subseteq X$, where $U'$, $V'$ are arbitrary open subsets of $Y$ such that $U = U' \cap X$ and $V = V' \cap X$.

A $p$-cover of $X$ is an open cover of $X$ possessing a finite subfamily which is dense in $X$ ([26]). A map $f: X \rightarrow Y$ is a $p$-mapping iff $f$ can be continuously extended to $k: kX \rightarrow kY$ ([26]).

A continuous mapping $f: X \rightarrow Y$ is a $p$-mapping iff $f$ can be continuously extended to $k: kX \rightarrow kY$ ([26]).

1.5. LEMMA. ([11], [22]). Let $X_{\alpha}$ be non-empty spaces for each $\alpha \in A$.

Then $k(PX_{\alpha}) = P kX_{\alpha}$ iff at least one of the following two conditions is satisfied.

a) $X_{\alpha}$ is H-closed for each $\alpha \in A$.

b) There exists $X_{\alpha}$ which is not H-closed. $X_{\alpha}$ is finite for all $\alpha \neq \alpha'$.

Moreover, all but finitely many $X_{\alpha}$'s have only one point.

In contrast of the above Lemma we show that the functor $k$ is continuous in some non-trivial cases i.e. that $k (\lim_{\alpha} X_{\alpha}) = \lim_{\alpha} kX_{\alpha}$.

Now we start with the key lemma of this Section.

1.6. LEMMA. Let $X = \{X_{\alpha}, f_{\alpha}, A\}$ be an inverse system of a Hausdorff spaces $X_{\alpha}, \alpha \in A$. Then:

i) if the mappings $f_{\alpha}$ are p-map then there exists inverse system $kX_{\alpha} = \{kX_{\alpha}, k f_{\alpha}, A\}$;

ii) if $\lim X$ is non-empty and if the projections $f_{\alpha}: \lim_{\alpha} X \rightarrow X_{\alpha}$, $\alpha \in A$, are p-map, then there exists a continuous mapping $K: k (\lim_{\alpha} X_{\alpha}) \rightarrow \lim_{\alpha} kX_{\alpha}$ which is an extension if the identity $i: \lim_{\alpha} X \rightarrow \lim_{\alpha} kX_{\alpha}$;

iii) if the projections $f_{\alpha}$ are p-map and onto, then $K$ is onto and $\lim_{\alpha} kX_{\alpha}$ is an H-closed extension of $\lim_{\alpha} X_{\alpha}$ such that $\lim_{\alpha} X$ is open in $\lim_{\alpha} kX_{\alpha}$.

Proof. (1) Apply Lemma 1.2. (iv).
(11) Now we have the p-map mappings $f_{\alpha} : \lim X \rightarrow kX_{\alpha}$, $\alpha \in A$. By virtue of Lemma 1.2. (iv) there exist a continuous mappings $k_{\alpha} : k(\lim X) \rightarrow kX_{\alpha}$, $\alpha \in A$. A family $\{k_{\alpha} : \alpha \in A\}$ induces a continuous mapping $K : k(\lim X) \rightarrow \lim kX$ [2:138]. The proof is completed.

Let us prove that $K$ is onto. For each $x \in \lim kX$ we consider a points $x = f_{\alpha}^{-1}(x)$, $\alpha \in A$, where $f_{\alpha} : \lim X \rightarrow kX_{\alpha}$, $\alpha \in A$, are the projections. For each $x_{\alpha}$ we have $\{x_{\alpha}\} = \{C1U : U$ is the open neighborhood of $x_{\alpha}\}$. A family $\{(k_{\alpha})^{-1}(U) : \alpha \in A\}$ is a centred family of open subsets in H-closed space $k(\lim X)$. This means that there exists a point $y \in \cap \{C1U : \alpha \in A\}$. Clearly $k_{\alpha}(y) = x_{\alpha}$ for each $\alpha \in A$. Thus, $K(y) = x$. This means that $K$ is onto and that $\lim kX$ is H-closed as a continuous image of H-closed space $k(\lim X)$. In order to complete the proof it suffices to prove that $\lim kX$ is dense in $X$. This is an immediate consequence of the definition of a base of the inverse limit space and the assumption that $f_{\alpha}$ are onto.

1.7.Lemma. Let $X = \{X_{\alpha}, f_{ab}, A\}$ be an inverse system with projections $f_{\alpha}$ which are onto p-map. For each $x \in k(\lim X) - \lim X$ there exists a $\alpha \in A$ such that $k_{\alpha}(x) \in kX_{\alpha} - X_{\alpha}$.

Proof. An immediate consequence of the fact that $x$ is free ultrafilter and the definition of a base on inverse limit space.

From Lemmas 1.3. and 11.7. we obtain the following.

1.8.Lemma. Let $X$ be an inverse system with projections $f_{\alpha}$ which are p-map onto. Then $\lim kX = k(\lim X)$ if and only if the following conditions are satisfied:
a) $\lim kX - \lim X$ is discrete in the topology induced by the topology on $\lim kX$,

b) each open subset $U \subseteq \lim X$ is r.o.-free in $\lim kX$.

A mapping $f:X \rightarrow Y$ is said to be $p$-perfect if $f$ is a p-map and $f(kX - X) = kY - Y$ [26].

1.9.lemma. Let $X$ be an inverse system with $p$-perfect onto mappings $f_{\alpha\beta}$ such that $f_{\alpha}$ are $p$-perfect and onto. Then $f_{\alpha}(\lim kX - \lim X) \subseteq kX_{\alpha} - X_{\alpha}$, $\alpha \in A$.

1.10.lemma. Let $X$ be an inverse system as in Lemma 1.9. A subspace $\lim kX - \lim X$ is discrete iff the following condition is satisfied: (D) For each point $x_{\alpha} \in kX_{\alpha} - X_{\alpha}$ there exists a $\beta \in A$, $\beta \geq \alpha$, such that for each $\gamma \in A$, $\alpha \leq \beta \leq \gamma$, the fiber $(k_{\beta_{\gamma}})^{-1}(x_{\beta})$ contains a single point for each $x_{\beta} \in (k_{\alpha\beta})^{-1}(x_{\alpha})$.

Proof. The "only if" part. Now the subspace $\lim kX - \lim X = Y$ of the space $\lim kX$ is the limit of inverse subsystem $Y = \{kX_{\alpha} - X_{\alpha}, k_{\alpha\beta} / (kX_{\beta} - X_{\beta}), A\}$. Each point $y \in Y$ is an open subset of $Y$. This means that $\{y\}$ contains the fiber $(k_{\alpha})^{-1}(U_{\alpha})$ for some
open subset $U$ of $k\alpha - X$. Thus (D) is satisfied.

The proof of the "if" part is similar.

1.11.THEOREM. Let $X = \{ X_\alpha, f_\alpha : \alpha \in A \}$ be an inverse system such that $f_\alpha$ are p-perfect mappings. If the projections $f_\alpha : \lim X \to X_\alpha$, $\alpha \in A$, are onto p-map, then $\lim kX$ and $k(\lim X)$ are homeomorphic iff $X$ satisfies the condition (D) and $\lim kX$ satisfies the condition (K).

Proof. The "if" part. By virtue of Lemmas 1.8. and 1.10. it follows that $\lim kX$ satisfies the conditions of Lemma 1.3. Thus, the mapping $K$ is a homeomorphism. The "only if" part follows from the fact that $kX$ satisfies the conditions of Lemma 1.3.

1.12.DEFINITION. A mapping $f:X \to Y$ is said to be $\theta$-continuous if for each $x \in X$ and each open $V \ni f(x)$ there is an open $U \ni x$ such that $f(CU) \subseteq CV$.

If $Y$ is regular, then each $\theta$-continuous mapping $f:X \to Y$ is continuous.

1.13.DEFINITION. A mapping $f:X \to Y$ is said to be $\theta$-homeomorphic if $f$ is $1-1$ onto such that $f$ and $f^{-1}$ are both $\theta$-continuous. We say that two extensions $Y$ and $Z$ of a space $X$ are $\theta$-equivalent if there exists a $\theta$-homeomorphism $H:Z \to Y$ which is the extension of identity $1:X \to X$.

1.14.LEmma. Let $X$ be an inverse system with p-perfect bonding mappings and proper onto projections. The space $\lim kX$ is $\theta$-equivalent to the space $k(\lim X)$ iff the condition (K) is satisfied.

Proof. The "if" part. Apply the Fomin modification $(\lim kX)$, [9:46] which is homeomorphic to $k(\lim X)$. Moreover, $(\lim kX)$ is $\theta$-homeomorphic to $\lim kX$ [2:46m Lemma 7.] since $K$ is 1-1.

The "only if" part is obvious since $K$ is $\theta$-homeomorphism.

For an inverse system of a regular spaces we have the following corollary of Theorem 1.11.

1.15.COROLLARY.Let $X$ be an inverse system of a regular spaces and perfect onto bonding mappings. The spaces $k(\lim X)$ and $\lim kX$ are equivalent iff the conditions (D) and (K) are satisfied.

Now we define some special kinds of the proper mappings.

A mapping $f:X \to Y$ is said to be skeletal (HJ) if for each open (regularly open) $U \subseteq X$ we have $\text{Int} f^{-1}(CU) \subseteq \text{Cl} f^{-1}(U)$ [17].

1.16.LEmma.[17]. Each HJ-mapping is a proper mapping.

A mapping $f:X \to Y$ is semi-open if $\text{Int} f(U)$ is non-empty for each non-empty open subset $U \subseteq X$.

Each semi-open mapping is HJ and proper. Each open mapping is semi-open.

We say that a mapping $f:X \to Y$ is irreducible if the set $f^*(U) = \{ y : f^{-1}(y) \subseteq U \}$ is non-empty for any non-empty open subset $U \subseteq X$.

Every closed irreducible mapping is a semi-open mapping.

A mapping $f:X \to Y$ has the inverse property if $f^{-1}(CIV) = \text{Cl} f^{-1}(V)$ for any open set $V \subseteq Y$.

Every open mapping has the inverse property and every mapping with the inverse property is HJ-mapping.

In the paper [15] it was proved the following theorem.

1.17.THEOREM.Let $X = \{ X_\alpha, f_\alpha : \alpha \in A \}$ be an inverse system with
HJ-mapping $f_{a \beta}$. If the projections $f_{\alpha} : \lim X \rightarrow X_\alpha, \alpha \in A$, are onto, then the projections $f_{\alpha}$ are HJ-mapping.

A mapping $f : X \rightarrow Y$ is absolutely closed if there do not exists a proper extension $T$ of $X$ and an extension $f : T \rightarrow Y$ of $f$.

1.18. LEMMA. [26]. Let $f : X \rightarrow Y$ be a continuous mapping. The following are equivalent:
(1) $f$ is absolutely closed.
(2) (a) If $A \subseteq X$ is regularly closed, then $f(A)$ is closed.
    (b) If $x \in kX - X$ and $y \in Y$, then there exists $U \in x$ such that $f^{-1}(y) \cap CIU = \emptyset$.

1.19. LEMMA. [26:211]. A p-mapping $f : X \rightarrow Y$ is p-perfect iff $f$ is absolutely closed.

Now we have the following corollary of Theorem 1.11.

1.20. COROLLARY. Let $X = \{X_\alpha, f_{a \beta} : A\}$ be an inverse system with absolutely closed HJ-mapping $f_{a \beta}$ and onto projections $f_{\alpha} : \lim X \rightarrow X_\alpha, \alpha \in A$. The space $k(\lim X)$ is equivalent to the space $\lim kX$ iff the conditions (K) and (D) are satisfied.

A special role play a closed irreducible mapping since we have the following

1.21. LEMMA. If $f : X \rightarrow Y$ is p-perfect closed irreducible mapping, then the restriction $kf/(kX - X)$ is one-to-one i.e. $kf/(kX - X)$ is a homeomorphism the space $kX - X$ onto $kY - Y$.

Proof. If $x = (U_\alpha : \alpha \in A)$ is a free ultrafilter, then $(f^\#(U) : U \in x)$ is a free ultrafilter. It is easy to prove that for $y = (V_\mu : \mu \in M)$ $y \neq x$ it follows that $(f^\#(U_\alpha) : \alpha \in A) \neq (f^\#(V_\mu) : \mu \in M)$. This means that $kf/(kX - X)$ is one-to-one. The proof is completed.

1.22. THEOREM. Let $X = \{X_\alpha, f_{a \beta} : A\}$ be an inverse system with perfect irreducible onto mapping $f_{a \beta}$. Then $\lim kX = k(\lim X)$.

Proof. The p-projections $f_{\alpha} : \lim X \rightarrow X_\alpha, \alpha \in A$, are perfect (=closed with compact fiber $f_{\alpha}^{-1}(x_\alpha)$). It is easy to prove that $f_{\alpha}, \alpha \in A$, are irreducible. We infer that $\lim kX - \lim X$ is homeomorphic to each $kX_\alpha - X_\alpha, \alpha A$. Thus the condition (D) is satisfied. Let us prove that the condition (K) is satisfied. Let $U, V$ be a pair of disjoint open subsets of $\lim X$. A sets $f^\#(U)$ and $f^\#(V)$ are disjoint open subsets of $X_\alpha, \alpha \in A$, since $f_{\alpha}, \alpha \in A$, are perfect and irreducible. Since $X_\alpha$ satisfies the condition (K) we have the following relation in $kX : Clf^{\#}(U) \cap Clf^{\#}(V) \subseteq X_\alpha$. By virtue of the irreducibility of $f_{\alpha}$ it follows that in $\lim kX$ we have the relation $ClU \cap ClV \subseteq \lim X$. The condition (K) is satisfied. By 1.11. the proof is completed.

We close this Section with theorems concerning the inverse systems of H-closed spaces. The "only if" part of the following
1.23. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}: \alpha, \beta \in A\}$ be an inverse system of $H$-closed spaces $X_\alpha$. A space $\lim X$ is $H$-closed iff the projections $f_\alpha$ are proper.

Proof. The "only if" part. If $\lim X$ is $H$-closed, then by Lemma 1.1. (ii) the projections $f_\alpha$ are proper.

The "if" part. Now $kX = \{kX_\alpha, kf_{\alpha\beta}, A\} = \{X_\alpha, f_{\alpha\beta}, A\} = X$ since the mapping defined in the proof of Lemma 1.6. We have $\lim X \subseteq K \subseteq \lim kX$. Since $\lim kX = \lim X$ we infer that $\lim X = K$. As a continuous image of $H$-closed space $k(\lim X)$ the space $K = \lim kX$ is $H$-closed. The proof is completed.

1.24. REMARK: The "if" part of Theorem 1.23. has been proved in the paper [3].

1.25. LEMMA. Let $X = \{X_\alpha, f_{\alpha\beta}: \alpha \in A\}$ be an inverse system of a completely Hausdorff spaces $X_\alpha$. A limit $\lim X$ is completely Hausdorff.

Proof. Trivial.

1.26. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}: \alpha \in A\}$ be an inverse system. A space $\lim X$ is nearly-compact iff the spaces $X_\alpha, \alpha \in A$, are nearly-compact and if the projections $f_\alpha, \alpha \in A$, are proper.

Proof. Apply Theorem 1.23. and Lemma 1.25.

2. FOMIN EXTENSION $\partial X$

Let $X$ be a Hausdorff space. We now define a topology on the set $X \cup F$ as follows. For each $U$ open in $X$ let $0_U$ be the union of $U$ and all ultrafilters of $F$ which contain $U$. It is easy to prove that

$$0_\cup \cap 0_\cap = 0_\cup \cap 0_\cap$$

This means that a family $\{0_\cup U: U \text{ is open in } X\}$ is a base for topology on $X \cup F$. We denote the set $X \cup F$ equipped with this topology by $\partial X$. The space $\partial X$ is called the Fomin extension of a space $X$ [9].

2.1. LEMMA. [9]. The space $\partial X$ is $H$-closed extension of a Hausdorff space $X$. If $Y$ is any $H$-closed extension of $X$, then there exists a $\varepsilon$-continuous extension of $f:X \to Y$ of the identity $i:X \to X \subseteq Y$.

Let $f:X \to Y$ be a continuous mapping. For each ultrafilter $x = (U_\alpha: U_\alpha \text{ is open in } X) \in \partial X - X$ we consider a
filter-base $\delta f(x) = \{V: V \text{ is open in } Y \text{ such that there exists a } U_{\alpha} \in x \text{ with } f(U_{\alpha}) \subseteq V\}$. It is easy to prove that if $f$ is a p-map, then $\delta f(x)$ is an open ultrafilter in $Y$. By virtue of $H$-closedness of $\partial Y$ the intersection $Z = \cap (CIV: V \in \delta f(\{1\}))$ is non-empty. As in the case of the Katetov extension $kX$ it is easy to prove the following Lemma.

2.2. **Lemma.** (a) If $f:X \rightarrow Y$ is a p-mapping, then $\delta f(x)$ contains a single point of $Y$,
(b) The mapping $\delta f: \partial X \rightarrow \partial Y$ is e-continuous,
(c) If $f$ is p-perfect then $\delta f$ is continuous,
(d) If $Y$ is regular and if $f$ is p-mapping then $\delta f$ is continuous.

By the proof similar to proof of Lemma 1.6. we obtain

2.3. **Lemma.** Let $X = \{X_{\alpha}, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces $X_{\alpha}$ and a $p$-mapping $f_{\alpha\beta}$. Then:
(1) There exists inverse system $\partial X = \{\partial X_{\alpha}, \partial f_{\alpha\beta}, A\}$ with e-continuous mappings $\partial f_{\alpha\beta}$
(11) If the mappings $f_{\alpha\beta}$ are p-perfect or if $X_{\alpha}, \alpha \in A$, are regular, then the mappings $\partial f_{\alpha\beta}$ are continuous. Moreover, there exists a continuous mapping $S: \partial(\lim X) \rightarrow \lim \partial X$. Moreover, there exists a continuous mapping $S: \partial(\lim X) \rightarrow \lim \partial X$.
(111) If in (11) $f_{\alpha\beta}$ are onto then $S$ is onto.

2.4. **Problem.** Under what conditions the mapping $S$ is a homeomorphism?

If the bonding mapping are perfect and irreducible then we have

2.5. **Theorem.** Let $X = \{X_{\alpha}, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces $X_{\alpha}$ and perfect closed irreducible onto mapping $f_{\alpha\beta}$. Then the mapping $S: \partial(\lim X) \rightarrow \lim \partial X$ is a homeomorphism.

**Proof.** By virtue of Lemma 1.21. the mapping $S$ is onto and 1-1. It remains to prove that $S$ is an open mapping. A subspace $\partial(\lim X) - \lim X$ is homeomorphic to each $\partial X_{\alpha} - X_{\alpha}, \alpha \in A$, since the projections $f_{\alpha\beta}$ are perfect onto and irreducible. This means that the subspace $\partial(\lim X) - \lim X$ is homeomorphic to the subspace $\lim \partial X$. The proof is completed.

3. **Absolutely of an Inverse Limit Space**

The space $\partial X$. Let $\partial X$ denotes a family of all open (fixed or free) ultrafilters on a Hausdorff space $X$. We introduce a topology into $\partial X$ in the following way. Let $O_U$ be the set of all ultra-filters that contain $U$, where $U$ is open in $X [9]$; $O_U$ is to be a base on $\partial X$. That this definition is correct it follows from the relation

$$O_U \circ = O_U \cup O_V$$

(2)

It is easy to prove that
This means that $O_U$ is open and closed subset of $eX$.

3.1. LEMMA. If $X$ is a Hausdorff space then $eX$ is zero-dimensional and compact.

Proof. See [9].

A space $X$ is called extremally disconnected if for each disjoint open sets $U, V \subseteq X$ we have $ClU \cap ClV = \emptyset$.

If $X$ is extremally disconnected and $Y$ is dense in $X$, then $Y$ is extremally disconnected [9].

3.2. LEMMA. [9:41]. If $X$ ia a Hausdorff space, then $eX$ is extremally disconnected zero-dimensional compact space. The equation $X = eX$ holds iff $X$ is a compact extremally disconnected Hausdorff space.

The absolute $wX$ of a space $X$. A subspace $wX$ of $eX$ containing all fixed open ultrafilters on $X$ is called the absolute (in the sense od Iliadis) of the space $X$ or the extremally disconnected resolution of the space $X$.

3.3. LEMMA. The absolute $wH$ is dense in $eX$ and, consequently, $wX$ is extremally disconnected.

Proof. See [9:41].

3.4. LEMMA. [9:44]. The absolute $wX$ is $e$-homeomorphic to $X$ iff $X$ is extremally disconnected. If $X$ is regular extremally disconnected, then $wX$ is homeomorphic to $X$.

For each $x \in wX$ we define a point $p(x)$ such that $p_x(x) = \{ClU : U \in X\}$.

3.5. LEMMA. [9:55]. The natural projection $p_x : wX \longrightarrow X$ is $e$-continuous, irreducible and perfect. It is continuous iff $X$ is regular.

3.6. THEOREM. [9:56]. Let $f : X \longrightarrow Y$ be a $e$-continuous irreducible perfect mapping of a Hausdorff space $X$ onto a Hausdorff space $Y$. Then there exists a homeomorphism $w : wX \longrightarrow wY$ onto $wY$ such that $fp_x = p_y f$.

The absolute $wX$ and the extensions of a space $X$.

3.7. LEMMA. [9:60]. Let $gX$ be an arbitrary extension of a Hausdorff space $X$. Then there exists a homeomorphism $h : e(gX) \longrightarrow eX$ such that $h(p_x^{-1}(x)) = p_x^{-1}(x)$ for each $x \in X$.

3.8. COROLLARY. [9]. $e(\beta X) = e(kX) = eX$.

3.9. COROLLARY. [9]. If $bX$ is an arbitrary extension of $X$, then $w(bX) = \beta(wX)$. In particular, $\beta(wX) = \beta(wX)$.

The absolute in the sense of Mioduszewski. Now we enlarge the Iliadis topology defined at the begin of this Section by adding sets of the form $p_x^{-1}(U)$, $U$ being an open subset of $X$. It is easy to verify that the sets of the form $O_U \cap p_x^{-1}(V)$ may be taken as a members of a topology on the set $wX$. We denote this space by $aX$.

3.10. LEMMA. The space $aX$ is extremally disconnected and the mapping $p_x : aX \longrightarrow X$ is continuous, irreducible and perfect.

The space $aX$ is minimal in the following sense:
3.11. Lemma. ([17:33]) For any extremally disconnected space $E$ and any HJ-mapping $h: E \longrightarrow X$ there exists a unique mapping $a_h: E \longrightarrow aX$ such that $h = p_x(a_h)$.

The following theorem plays a special role in our investigation of the absolute of a limit space.

3.12. Theorem. Let $f: X \longrightarrow Y$ be a continuous mapping. A mapping $f$ has a unique absolute $a_f$ such that $p_v a_f = f p_x$ iff the mapping $f$ is HJ.

3.13. Remark. A) The "if" part of Theorem 3.12. has been proved in the paper [17:24] and the "only if" part in the paper: Shapiro L.B., Ob absoljutah topologiceskih prostranstv i nepryevnyh otobrazenijah, DAN SSSR 226:3(1976), 523-526.

B) Let us note that the absolute of a continuous mapping always exists but need not be unique.

C) From the proof of the "if" part of Theorem 3.12. it follows that the "if" part holds for the absolute $a_X$ in the sense of Iliadis.

D) Another construction of the absolute for regular spaces can be found from [1:363-370].

3.14. Lemma. ([18:124] or [1:363-370]). Let $f: X \longrightarrow Y$ be a continuous mapping. Then there exists the absolute $a_f: aX \longrightarrow aY$.

Moreover:

a) If $f$ is bicom pact, then $a_f$ is bicom pact;

b) If $f$ is irreducible and perfect (into, onto) $Y$, then $a_f$ is a homomorphism (into, onto) $aY$.

3.15. Lemma. ([18]). If $f: X \longrightarrow Y$ is an open onto mapping, then $a_f: aX \longrightarrow aY$ is onto.

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Now we apply this expository material to the inverse systems and their limits.

3.16. Theorem. Let $X = \{X_a, f_a\_a, A\}$ be an inverse system of a Hausdorff spaces $X_a$. If the mappings $f_a\_a$ are HJ-mapping then there exists an inverse system $wX = \{wX_a, w f_a\_a, A\}$ and a mapping $W: w X \rightarrow wX$.

Proof. Apply Remark 3.13. C) and the fact that the projections $f_a\_a$ are HJ-mappings. Then modify the proof of Theorem 1.6.

3.17. Remark. A) Similarly from Theorem 3.12. it follows that there exists an inverse system $aX = \{aX_a, a f_a\_a, A\}$ for an inverse system as in Theorem 3.16.

B) There exists inverse system $eX = \{eX_a, e f_a\_a, A\}$ if $X$ is the inverse system of Hausdorff spaces and HJ bonding mappings.

C) If $X$ is an inverse sequence then by total induction on can construct the inverse systems $wX (aX, eX)$ without the assumption that the absolute $w f (a f, e f)$ are unique.

3.18. Theorem. Let $X = \{X_a, f_a\_a, A\}$ be an inverse system of a Hausdorff spaces $X_a$ with irreducible perfect mappings $f_a\_a$ such
that the projections $f_\alpha$ are onto. Then the mapping $w: w(\lim X) \mapsto \lim wX$ is a homeomorphism.

Proof. From Theorem 3.6, it follows that $w^{f_{\alpha \beta}}$ are homeomorphisms. Similarly, we infer that $w^{f_\alpha}$ are homeomorphisms. This means that the spaces $w(\lim X)$ and $\lim wX$ are homeomorphic to $\lim X_\alpha$, $\alpha \in A$. The proof is completed.

3.19. THEOREM. Let $X = \{X_\alpha, f_{\alpha \beta}, A\}$ be an inverse system of Hausdorff spaces $X_\alpha$ and HJ bonding mappings $f_{\alpha \beta}$ such that the projections $f_\alpha$ are onto. Then the spaces $e(\lim X)$ and $\lim eX$ are homeomorphic if the following condition (S) is satisfied:

(S) For each two disjoint open subsets $U$ and $V$ of $\lim X$ there is a $\alpha \in A$ such that $f_\alpha(U)$ and $f_\alpha(V)$ have a disjoint neighborhood.

Proof. The "if" part. Let $x$ and $y$ be two distinct points in the space $e(\lim X)$. This means that there exists a pair $U, V$ of disjoint open subsets of $\lim X$ such that $U \in x$, $V \in y$. From the condition (S) it follows that $e^{f_\alpha(x)} = \{W : W$ open in $X_\alpha$ and there exists $U' \in x$ such that $f_\alpha(U') \subseteq W\}$ is not equal to $e^{f_\alpha(y)} = \{W : W$ is open in $X$ and there exists $V' \in y$ such that $f_\alpha(V') \subseteq W\}$. This means that the mapping $e: e(\lim X) \mapsto \lim eX$ is $1-1$. Since $e$ is onto and $e(\lim X)$ is compact, we infer that $e$ is a homeomorphism. The proof of the "if" part is completed. The proof of the "only if" part is similar.

3.20. LEMMA. The condition (S) is satisfied: (a) If the projections $f_\alpha$ are closed irreducible or (b) if for each open subset $U \subseteq \lim X$ there exists a $\alpha \in A$ and an open subset $U_\alpha$ of $X$ such that $f_\alpha^{-1}(U_\alpha) = U$.

Proof. Obvious.

3.21. THEOREM. Let $X = \{X_\alpha, f_{\alpha \beta}, A\}$ be an inverse system of a Hausdorff spaces $X_\alpha$ such that $f_{\alpha \beta}$ are closed irreducible and the projections $f_\alpha$ are closed onto (or the condition (b) of 3.20. is satisfied), then $e(\lim X) = \lim eX$.

Proof. Apply Lemma 3.20. and Theorem 3.19.

3.22. COROLLARY. Let $X = \{X_n, f_{nm}, N\}$ be an inverse sequence of a Hausdorff spaces $X_n$ with closed irreducible onto mappings $f_{nm}$. Then the spaces $e(\lim X)$ and $\lim eX$ are homeomorphic.

Proof. It is well known that the projections $f_{nm}$ are closed and irreducible onto mappings. Now apply Theorem 3.21.

If the mappings $f_{\alpha \beta}$ and the projections $f_\alpha$ in Theorem 3.21. are p-perfect then a restrictions of $e^{f_{\alpha \beta}} / wX_\beta, \beta \in A$, are identical with $w^{f_{\alpha \beta}}$. Similarly, a restriction of $e^{f_\alpha} / w(\lim X)$ is identical with $W^{f_\alpha}$. Thus we have
3.23. **THEOREM.** Let $X = (X_\alpha, f_{\alpha\beta}, A)$ be an inverse system as in 3.21. If the mappings $f_{\alpha\beta}$ and the projections $f_\alpha$ are p-perfect, then the spaces $w(\lim X)$ and $\lim wX$ are homeomorphic.

If the bonding mappings $f_{\alpha\beta}$ are perfect irreducible then from 3.23. holds Theorem 3.18.

For the absolute $aX$ in the sense of Mioduszewski we now prove 3.24. **THEOREM.** Let $X = (X_\alpha, f_{\alpha\beta}, A)$ be an inverse system of a Hausdorff spaces $X_\alpha$. If the spaces $w(\lim X)$ and $\lim wX$ are homeomorphic, then the spaces $a(\lim X)$ and $\lim aX$ are homeomorphic.

**Proof.** Let $G$ be any open neighborhoods of $x \in a(\lim X)$. By the definition of a base in $a(\lim X)$ there exist a neighborhood of $x$ of the form $O \cap p_{\lim^{-1}}(V)$ contained in $G$, where $V$ is open in $\lim X$ and $O_\cup$ is open in $w(\lim X)$. From the relations $w(\lim X) = \lim wX$ and $x \in O_\cup$ it follows there exists an open $U_\alpha \subset X_\alpha$ such that a set $(w_{\alpha}^{-1}(O))$ is a neighborhood of $x$ contained in $O_\cup$. Similarly there exists an open $V_\alpha \subset X$ such that $f_{\alpha}^{-1}(V) \subset V$ is a neighborhood of $x$. This means that a set $p_{\lim^{-1}} f_{\alpha}^{-1}(V) \cap (w_{\alpha}^{-1}(O))$ is a neighborhood of $x$ which is contained in $O_\cup \cap p_{\lim^{-1}}(V)$. From the relation $p_{\lim^{-1}} f_{\alpha}^{-1}(V_\alpha) = (w_{\alpha}^{-1})^{-1} p_{\alpha}^{-1}(V_\alpha)$ we infer that there exists a neighborhood $p_{\alpha}^{-1}(V) \cap O_\alpha = G_\alpha \subset aX_\alpha$ such that $(w_{\alpha}^{-1}(G) = (af_{\alpha}^{-1})$ is contained in $G$. This means that $G$ is open in $\lim aX$. Thus the mapping $A:a(\lim X) \longrightarrow \lim aX$ is 1-1 continuous and open mapping onto $\lim aX$ i.e. $A$ is a homeomorphism. The proof is completed.

We closed this Section with some theorems concerning the non-emptiness of the inverse limit space.

3.25. **LEMMA.** A Hausdorff space is H-closed iff $eX = wX$.

**Proof.** If $X$ is H-closed, then each open ultrafilter on $X$ is fixed. Thus $eX = wX$. Conversely, if $eX = wX$, then $X$ is H-closed since the mapping $p_x : wX \rightarrow eX \longrightarrow X$ is $e$-continuous and $eX$ is compact. The proof is completed.

3.26. **THEOREM.** Let $X = (X_\alpha, f_{\alpha\beta}, A)$ be an inverse system of H-closed spaces $X_\alpha$ and HJ-mapping $f_{\alpha\beta}$. The space $\lim X$ is non-empty iff the spaces $X_\alpha$, $\alpha \in A$, are non-empty. Moreover, if the mappings $f_{\alpha\beta}$ are onto, then the projections $f_\alpha$ are onto.

**Proof.** By Theorem 3.16. we obtain the inverse system $wX = (wX_\alpha, w_{\alpha\beta}, A)$ which is the inverse system of compact spaces $wX_\alpha = eX_\alpha$.

It is well known that $\lim wX$ is non-empty. This means that $\lim X$ is non-empty since there is a mapping $p : wX \longrightarrow X$, $p = (p_{x\alpha} : \alpha \in A)$.

3.27. **COROLLARY.** Let $X = (X_\alpha, f_{\alpha\beta}, A)$ be an inverse system of a Hausdorff spaces $X_\alpha$ and $p$-maps $f_{\alpha\beta}$ such that $kf_{\alpha\beta} : kX_\beta \longrightarrow kX_\alpha$ are
3.28. COROLLARY. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces $X_\alpha$ such that for each $x_\alpha \in X_\alpha$ and each $\beta \geq \alpha$ $f_{\alpha\beta}^{-1}(x_\alpha) = Y_\beta$ is non-empty $H$-closed subspace of $X_\beta$. If the restrictions $f_{\beta\gamma} / Y_\gamma$ are $H$-closed, then $\lim X$ is non-empty.

If the mappings $f_{\alpha\beta}$ are open, then the restrictions $f_{\beta\gamma} / Y_\gamma$ are open [2:95]. Thus we have

3.29. COROLLARY. If $X$ is an inverse system of a Hausdorff spaces $X_\alpha$ and open onto mappings $f_{\alpha\beta}$ such that each $f_{\alpha\beta}^{-1}(x_\alpha)$ is $H$-closed, then $\lim X$ is non-empty.

4. ALMOST REALCOMPACTIFICATION $rX$

A class of almost realcompact spaces was introduced by Frolik (see [26]).

We say that an open ultrafilter $U = \{U : \mu \in M, U \subseteq X\}$ is countably almost centred if each countable subfamily $\{U_1, \ldots, U_n, \ldots\}$ of $U$ has the property that $\cap \{Cl U : i \in N\}$ is non-empty.

4.1. DEFINITION. A Hausdorff space $X$ is almost realcompact if each countably almost centred open ultrafilter on $X$ is fixed.

Frolik has been proved the following theorems.

4.2. THEOREM. The Cartesian product of almost realcompact spaces is almost realcompact.

4.3. THEOREM. Each closed subset of a regular almost realcompact space $X$ is almost realcompact.

It is well-known that an inverse limit of a Hausdorff spaces is closed in the Cartesian product [2]. Thus we have the following theorem.

4.4. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a regular almost realcompact spaces $X_\alpha$, then $\lim X$ is a regular almost realcompact space.

4.5. THEOREM. [26]. For each completely regular space $X$ there exists an almost realcompact space $rX$ with the following properties:

a) $X \subseteq rX \subseteq \beta X$, where $\beta X$ is the Stone-Cech's compactification of $X$;

b) If $f : X \rightarrow Y$ is a mapping into any almost realcompact completely regular space, then there exist $rf : rX \rightarrow Y$ such that $f = rf/X$.

Let us note that $rf$ is the restriction of $\beta f$ onto $rX$.

4.6. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a completely regular spaces $X_\alpha$ such that the projections $f_\alpha$ are onto. If $\beta(\lim X) = \lim \beta X$ then $r(\lim X) = \lim rX$.

Proof. From the properties of the Stone-Cech's compactification and from Theorem 4.5. b) it follows that there exist inverse systems $\beta X = \{X_\alpha, \beta f_{\alpha\beta}, A\}$ and $rX = \{rX_\alpha, rf_{\alpha\beta}, A\}$. The inverse system $rX$ is the subsystem of the system $\beta X$. By virtue of the
surjectivity of the mappings $f_{\alpha \beta}$ we infer that $\lim X$ is densely embedded in $\lim X$ and $\lim X$. On can also construct a mappings $R: \lim X \to \lim X$ and $B: \beta(\lim X) \to \lim X$. Moreover, $R$ is the restriction of $B$ onto $\lim X$. It is clear that if $B$ is the homeomorphism, then $R$ is the homeomorphism. The proof is completed.

4.7. Remark. The notion of the almost realcompactification is a generalization of the Hewitt realcompactification $\nu X$ of a completely regular space $X$ [2:277]. The space $\nu X$ is the subspace of $\beta X$ such that each real-valued function $f: X \to R$ has an extension on $X$. It is evident that Theorem 4.7. holds also for the spaces $\nu(\lim X)$ and $\lim \nu X$.

4.8. Theorem. Let $X$ be an inverse system as in 4.6. The spaces $\lim X$ and $\lim x X$ are homeomorphic if the following condition (C5) is satisfied:

(C5) For every pair $F_1, F_2$ of completely separated subsets of $\lim X$ there exists a $\alpha \in A$ such that $f_{\alpha 1}^{-1}(F_1)$ and $f_{\alpha 2}^{-1}(F_2)$ are completely separated subsets of $X_{\alpha}$.

Proof. Apply theorem 4.6. and Lemma 1.1. of the paper [7].

If the spaces $\lim X$ and $X_{\alpha}$, $\alpha \in A$, are normal then each pair of a closed subsets of these spaces are completely separated. Thus the condition (C5) can be replaced by the following condition:

(S) For each pair $F_1, F_2$ of disjoint closed subsets of $\lim X$ there exists a $\alpha \in A$ such that $C_{\alpha 1}f^{-1}(F_1) \cap C_{\alpha 2}f^{-1}(F_2) = \emptyset$.

There condition (C5) is satisfied if the inverse system $X$ is a factorizable or f-system [17]. This means that for each real-valued function $f: \lim X \to R$ there exists a $\alpha \in A$ and a real-valued function $g_{\alpha}: X_{\alpha} \to R$ such that $f = g_{\alpha}f_{\alpha}$.

4.9. Theorem. If $X$ is an f-system with onto projections $f_{\alpha}: \lim X \to X_{\alpha}$, $\alpha \in A$, then $\lim x X = \lim X$.

Proof. Each f-system satisfies the condition (C5). Apply Theorem 4.8.

4.10. Theorem. Let $X$ be an $\delta$-directed inverse system with onto projections $f_{\alpha}$ such that a space $\lim X$ is a Lindelof space. Then $\lim x X = \lim X$.

Proof. From [17: Theorem 1.10] it follows that $\beta(\lim X) = \lim \beta X$. Apply Theorem 4.8.

4.11. Theorem. Let $X = \{X_n, f_{nm}, N\}$ be an inverse sequence of a normal spaces $X_n$ and onto bonding mappings $f_{nm}$. If a space $\lim X$ is countably compact, then $\lim x X = \lim X$.

Proof. By virtue of Theorem 1.3. of the paper [17] it follows that $\beta(\lim X) = \lim \beta X$. Apply Theorem 4.8. replacing the condition (C5) by the condition (S).

If the spaces $X_n$ are countably compact and if the mappings $f_{nm}$ are closed, then $\lim X$ is countably compact [13]. Thus we have
4.12. THEOREM. Let $X = \{X_n, f_n^m, N\}$ be an inverse sequence of a normal countably compact spaces $X_n$ and a closed onto mappings $f_n^m$. Then $r(\lim X) = \lim rX$.

By the same method of proof one can prove for a sequentially compact (strongly countably compact, D-compact) spaces the following:

4.13. THEOREM. Let $X$ be an inverse sequence of a normal sequentially compact (strongly countably compact, D-compact) spaces. Then $r(\lim X) = \lim rX$.

4.14. THEOREM. Let $X = \{X_\alpha, f_\alpha^\beta, A\}$ be an inverse system with perfect fully closed onto mappings $f_\alpha^\beta$. If the spaces $X_\alpha, \alpha \in A$, are normal countably compact, then $r(\lim X) = \lim rX$.

Proof. Let us recall that a mapping $f: X \rightarrow Y$ [17] is fully closed if for each point $y \in Y$ and each finite open cover $\{U_i, i = 1, \ldots, s\}$ of $f^{-1}(y)$ by open sets $U_i, 1 = 1, \ldots, s$, the set $\{y\} \setminus \bigcup f(U_i)$ is an open set in $Y$. Now from Theorem 4.8. of [17] it follows that $\beta(\lim X) = \lim \beta X$. Theorem 4.8. completes the proof.

We say that a Hausdorff space $X$ is $m$-compact, $m \in \mathbb{N}$, if each open cover $U$ of $X$ has a subcover $W$ of the cardinality $|W| < m$.

Each countably compact space $X$ is an $s_m$-compact space.

4.15. THEOREM. Let $X = \{X_\alpha, f_\alpha^\beta, A\}$ be an well-ordered inverse system of $s_m$-compact normal spaces $X_\alpha$ such that $f_\alpha^\beta$ are closed onto mappings and $\text{cf}(A) < s_m^s$. Then $r(\lim X) = \lim rX$.

Proof. Let us recall that $\text{cf}(A)$ is the smallest ordinal number which is cofinal in $A$. Now the condition ($S$) is satisfied [13]. Theorem 4.8. completes the proof.

4.16. REMARK. By the same method of proof one can see that Theorems 4.6. - 4.15. holds for the realcompactification $\nu(\lim X)$.

We close this Section by the consideration of the almost realcompactification $r(\lim X)$ of an inverse system of a Hausdorff spaces.

If $X$ is a Hausdorff space then an almost realcompactification $rX$ has been defined by Liu and Strecker [12] as follows. Let $rX$ be a subspace of the Katetov extension $kX$ containing a points of $X$ and all countably almost centred open ultrafilters on $X$. The topology on $rX$ is the subspace topology.

Liu and Strecker was proved the following lemma.

4.17. LEMMA. [12]. a) The space $rX$ is the almost realcompact Hausdorff space in which $X$ is densely embedded.

b) If $Y$ is any almost realcompactification of $X$ then there exists an extension $f: rX \rightarrow Y$ of the identity $i: X \rightarrow Y$.

4.18. THEOREM. Let $X = \{X_\alpha, f_\alpha^\beta, A\}$ be an inverse system of a Hausdorff spaces $X_\alpha$ and $p$-perfect onto $f_\alpha^\beta$ such that the mapping $r_\alpha^\beta$ are onto. If $k(\lim X) = \lim kX$ then $r(\lim X) = \lim rX$. 

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Proof. There exists the inverse system $\kappa X$ since $f_{\alpha\beta}$ are perfect. The inverse system $rX$ is the subsystem of $\kappa X$. Clearly, if the spaces $\kappa(\lim X)$ and $\lim \kappa X$ are homeomorphic, then the spaces $r(\lim X)$ and $\lim rX$ are homeomorphic. The proof is completed.

4.19. REMARK. Now on cannot be proved that the inverse limit of any almost realcompact spaces is almost realcompact since a closed subset of any nonregular almost realcompact space need not be almost realcompact.

4.20. THEOREM. Let $X$ be an inverse system as in theorem 4.18. If the spaces $X_\alpha$, $\alpha \in A$, are almost realcompact, then $\lim X$ is almost realcompact.

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Lončar I. H-zatvorena proširenja i apsolut inverznog limesa

SA D RŽ A J

U radu su istraživana H-zatvorena proširenja inverznog limesa. Pri tome je posebna pažnja posvećenja nužnim i dovoljnim uvjetima koje mora ispunjavati inverzni sistem da bi Katetovljeo proširenje k(limX) bilo ekvivalentno limesu inverznog sistema kX (Theorem 1.11.). Pomoću ovog teorema dobiveni su neki teoremi za H-zatvorenost i blisku kompaktnost inverznog limesa (Theoremi 1.23. - 1.26.). Za Fominovo proširenje 8(limX) dobiven je Teorem 2.5. Teorem 3.19. daje nužne i dovoljne uvjete da bi apsolut inverznog limesa bio ekvivalentan inverznom limesu apsoluta. Pomoću pridruženog inverznog sistema αX moguće je dobiti neke teoreme za nepraznost inverznog limesa (Teoremi 3.25. - 3.29.)