ON CERTAIN FUNCTIONAL EQUATION ARISING FROM
\((m, n)\)-JORDAN CENTRALIZERS IN PRIME RINGS

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Abstract. The purpose of this paper is to prove the following result.
Let \( m \geq 1, n \geq 1 \) be some fixed integers and let \( R \) be a prime ring with \( \text{char}(R) = 0 \) or \((m + n)^2 < \text{char}(R)\). Suppose there exists an additive mapping \( T : R \to R \) satisfying the relation
\[
2(m + n)T(x^3) = m(2m + n)x^2T(x) + 2mnxT(x)x + n(2n + m)x^2T(x)
\]
for all \( x \in R \). In this case \( T \) is a two-sided centralizer.

Throughout, \( R \) will represent an associative ring with center \( Z(R) \). Given an integer \( n \geq 2 \), a ring \( R \) is said to be \( n \)-torsion free, if for \( x \in R \), \( nx = 0 \) implies \( x = 0 \). As usual the commutator \( xy - yx \) will be denoted by \( [x, y] \). We shall use the commutator identities \( [xy, z] = [x, z]y + x[y, z] \) and \( [x, yz] = [x, y]z + y[x, z] \) for all \( x, y, z \in R \). Recall that a ring \( R \) is prime if for \( a, b \in R \), \( aRb = (0) \) implies that either \( a = 0 \) or \( b = 0 \) and is semiprime in case \( aRa = (0) \) implies \( a = 0 \). We denote by \( \text{char}(R) \) the characteristic of a prime ring \( R \). An additive mapping \( D : R \to R \), where \( R \) is an arbitrary ring, is called a derivation if \( D(xy) = D(x)y + xD(y) \) holds for all pairs \( x, y \in R \), and is called a Jordan derivation in case \( D(x^2) = D(x)x +xD(x) \) is fulfilled for all \( x \in R \). A derivation \( D \) is inner in case there exists \( a \in R \), such that \( D(x) = [a, x] \) holds for all \( x \in R \). Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein ([10]) asserts that any Jordan derivation on a prime ring with \( \text{char}(R) \neq 2 \) is a derivation. A brief proof of Herstein’s result can be found in [3].

2010 Mathematics Subject Classification. 16W10, 46K15, 39B05.

Key words and phrases. Ring, prime ring, semiprime ring, Banach space, Hilbert space, algebra of all bounded linear operators, standard operator algebra, derivation, Jordan derivation, left (right) centralizer, two-sided centralizer, left (right) Jordan centralizer, \((m, n)\)-Jordan centralizer.

This research has been supported by the Research Council of Slovenia.
((8)) generalized Herstein’s result to 2−torsion free semiprime rings (see also [4] for an alternative proof).

We denote by \( Q_r, Q_l, Q_s, C \) and \( RC \) the right, left, symmetric Martindale ring of quotients, extended centroid and central closure of a semiprime ring \( R \), respectively. For the explanation of \( Q_r, Q_l, Q_s, C \) and \( RC \) we refer the reader to [1]. An additive mapping \( T : R \to R \) is called a left centralizer in \([8]\) case \( T(xy) = T(x)y \) holds for all pairs \( x, y \in R \). In case \( R \) has the identity element \( T : R \to R \) is a left centralizer if \( T \) is of the form \( T(x) = ax \) for all \( x \in R \), where \( a \in R \) is some fixed element. For a semiprime ring \( R \) all left centralizers are of the form \( T(x) = qx \) for all \( x \in R \), where \( q \in Q_r \) is some fixed element (see Chapter 2 in [1]). An additive mapping \( T : R \to R \) is called a left Jordan centralizer if \( T(x) = T(x)x \) holds for all \( x \in R \). The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call \( T : R \to R \) a two-sided centralizer in \( R \) is both a left and a right centralizer. In case \( T : R \to R \) is a two-sided centralizer, where \( R \) is a semiprime ring with extended centroid \( C \), then there exists an element \( \lambda \in C \) such that \( T(x) = \lambda x \) for all \( x \in R \) (see Theorem 2.3.2 in [1]). Zalar ([20]) has proved that any left (right) Jordan centralizer on a semiprime ring is a left (right) centralizer. For results concerning centralizers in rings and algebras we refer to [11, 15, 17–20] where further references can be found. A mapping \( F \), which maps a ring \( R \) into itself, is called centralizing on \( R \) in case \( [F(x), x] \in Z(R) \) holds for all \( x \in R \). A classical result of Posner ([13]) (Posner’s second theorem) states that the existence of a nonzero centralizing derivation on a prime ring \( R \) with \( \text{char}(R) \neq 2 \) forces the ring to be commutative. Let \( X \) be a real or complex Banach space and let \( L(X) \) and \( F(X) \) denote the algebra of all bounded linear operators on \( X \) and the ideal of all finite rank operators in \( L(X) \), respectively. An algebra \( A(X) \subset L(X) \) is said to be standard in case \( F(X) \subset A(X) \). Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. Let \( m \geq 0, n \geq 0 \) be fixed integers with \( m + n \neq 0 \) and let \( R \) be an arbitrary ring. An additive mapping \( T : R \to R \) is called an \((m, n)\)-Jordan centralizer in case

\[
(m + n)T(x^2) = mT(x)x + nxT(x)
\]

holds for all \( x \in R \). The concept of \((m, n)\)-Jordan centralizer was introduced by Vukman ([19]). Obviously, \((1,0)\)-Jordan centralizer is a left Jordan centralizer, \((0,1)\)-Jordan centralizer is a right Jordan centralizer, and in case \((1,1)\)-Jordan centralizer we have the relation

\[
2T(x^2) = T(x)x + xT(x), x \in R.
\]

Vukman ([15]) has proved that in case there exists an additive mapping \( T : R \to R \), where \( R \) is a \( 2 \)-torsion free semiprime ring, satisfying the relation (2), then \( T \) is a two-sided centralizer. Vukman ([19]) conjectured
that any \((m, n)\)-Jordan centralizer on a semiprime ring with suitable torsion restrictions, where \(m \geq 1, n \geq 1\) are some fixed integers, is a two-sided centralizer. Vukman ([19]) has proved the following result which proves a special case of the conjecture we have just mentioned above.

**Theorem 1.** Let \(m \geq 1, n \geq 1\) be some fixed integers and let \(R\) be a prime ring with \(\text{char}(R) \neq 6mn(m + n)\). Suppose \(T : R \to R\) is an \((m, n)\)-Jordan centralizer. If \(Z(R)\) is nonzero, then \(T\) is a two-sided centralizer.

One can easily prove that any \((m, n)\)-Jordan centralizer \(T : R \to R\), where \(R\) is an arbitrary ring, satisfies the relation

\[
2(m + n)^2 T(x^3) = m(2m + n)T(x)x^2 + 2mnxT(x)x + n(2n + m)x^2T(x)
\]

for all \(x \in R\) (see [19] for the details). Recently, Vukman ([19]) considered the above relation in standard operator algebras on a real or complex Hilbert space. It is our aim in this paper to prove the following result.

**Theorem 2.** Let \(m \geq 1, n \geq 1\) be some fixed integers and let \(R\) be a prime ring with \(\text{char}(R) = 0\) or \((m + n)^2 < \text{char}(R)\) and let \(T : R \to R\) be an additive mapping satisfying the relation

\[
(3) \quad 2(m + n)^2 T(x^3) = m(2m + n)T(x)x^2 + 2mnxT(x)x + n(2n + m)x^2T(x)
\]

for all \(x \in R\). In this case \(T\) is a two-sided centralizer.

For the proof of Theorem 2 we need Theorem 3 below, which might be of independent interest. As the main tool in this paper we use the theory of functional identities (Brešar-Beidar-Chebotar theory). We refer the reader to [6] for introductory account of functional identities and to [7] for full treatment of this theory. Let \(R\) be an algebra over a commutative ring \(\phi\). Further let

\[
p(x_1, x_2, x_3) = \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}
\]

be a fixed multilinear polynomial in noncommuting indeterminates \(x_i\) over \(\phi\). Here \(S_3\) stands for the symmetric group of order 3 and \(e \in S_3\) for its identity element. Further, let \(L\) be a subset of \(R\) closed under \(p\), i.e., \(p(\bar{x}_3) \in L\) for all \(x_1, x_2, x_3 \in L\), where \(\bar{x}_3 = (x_1, x_2, x_3)\). We shall consider a mapping \(T : L \to R\) satisfying

\[
(4) \quad 2(m + n)^2 T(p(\bar{x}_3)) = m(2m + n) \sum_{\pi \in S_3} T(x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}
\]

\[
+ 2mn \sum_{\pi \in S_3} x_{\pi(1)}T(x_{\pi(2)}x_{\pi(3)} + n(2n + m) \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}T(x_{\pi(3)})
\]

for all \(x_1, x_2, x_3 \in L\). In the first step of the proof of the following theorem we derive a functional identity from (5). Let us mention that the idea of considering the expression \([p(\bar{x}_3), p(\bar{y}_3)]\) in its proof is taken from [2].
Theorem 3. Let \( R \) be an algebra over \( \phi \). Suppose that \( \mathcal{L} \) is a \( 6 \)-free Lie subalgebra of \( R \) closed under \( p \). If \( T : \mathcal{L} \to R \) is an additive map satisfying (3) for all \( x \in \mathcal{L} \), then there exists \( p \in C(\mathcal{L}) \) and \( \lambda : \mathcal{L} \to C(\mathcal{L}) \) such that \( 2(m(2m + n)(m + n)^2 T(x) = px + \lambda(x) \) for all \( x \in \mathcal{L} \), where \( C(\mathcal{L}) \) is extended centroid of \( \mathcal{L} \).

Proof. A complete linearization of (3) gives us (5). Note that for any \( a \in \mathcal{L} \) and \( \bar{x}_3 \in \mathcal{L}^3 \) we have
\[
[p(\bar{x}_3), a] = \sum_{\pi \in S_3} [x_{\pi(1)} x_{\pi(2)} x_{\pi(3)}, a]
\]
\[
= \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} a - a \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)}
\]
\[
= \sum_{\pi \in S_3} x_{\pi(1)} a x_{\pi(2)} x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} a x_{\pi(3)} - \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} a x_{\pi(3)}
\]
\[
= \sum_{\pi \in S_3} x_{\pi(1)} a x_{\pi(2)} x_{\pi(3)} - a \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} a x_{\pi(3)}
\]
\[
= \sum_{\pi \in S_3} [x_{\pi(1)}, a] x_{\pi(2)} x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)} [x_{\pi(2)}, a] x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)} x_{\pi(2)} [x_{\pi(3)}, a]
\]
\[
= p([x_1, a], x_2, x_3) + p(x_1, [x_2, a], x_3) + p(x_1, x_2, [x_3, a]).
\]
Using this in (5) we obtain
\[
2(m + n)^2 T([p(\bar{x}_3), a]) = 2(m + n)^2 T(p([x_1, a], x_2, x_3))
\]
\[
+ 2(m + n)^2 T(p(x_1, [x_2, a], x_3)) + 2(m + n)^2 T(p(x_1, x_2, [x_3, a])).
\]
It follows that
\[
2(m + n)^2 T([p(\bar{x}_3), a]) = \sum_{\pi \in S_3} m(2m + n) T([x_{\pi(1)}, a]) x_{\pi(2)} x_{\pi(3)}
\]
\[
+ \sum_{\pi \in S_3} m(2m + n) T(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, a] + \sum_{\pi \in S_3} 2mn [x_{\pi(1)}, a] T(x_{\pi(2)}) x_{\pi(3)}
\]
\[
+ \sum_{\pi \in S_3} 2mn x_{\pi(1)} T([x_{\pi(2)}, a]) x_{\pi(3)} + \sum_{\pi \in S_3} 2mn x_{\pi(1)} T(x_{\pi(2)}) [x_{\pi(3)}, a]
\]
\[
+ \sum_{\pi \in S_3} n(2n + m) [x_{\pi(1)} x_{\pi(2)}, a] T(x_{\pi(3)})
\]
\[
+ \sum_{\pi \in S_3} n(2n + m) x_{\pi(1)} x_{\pi(2)} T([x_{\pi(3)}, a]).
\]
Further, let \( s : \mathbb{Z} \to \mathbb{Z} \) be a mapping defined by \( s(i) = i - 3 \). For each \( \sigma \in S_3 \) the mapping \( s^{-1}\sigma : \{4, 5, 6\} \to \{4, 5, 6\} \) will be denoted by \( \overline{\sigma} \). Then we have in particular, where \( a = p(x_4, x_5, x_6) \)

\[
2(m + n)^2 T([p(x_1, x_2, x_3), p(x_4, x_5, x_6)]) = \sum_{\pi \in S_3} m(2m + n) T([x_{\pi(1)}, p(x_4, x_5, x_6)]) x_{\pi(2)} x_{\pi(3)}
\]

\[
+ \sum_{\pi \in S_3} m(2m + n) T(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, p(x_4, x_5, x_6)]
\]

\[
+ \sum_{\pi \in S_3} 2mn [x_{\pi(1)}, p(x_4, x_5, x_6)] T(x_{\pi(2)}) x_{\pi(3)}
\]

\[
(6) + \sum_{\pi \in S_3} 2mn x_{\pi(1)} T([x_{\pi(2)}, p(x_4, x_5, x_6)]) x_{\pi(3)}
\]

\[
+ \sum_{\pi \in S_3} 2mn x_{\pi(1)} T(x_{\pi(2)}) [x_{\pi(3)}, p(x_4, x_5, x_6)]
\]

\[
+ \sum_{\pi \in S_3} n(2n + m) [x_{\pi(1)} x_{\pi(2)}, p(x_4, x_5, x_6)] T(x_{\pi(3)})
\]

\[
+ \sum_{\pi \in S_3} n(2n + m) x_{\pi(1)} x_{\pi(2)} T([x_{\pi(3)}, p(x_4, x_5, x_6)])
\]

and

\[
2(m + n)^2 T([x_{\pi(1)}, p(x_4, x_5, x_6)]) = -2(m + n)^2 T([p(x_4, x_5, x_6), x_{\pi(1)}])
\]

\[
= \sum_{\sigma \in S_3} m(2m + n) T([x_{\sigma(1)}, x_{\sigma(1)}], x_{\sigma(2)}, x_{\sigma(3)})
\]

\[
+ \sum_{\sigma \in S_3} m(2m + n) T(x_{\sigma(1)}) [x_{\sigma(2)}, x_{\sigma(3)}]
\]

\[
+ \sum_{\sigma \in S_3} 2mn [x_{\sigma(1)}, x_{\sigma(1)}] T(x_{\sigma(2)}) x_{\sigma(3)}
\]

\[
+ \sum_{\sigma \in S_3} 2mn x_{\sigma(1)} T(x_{\sigma(2)}) [x_{\sigma(3)}]
\]

\[
+ \sum_{\sigma \in S_3} n(2n + m) [x_{\sigma(1)}, x_{\sigma(1)} x_{\sigma(2)}] T(x_{\sigma(3)})
\]

\[
+ \sum_{\sigma \in S_3} n(2n + m) x_{\sigma(1)} x_{\sigma(2)} T([x_{\sigma(1)}, x_{\sigma(3)}]),
\]

for all \( x_1, \ldots, x_6 \in \mathcal{L} \). We shall write

\[
\varphi(x_{\pi(1)}) = 2(m + n)^2 T([x_{\pi(1)}, p(x_4, x_5, x_6)]).
\]
Similarly we define

\[ \phi(x_{\pi(2)}) = 2(m + n)^2 T([x_{\pi(2)}, p(x_4, x_5, x_6)]) , \]

and

\[ \phi(x_{\pi(3)}) = 2(m + n)^2 T([x_{\pi(3)}, p(x_4, x_5, x_6)]) . \]

Using this together with (6) we obtain

\[
(2(m + n)^2)^2 T([p(x_1, x_2, x_3), p(x_4, x_5, x_6)]
= \sum_{\pi \in S_3} m(2m + n) \phi(x_{\pi(1)}) x_{\pi(2)} x_{\pi(3)}
+ \sum_{\pi \in S_3} 2m(2m + n)(m + n)^2 T(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, p(x_4, x_5, x_6)]
+ \sum_{\pi \in S_3} 4mn(m + n)^2 [x_{\pi(1)}, p(x_4, x_5, x_6)] T(x_{\pi(2)}) x_{\pi(3)}
\]

(7)

\[
+ \sum_{\pi \in S_3} 2mn x_{\pi(1)} \phi(x_{\pi(2)}) x_{\pi(3)}
+ \sum_{\pi \in S_3} 4mn(m + n)^2 x_{\pi(1)} T(x_{\pi(2)}) [x_{\pi(3)}, p(x_4, x_5, x_6)]
+ \sum_{\pi \in S_3} 2n(2n + m)(m + n)^2 [x_{\pi(1)} x_{\pi(2)}, p(x_4, x_5, x_6)] T(x_{\pi(3)})
+ \sum_{\pi \in S_3} n(2n + m) x_{\pi(1)} x_{\pi(2)} \phi(x_{\pi(3)}).
\]

Since

\[
[x_{\pi(1)} x_{\pi(2)}, p(x_4, x_5, x_6)] = \sum_{\sigma \in S_3} [x_{\pi(1)} x_{\pi(2)}, x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}] ,
\]

then (7) reduces to

\[
(2(m + n)^2)^2 T([p(x_1, x_2, x_3), p(x_4, x_5, x_6)]
= \sum_{\pi \in S_3} \sum_{\sigma \in S_3} m(2m + n) \phi(x_{\pi(1)}) x_{\pi(2)} x_{\pi(3)}
+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 2m(2m + n)(m + n)^2 T(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)}, x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}]
+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 4mn(m + n)^2 [x_{\pi(1)}, x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}] T(x_{\pi(2)}) x_{\pi(3)}
\]

(8)

\[
+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 2mn x_{\pi(1)} \phi(x_{\pi(2)}) x_{\pi(3)}
+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 4mn(m + n)^2 x_{\pi(1)} T(x_{\pi(2)}) [x_{\pi(3)}, x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}] .
\]
If we replace the roles of denotations $\pi$ and $\sigma$, then from (8) we get

$$(2(m + n))^2 T([p(x_1, x_2, x_3), p(x_4, x_5, x_6)])$$

$$= \sum_{\pi \in S_3} \sum_{\sigma \in S_3} m(2m + n) \hat{\varphi}(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}$$

$$+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 2m(2m + n)(m + n)^2 T(x_{\pi(1)}) \left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, x_{\pi(2)}x_{\pi(3)}\right]$$

$$+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 4mn(m + n)^2 \left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, x_{\pi(1)}\right] T(x_{\pi(2)})x_{\pi(3)}$$

$$(9) + \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 2mn x_{\pi(1)} \hat{\varphi}(x_{\pi(2)})x_{\pi(3)}$$

$$+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 4mn(m + n)^2 x_{\pi(1)} T(x_{\pi(2)}) \left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, x_{\pi(3)}\right]$$

$$+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} 2n(2m + n)(m + n)^2 \left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, x_{\pi(1)}x_{\pi(2)}\right] T(x_{\pi(3)})$$

$$+ \sum_{\pi \in S_3} \sum_{\sigma \in S_3} n(2m + n)x_{\pi(1)}x_{\pi(2)} \hat{\varphi}(x_{\pi(3)}).$$

for all $x_1, ..., x_6 \in \mathcal{L}$, where

$$\hat{\varphi}(x_{\pi(i)}) = 2(m + n)^2 T([p(x_1, x_2, x_3), x_{\pi(i)}])$$

for $i = 1, 2, 3$. We obtain that

$$\hat{\varphi}(x_{\pi(i)}) = -\varphi(x_{\pi(i)})$$

for $i = 1, 2, 3$. Comparing (8) and (9) we obtain the following identity

$$0 = \sum_{\pi \in S_3} \sum_{\sigma \in S_3} \left( m(2m + n)\varphi(x_{\pi(1)})x_{\pi(2)} + 2mnx_{\pi(1)}\varphi(x_{\pi(2)}) \right)$$

$$+ 4mn(m + n)^2 x_{\pi(1)}T(x_{\pi(2)})x_{\pi(3)}x_{\pi(1)}x_{\pi(2)}$$

$$- 4mn(m + n)^2 x_{\pi(1)}T(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}$$

$$+ 2m(2m + n)(m + n)^2 T(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}x_{\pi(1)}x_{\pi(2)}$$

$$- 2m(2m + n)(m + n)^2 T(x_{\pi(1)})x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(2)}x_{\pi(3)}.$$
Define maps $E, F : \mathcal{L}^5 \rightarrow R$ by the rule

$$E(u_1, u_2, u_3, u_4, u_5) = m(2m + n)\varphi(u_1)u_2 + 2mn u_1 \varphi(u_2)$$

$$+ 4mm(m + n)^2 (u_2 T(u_4) u_5 u_1 u_2 - u_1 T(u_2) u_3 u_4 u_5)$$

$$+ 2m(2m + n)(m + n)^2 (T(u_3) u_4 u_5 u_1 u_2 - T(u_1) u_3 u_4 u_5 u_2)$$

and

$$F(u_1, u_2, u_3, u_4, u_5) = n(2n + m) u_1 \varphi(u_2)$$

$$- 4m(m + n)^2 (u_1 u_2 u_3 T(u_4) u_5 - u_3 u_4 u_5 T(u_1) u_2)$$

$$- 2m(2m + n)(m + n)^2 (u_1 u_2 u_3 u_4 T(u_5) - u_1 u_3 u_4 u_5 T(u_2))$$

for all $u_5 \in \mathcal{L}^5$. 

for all $x_1, \ldots, x_6 \in \mathcal{L}$. 

Define maps $E, F : \mathcal{L}^5 \rightarrow R$ by the rule
Accordingly, (10) can be rewritten as

$$0 = \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} \sum_{i \in S_{3}} E(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)} \pi(3)) x_{\pi(3)}$$

$$+ \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} \sum_{i \in S_{3}} E(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)} \pi(3)) x_{\sigma(3)}$$

$$+ \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} \sum_{i \in S_{3}} F(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)} \pi(3))$$

$$+ \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} \sum_{i \in S_{3}} F(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)} \pi(3))$$

and hence

$$0 = \sum_{i=1}^{3} \left( \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} E(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)} \pi(3)) \right) x_{i}$$

$$+ \sum_{i=4}^{6} \left( \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} E(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)} \pi(3)) \right) x_{i}$$

$$+ \sum_{j=1}^{3} x_{j} \left( \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} F(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)} \pi(3)) \right)$$

$$+ \sum_{j=4}^{6} x_{j} \left( \sum_{\pi \in S_{3}} \sum_{\sigma \in S_{3}} F(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)} \pi(3)) \right)$$

for all $x_{1}, ..., x_{6} \in \mathcal{L}$. Then we have that

$$\sum_{i=1}^{6} E_{i}(\bar{x}_{6}) x_{i} + \sum_{j=1}^{6} F_{j}(\bar{x}_{6}) x_{j} = 0$$

for all $\bar{x}_{6} \in \mathcal{L}^{6}$, where $E_{i}, F_{j} : \mathcal{L}^{5} \rightarrow R$ and $E^{j}, F^{j} : \mathcal{L}^{6} \rightarrow R$ are mappings

$$E^{j}(\bar{x}_{6}) = E(x_{1}, ..., x_{i-1}, x_{i}, ..., x_{6})$$

and

$$F^{j}(\bar{x}_{6}) = F(x_{1}, ..., x_{i-1}, x_{i}, ..., x_{6})$$

Now we simply apply the definition of 6-freeness $\mathcal{L}$. There exists maps $p_{6,j} : \mathcal{L}^{4} \rightarrow \mathcal{L}, j = 1, ..., 5$ and $\lambda_{6} : \mathcal{L}^{5} \rightarrow \mathcal{C}(\mathcal{L})$ such that

$$\sum_{i \in S_{3}} \sum_{\sigma(3) = \pi(3)} E(x_{\pi(1)}, x_{\pi(2)}, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \sum_{j=1}^{5} x_{j} p_{6,j}(\bar{x}_{5}) + \lambda_{6}(\bar{x}_{5})$$
for all $x_5 \in L^5$. Recalling the definition of map $E$ and after some steps we arrive at
\begin{equation}
2m(2m+n)(m+n)^2T(x) = xp + \lambda(x),
\end{equation}
for all $x \in L$, where $p \in L$ and $\lambda : L \to C(L)$. The symmetric analogue in which maps $F$ are involved, is clearly proved in the same way. Therefore
\begin{equation}
2n(2n+m)(m+n)^2T(x) = \bar{p}x + \bar{\lambda}(x)
\end{equation}
for all $x \in L$ and some $\bar{p} \in L$ and $\bar{\lambda} : L \to C(L)$. The symmetric analogue in which maps $F$ are involved, is clearly proved in the same way. Therefore
\begin{equation}
2mn(2m+n)(m+n)^2T(x) = p(x) + m(2m+n)\bar{p}x + \bar{\lambda}(x)
\end{equation}
for all $x \in L$. Comparing this two identities we arrive at
\begin{equation}
\bar{p}x = \bar{\lambda}(x)
\end{equation}
for all $x \in L$. It follows that $n(2n+m)p = m(2m+n)\bar{p} \in C(L)$, which yields $p, \bar{p} \in C(L)$. Therefore the proof is completed.

We are now in a position to prove Theorem 2.

Proof of Theorem 2. The complete linearization of (3) gives us (5). Assume first that $R$ is not a PI ring. According to Theorem 3 there exist $p \in C$ and $\lambda : R \to C$ such that
\begin{equation}
2m(2m+n)(m+n)^2T(x) = xp + \lambda(x).
\end{equation}
Then we have
\begin{equation}
x^2((m+n)^2xp + 2(m+n)^2\lambda(x)) = (m+n)^2\lambda(x^3),
\end{equation}
which yields
\begin{equation}
x^2(xp + 2\lambda(x)) = \lambda(x^3),
\end{equation}
for all $x \in R$. A complete linearization of this identity leads to
\begin{equation}
\sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}(x_{\pi(3)}p + 2\lambda(x_{\pi(3)})) = \lambda(p(x_3))
\end{equation}
for all $x_1, x_2, x_3 \in R$. Since $R$ is not a PI ring it follows that
\begin{equation}
xp + 2\lambda(x) = 0
\end{equation}
for all $x \in R$. Now our aim is to show that $\lambda = 0$. Thus $[xp, y] = 0$ for all $x, y \in R$. Then we have $[x, y]zp = 0$ for all $x, y, z \in R$. It follows that $R$ is commutative or $p = 0$. If $p = 0$, then $\lambda(x) = 0$ for all $x \in R$ by (12). If $[x, y] = 0$, then from (12) follows that $\lambda(x)y - \lambda(y)x = 0$ for all $x, y \in R$. Consequently $\lambda = 0$. 

We are now in a position to prove Theorem 2.
Now suppose that $R$ is a PI ring. It is well-known that in this case $R$
has a nonzero center (see [14]). Let $c$ be a nonzero central element. Pick any
\(x \in R\) and set \(x_1 = x_2 = cx\) and \(x_3 = x\) in (5) we get
\[
\begin{align*}
12(m + n)^2T(c^2x^3) &= 2m(2m + n)(2T(c)x^2 + T(x)x^3)c \\
&\quad + 4mn(2xT(cx)x + xT(x)xc)c + 2n(2n + m)(2x^2T(cx) + x^2T(x)c)c.
\end{align*}
\]
(13)

Next, setting \(x_1 = x_2 = c\) and \(x_3 = x^3\) in (5), we have
\[
\begin{align*}
12(m + n)^2T(c^2x^3) &= 2m(2m + n)(2T(c)x^3 + T(x^3)c)c \\
&\quad + 4mn(x^3T(c) + T(c)x^3 + T(x^3)c)c + 2n(2n + m)(2x^3T(c) + T(x^3)c)c.
\end{align*}
\]
(14)

Comparing both identities we obtain
\[
\begin{align*}
m(2m + n)T(cx)x^2 + 2mnxT(cx)x + n(2n + m)x^2T(cx) \\
&\quad = 2m(m + n)T(c)x^3 + 2n(n + m)x^3T(c)
\end{align*}
\]
(15)

for all \(x \in R\). If \(x = c\) we have
\[
T(c^2) = T(c)c.
\]
(16)

Setting \(x_1 = x\) and \(x_2 = x_3 = c\) in the complete linearization of (15) we get
\[
\begin{align*}
(m + n)T(cx) &= mT(c)x + nxT(c)
\end{align*}
\]
(17)

for all \(x \in R\). Multiplying (17) by \(c^2\) we get
\[
\begin{align*}
(m + n)T(cx)c^2 &= mT(c^2)xc + nxT(c^2)c
\end{align*}
\]
and substituting \(x\) by \(cx\) in (17) we get
\[
\begin{align*}
(m + n)T(c^2x)c &= mT(c^2)x + nxT(c^2)c.
\end{align*}
\]
Comparing the last two identities we see that
\[
\begin{align*}
T(c^2x) &= T(cx)c.
\end{align*}
\]
(18)

Setting \(x_1 = x\) and \(x_2 = x_3 = c\) in (5) we have
\[
\begin{align*}
12(m + n)^2T(c^2x) &= 2m(2m + n)(2T(c)x^2 + 4T(c)xc) \\
&\quad + 2mn(2T(x)c^2 + 2T(c)xc + 2xT(c)c) \\
&\quad + 2n(2n + m)(2c^2T(x) + 4xT(c)xc)
\end{align*}
\]
(19)

and so
\[
\begin{align*}
T(cx) &= T(x)c = cT(x).
\end{align*}
\]
(20)

Setting \(x_1 = x_2 = x\) and \(x_3 = c\) in the complete linearization of (15) and
using (20) we get
\[
T(c)x^2 + x^2T(c) = 2xT(c)x.
\]
This can be rewritten as
\[
[[T(c), x], x] = 0
\]
for all \( x \in R \). From Posner’s second theorem it follows that \([T(c), x] = 0\) for all \( x \in R \). From (17) consequently we get
\[
T(cx) = T(c)x = xT(c).
\]
(21)

Next we replace \( x \) for \( xy \) in (17) we obtain
\[
(m + n)T(xy)c = (mT(c)x)y + x(nyT(c))
\]
\[
= (m + n)T(x)yc + (m + n)xT(y)c - (m + n)xT(c)y.
\]
(22)

Multiplying this identity on the left by \( z \) we get
\[
(m + n)zT(xy)c = (m + n)zT(x)yc
\]
\[
+ (m + n)zxT(y)c - (m + n)zxT(c)y.
\]
(23)

Then substituting \( x \) for \( zx \) in (22) we have
\[
(m + n)T(zxy)c = (m + n)T(zx)yc
\]
\[
+ (m + n)zxT(y)c - (m + n)zxT(c)y.
\]
(24)

From (23) and (24) we obtain
\[
T(zxy) = zT(xy) + T(zx)y - zT(xy).
\]
(25)

We use the last identity and (21) to get
\[
T(zxy) = zT(xy) + T(zc)y - zT(xy).
\]

Then by (21) we have
\[
T(zyx) = zT(xy) + zT(c)y - zT(c)y,
\]
and so
\[
T(zyx)c = zT(y)c
\]
which yields \( T(zy) = zT(y) \). Similarly we get \( T(zy) = T(z)y \) for all \( y, z \in R \) and \( T \) is two-sided centralizer. Therefore the proof is completed.

The relation (25) leads to the following relation
\[
F(xyx) = F(xy)x - xF(y)x + xF(yx)
\]
for all \( x, y \in R \), where \( F \) is an additive mapping which maps a ring \( R \) into itself. The question arises about the solution of the above equation. Let us consider some relations which are similar to the above relation. An additive mapping \( D : R \to R \), where \( R \) is an arbitrary ring, is called a Jordan triple derivation in case
\[
D(xyx) = D(xy)x + xD(y)x + xyD(x)
\]
holds for all pairs \( x, y \in R \). One can easily prove that any Jordan derivation on a 2−torsion free ring is a Jordan triple derivation (see [3] for the details). Brešar ([5]) has proved the following result.

**Theorem 4.** Let \( R \) be a 2−torsion free semiprime ring and let \( D : R \to R \) be a Jordan triple derivation. In this case \( D \) is a derivation.
Since, as we have mentioned above, any Jordan derivation on a 2−torsion
free ring is a Jordan triple derivation, Theorem 4 generalizes Cusack’s
generalization of Herstein’s theorem. Brešar’s result above has been recently
generalized by Liu and Shiue ([12]). Motivated by Theorem 4 Vukman,
Eremita and Kosi-Ubl (16]) have proved the following result (see also [9]).

**Theorem 5.** Let $R$ be a 2−torsion free semiprime ring and let $T : R \to R$
be an additive mapping satisfying the relation
\begin{equation}
T(xy) = T(x)yx - xT(y)x + xyT(x)
\end{equation}
for all pairs $x, y \in R$. In this case $T$ is of the form $2T(x) = qx + xq$ for all
$x \in R$, where $q \in Q_S$ is some fixed element.

We proceed with the following result.

**Proposition 6.** Let $R$ be a 2−torsion free semiprime ring with the
identity element $e$ and let $F : R \to R$ be an additive mapping satisfying
the relation
\begin{equation}
F(xy) = F(xy)x - xF(y)x + xF(yx)
\end{equation}
for all pairs $x, y \in R$. In this case $F$ is of the form
\begin{equation}
2F(x) = D(x) + F(e)x + xF(e),
\end{equation}
where $D : R \to R$ is a derivation.

**Proof.** Putting in the relation (27) $y = e$ we obtain
\begin{equation}
F(x^2) = F(x)x - xF(e)x + xF(x)
\end{equation}
for all $x \in R$. Let us denote $2F(x) - F(e)x - xF(e)$ by $D(x)$. Then applying
the relation (28) a simple calculation shows that
\begin{equation}
D(x^2) = D(x)x + xD(x)
\end{equation}
holds for all $x \in R$. We have an additive mapping $D : R \to R$ satisfying the
relation (29) for all $x \in R$. In other words, $D$ is a Jordan derivation on $R$.
Applying Cusack’s generalization of Herstein’s theorem one concludes that $D$
is a derivation, which completes the proof.

Proposition 6 together with Theorem 5 leads to the following conjecture.

**Conjecture 7.** Let $R$ be a 2−torsion free semiprime ring and let $F : R \to R$ be an additive
mapping satisfying the relation
\begin{equation}
F(xy) = F(xy)x - xF(y)x + xF(yx)
\end{equation}
for all pairs $x, y \in R$. In this case $F$ is of the form $2F(x) = D(x) + qx + xq$
for all $x \in R$, where $D : R \to R$ is a derivation and $q \in Q_S$ is some fixed
element.

**Acknowledgements.**
The authors are thankful to the referee for his valuable comments.
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