A NOTE ON CHARACTER SQUARE

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Abstract. We study the finite groups with an irreducible character $\chi$ satisfying the following hypothesis: $\chi^2$ has exactly two distinct irreducible constituents, and one of which is linear, and then obtain a result analogous to the Zhmud’s ([8]).

1. Introduction

Let $\chi$ be a character of a finite group $G$. Clearly $\chi^2$ is a character of $G$, and if $\chi^2$ has few irreducible constituents then the group structure is necessarily restricted. For example, Isaacs and Zisser ([5]) have studied the groups $G$ with a faithful irreducible character $\chi$ such that $\chi^2 = a\psi$ or $\chi^2 = a\psi + b\overline{\psi}$, where $a, b$ are positive integers, $\psi \in \text{Irr}(G)$ and $\overline{\psi}$ is the complex conjugate of $\psi$; while Zhmud ([8]) has considered the finite groups with a faithful irreducible character $\chi$ satisfying the following hypothesis.

Hypothesis 1. $\chi$ is real, and $\chi^2$ has just two distinct irreducible constituents.

Theorem A ([8]). Let $G$ be a finite group with a faithful irreducible character $\chi$ satisfying Hypothesis 1. Then $G \in \{ \text{SL}(2,3), \hat{S}_4, \text{SL}(2,5) \}$, where $\hat{S}_4$ is one of the two representation groups of the symmetric group $S_4$ with generalized quaternion Sylow 2-subgroup.

Observe that $\chi \in \text{Irr}(G)$ is real if and only if $[\chi^2, 1_G] = 1$. Hence the above hypothesis is equivalent to the following one, there exist a positive integer $b$ and an irreducible character $\psi$ of $G$ such that $\chi^2 = 1_G + b\psi$.

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In this note, we will study the finite groups $G$ with a faithful $\chi \in \text{Irr}(G)$ satisfying the following hypothesis.

**Hypothesis 2.** $\chi^2$ has just two distinct irreducible constituents, and one of which is linear.

**Theorem B.** A finite group $G$ has a faithful irreducible character $\chi$ satisfying Hypothesis 2 if and only if $G' \cap Z(G) \cong \mathbb{Z}_2$ and $G$ has a faithful and primitive irreducible character of degree 2.

Note that if a finite group $G$ has a faithful and primitive irreducible character of degree 2, then $Z(G)$ is cyclic and (see [1, Theorem 18.1]) $G/Z(G)$ is isomorphic to $A_4$, $S_4$ or $PSL(2, 5)$, also $G$ can be viewed as a finite subgroup of the general linear group $GL(2, \mathbb{C})$. A classification of such groups can be found in [2, Section 3] or [3] for example. It is clear that there are many groups satisfying the conditions in Theorem B.

**Example 1.1.** Let $H \cong GL(2, 3)$ be one of the two representation groups of $S_4$ with semidihedral Sylow 2-subgroup. Let us present an irreducible character $\chi$ of $H$ satisfying Hypothesis 1.2. It follows from the character table of $H$ that it has three irreducible characters $\chi, \lambda$ and $\psi$, taking on the corresponding $G$-classes the following values:

\[
\begin{align*}
\chi &: 2, 0, -1, 0, -2, i\sqrt{2}, 1, -i\sqrt{2}, \\
\lambda &: 1, -1, 1, 1, -1, 1, -1, \\
\psi &: 3, 1, 0, -1, 3, -1, 0, -1.
\end{align*}
\]

Then $\chi^2 : 4, 0, 1, 0, 4, -2, 1, -2$, and so $\chi^2 = \lambda + \psi$.

Throughout this note, $G$ always denotes a finite group, all characters are complex characters. For a character $\chi$ of a finite group $G$, we denote by $\text{Irr}(\chi)$ the set of irreducible constituents of $\chi$. In general, we use Isaacs [4] as a source for standard notations and results from character theory.

2. **Proofs**

For any irreducible character $\chi$ of $G$, we define $\chi^{(2)}$ by $\chi^{(2)}(g) = \chi(g^2)$ for any $g \in G$. It is known that $\chi^{(2)}$ is a generalized character of $G$ and that $\text{Irr}(\chi^{(2)}) \subseteq \text{Irr}(\chi^2)$ (see [1, §4.6]).

**Lemma 2.1** ([7, Lemma 1.1]). Let $\chi \in \text{Irr}(G)$ and $g$ be a $p$-element of $G$ for some prime $p$. If $|\chi(g)|^2 = a$ is a rational integer, then $p$ divides $\chi(1)^2 - a$. In particular, if $\chi(g) = 0$ then $p$ divides $\chi(1)$; and if $|\chi(g)| = 1$ then $p$ does not divide $\chi(1)$.

**Lemma 2.2.** Let $G$ be a finite group with an irreducible character $\chi$ (not necessarily faithful) satisfying Hypothesis 2. Then the following statements hold.
(1) \( \chi^2 = \lambda + \psi \), where \( \lambda, \psi \in \text{Irr}(G) \), and \( \lambda(1) = 1 \).

(2) The restriction of \( \chi \) on \( G' \) is irreducible.

**Proof.** (1) By the hypothesis, there exist positive integers \( a, b \) and \( \lambda, \psi \in \text{Irr}(G) \) such that
\[
\chi^2 = a\lambda + b\psi, \quad \lambda(1) = 1.
\]
Since \( \lambda \) is linear and \( 0 < [\chi^2, \lambda] = [\chi, \lambda^n] \), we have \( a = [\chi^2, \lambda] = [\chi, \lambda^n] = 1 \).

Clearly \( \chi \) is nonlinear, and it follows by Burnside’s Theorem ([4, Theorem 3.15]) that \( \chi(g_0) = 0 \) for some \( g_0 \in G \). Then \( 0 = \chi^2(g_0) = \lambda(g_0) + b\psi(g_0) \), hence the algebraic integer
\[
\lambda(g_0)\psi(g_0) = -\lambda(g_0)\lambda(g_0)/b = -1/b
\]
and this implies that \( b = 1 \), so \( \chi^2 = \lambda + \psi \) as desired.

(2) Suppose that the restriction of \( \chi \) on \( G' \) is reducible. By [4, Theorem 6.22], there exists a normal subgroup \( M \) of \( G \) such that \( |G : M| \) is a prime \( p \) and \( \chi \) is induced by some \( \chi_1 \in \text{Irr}(M) \). Since \( M \unlhd G \), by the definition of induced character, we have \( \chi(g) = 0 \) whenever \( g \in G - M \), and hence
\[
\psi(g) = \chi^2(g) - \lambda(g) = -\lambda(g)
\]
is a root of unit. Let \( \theta = \psi_M \). Then
\[
[\theta, \theta] = \frac{\sum_{g \in G} |\psi(g)|^2 - \sum_{g \in G - M} |\psi(g)|^2}{|M|} = \frac{|G| - (|G| - |M|)}{|M|} = 1,
\]
hence \( \theta \) is irreducible. Write \( \chi_M = \chi_1 + \cdots + \chi_p \), where \( \chi_1, \cdots, \chi_p \) are distinct \( G \)-conjugates of \( \chi_1 \). Observe that
\[
\bigcup_{1 \leq i, j \leq p} \text{Irr}(\chi_i \chi_j) = \text{Irr}(\chi_M^2) = \text{Irr}(\chi_M) = \text{Irr}(\lambda_M + \psi_M) = \{ \lambda_M, \theta \}.
\]
It follows that for some \( i, j \in \{1, \cdots, p\} \), \( [\theta, \chi_i \chi_j] > 0 \), and then
\[
\psi(1) = \theta(1) = \chi_i(1)\chi_j(1) = \chi(1)^2/p^2 = (1 + \psi(1))/p^2,
\]
a contradiction. Hence \( \chi_M \) is necessary irreducible.

The following result is due to E.M. Zhmud (see [8, page 89]), we restate its short proof for the reader’s convenience.

**Lemma 2.3 ([8]).** Suppose that a finite group \( G \) has an irreducible character \( \chi \) such that \( \chi^2 = \lambda + \psi \) for some \( \psi \in \text{Irr}(G) \). Then \( \chi(1) = 2 \).

**Proof.** As \( \text{Irr}(\chi^2) \subseteq \text{Irr}(\chi^2) \), we have \( \chi^2 = m\lambda_G + n\psi \), \( m, n \in \mathbb{Z} \).

Clearly \( \chi \) is real, hence \( m = [\chi^2, 1_G] = \pm 1 \) by Theorem 4.6.2 in [1]. Then
\[
1 + \psi(1) = \chi^2(1) = (\chi^2(1))^2 = (m + n\psi(1))^2 = 1 + 2mn\psi(1) + n^2\psi^2(1),
\]
since \( 2mn + n^2\psi(1) = 1 \). It follows that \( n = \pm 1 \), hence \( \psi(1) = (1-2mn)/n^2 = 3 \), and \( \chi(1) = 2 \).

**Lemma 2.4.** Let \( G \) be a finite group with an irreducible character \( \chi \) (not necessary faithful) satisfying Hypothesis 2. Then \( \chi(1) = 2 \).
Proof. By Lemma 2.2,

\[ \chi^2 = \lambda + \psi, \]

where \( \lambda, \psi \in \text{Irr}(G) \), and \( \lambda(1) = 1 \). We work by induction first on \( |G : \ker \lambda| \) and second on \( |G| \).

If \( \lambda = 1_G \), then by Lemma 2.3, \( \chi(1) = 2 \) and we are done.

Now suppose that \( \lambda \neq 1_G \). Let \( N = \ker \lambda \), then \( N < G \). Write \( |G/N| = 2^k v \), where \( v \) is odd. Let

\[ e = (v - 1)/2, \quad \chi_0 = \chi\lambda^e, \quad \lambda_0 = \lambda^v, \quad \psi_0 = \psi\lambda^{2e}. \]

Then

\[ \lambda^{2e+1} = \lambda_0, \quad \chi_0^2 = (\chi\lambda^e)^2 = \chi^2\lambda^{2e} = \lambda^{2e+1} + \psi\lambda^{2e} = \lambda_0 + \psi_0. \]

Hence we may replace \( \chi, \lambda, \psi \) by \( \chi_0, \lambda_0, \psi_0 \) respectively. Since \( |G : \ker \lambda_0| = 2^k \), by induction we may assume that \( G/N \) is a cyclic 2-group. Let \( M \) be a maximal subgroup of \( G \) with \( N \leq M < G \), and let \( t \) be a 2-element outside \( M \).

We claim that \( \psi_M \) is irreducible. Suppose that this is not true. Since \( |G : M| = 2 \), \( \psi \) vanishes on \( G - M \) by [4, Lemma 2.29], and in particular \( \psi(t) = 0 \). Then \( \chi^2(t) = \lambda(t) + \psi(t) = \lambda(t) \). Now \( \chi(t) \) is a root of unit, it follows by Lemma 2.1 that \( \chi(1)^2 - 1 \) is even, so \( \chi(1) \) is odd. By [6, Theorem A], we may take an odd order element \( g \in G \) (and hence \( g \in N \)) such that \( \chi(g) = 0 \). Observe that \( \chi(g) = 0 \) is equivalent to \( \chi(g^2) = 0 \) for the odd order element \( g \). Since \( \text{Irr}(\chi^2) \subseteq \text{Irr}(\chi_0^2) \), we may write the generalized character \( \chi^{(2)} \) as

\[ \chi^{(2)} = m\lambda + n\psi, \]

where \( m, n \) are integers. Then

\[ 0 = \chi(g^2) = \chi^{(2)}(g) = m\lambda(g) + n\psi(g) = m + n\psi(g), \]

\[ 0 = \chi^2(g) = \lambda(g) + \psi(g) = 1 + \psi(g). \]

Now we have

\[ m = n, \quad \chi(1) = \chi^{(2)}(1) = m + n\psi(1), \quad m \geq 1. \]

However \( \chi^2(1) = 1 + \psi(1) \), and this implies that \( \chi^2(1) \leq \chi(1) \), a contradiction. Hence \( \psi_M \) is irreducible as claimed.

Now

\[ (\chi_M)^2 = \chi_M^2 = \lambda_M + \psi_M. \]

Since \( \chi_M \) is irreducible by Lemma 2.2, we may replace \( G, \chi, \lambda, \psi \) by \( M, \chi_M, \lambda_M, \psi_M \) respectively. Clearly \( |M : \ker \lambda_M| \leq |G : \ker \lambda| \) and \( |M| < |G| \), it follows by induction that \( \chi(1) = \chi_M(1) = 2 \), and we are done. \( \square \)
In what follows, we will use the following easy results: (1) cd($A_4$) = $\{1, 3\}$, cd($S_4$) = $\{1, 2, 3\}$, cd($PSL(2, 5)$) = $\{1, 3, 4, 5\}$; (2) if $D$ is a normal subgroup of $G$ with $G' \cap D = 1$, then any irreducible (linear) character of $D$ is extendible to $G$.

**Proof of the necessity in Theorem B.** Let $\chi$ be a faithful irreducible character of $G$ satisfying Hypothesis 2. By Lemma 2.2 and 2.4, $\chi(1) = 2$ and $\chi_{\tau}$ is irreducible.

We claim that $\chi$ is primitive. Otherwise, $\chi = \vartheta^G$, where $\vartheta \in \text{Irr}(H)$, $H < G$. Since $\chi(1) = 2$, we have $|G : H| = 2$ and hence $G' \leq H$. Since $\chi_{\tau}$ is irreducible, $\chi_{\vartheta}$ is also irreducible, this is a contradiction. Thus $\chi$ is primitive as claimed.

Now by [1, Theorem 18.1] $G/Z(G)$ is isomorphic to $A_4$, $S_4$ or $PSL(2, 5)$. We first assume that $G' \cap Z(G) = 1$. Suppose that $G/Z(G) \cong A_4$. Then $G' \cong (G' \times Z(G))/Z(G) = (G/Z(G))'$ is abelian, and this contradicts the fact that $\chi_{\vartheta}$ is irreducible and of degree 2. Suppose that $G/Z(G) \cong S_4$. Since $(G/Z(G))' = A_4$, we have $G' = A_4$. But cd($A_4$) = $\{1, 3\}$, this violates the fact that $\chi(1) = 2$ and $\chi_{\tau}$ is irreducible. Suppose that $G/Z(G) \cong PSL(2, 5)$, then $G = PSL(2, 5) \times Z(G)$, this is impossible since $\chi(1) = 2$. Therefore $G' \cap Z(G) > 1$.

Let $\chi_{G' \cap Z(G)} = 2\mu$ where $\mu \in \text{Irr}(G' \cap Z(G))$, and let $1 \neq g \in G' \cap Z(G)$. Clearly $Z(G)$ is cyclic and $\mu$ is faithful. Then $4 = |\chi^2(g)| = |\lambda(g) + \psi(g)| = [1 + \psi(g)]$. Since $\psi(g)$ is the sum of three roots of unity, we have $\psi(g) = 3$ and $\chi(g) = -2$. So $\mu(g) = -1$, and hence $g^2 \in \ker \mu = 1$, that is, $G' \cap Z(G) \cong \mathbb{Z}_2$.

**Proof of the sufficiency in Theorem B.** Suppose that a finite group $G$ has a faithful and primitive irreducible character $\chi$ of degree 2 and that $G' \cap Z(G) \cong \mathbb{Z}_2$. Then $Z(G)$ is cyclic (see [4, Theorem 2.32]) and $G/Z(G)$ is isomorphic to $A_4$, $S_4$ or $PSL(2, 5)$ (see [1, Theorem 18.1]). Write $Z(G) = Z$, $G' \cap Z = D$, and let $\chi_{\tau} = 2\mu$ where $\mu \in \text{Irr}(Z)$ is faithful. Clearly $\chi_{\tau}^2 = 4\mu^2$ and $\mu^2$ is a faithful linear character of $Z/D$. Let $\tau$ be any irreducible constituent of $\chi^2$. Clearly $\tau$ is an irreducible constituent of $(\mu^2)^G$. Since $(G/D)' \cap (Z/D) = G'/D \cap Z/D = (G' \cap Z)/D = 1$, $\mu^2$ is extendible to $\mu_0 \in \text{Irr}(G/D) \subset \text{Irr}(G)$. It follows by [4, Corollary 6.17] that $\tau = \mu_0 \sigma$ for some $\sigma \in \text{Irr}(G/Z)$.

We claim that this $\chi$ meets the requirement. Otherwise, $\chi^2$ has no irreducible constituent of degree 3. Assume that $G/Z(G) \cong A_4$ or $S_4$. Since cd($A_4$) = $\{1, 3\}$ and cd($S_4$) = $\{1, 2, 3\}$, we see that any irreducible constituent $\tau$ of $\chi^2$ is a product of $\mu_0$ and an irreducible character $\sigma$ of $G/Z$ with degree at most 2. Note that in the group $A_4$ or $S_4$, the Klein four group $K_4$ is contained in the kernel of any irreducible character of degree at most 2. Hence the Fitting subgroup $F(G)$ of $G$ is contained in the kernel of $\sigma$. By
the arbitrariness of $\tau$, it follows that $F(G) \leq Z(\chi^2) \leq Z(G)$, a contradiction. Thus this $\chi$ satisfies Hypothesis 2. Assume that $G/Z \cong PSL(2,5)$. Since $\text{cd}(PSL(2,5)) = \{1,3,4,5\}$, we get that $\chi^2 = \mu_0\sigma \in \text{Irr}(G)$, where $\sigma \in \text{Irr}(G/Z)$ is of degree 4. Since $\text{Irr}(\chi^{(2)}) \subseteq \text{Irr}(\chi^2)$, we deduce the following contradiction that $\chi^{(2)} = m\chi^2$ for some integer $m$. Hence the $\chi$ satisfies Hypothesis 2. 

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REFERENCES


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