ALTERNATE PROOF OF THE REINHOLD BAER THEOREM ON 2-GROUPS WITH NONABELIAN NORM

Yakov Berkovich
University of Haifa, Israel

Abstract. We present a new easy proof of the classical theorem due to Reinhold Baer asserting that the nonabelian norm of a 2-group \( G \) coincides with \( G \), i.e., \( G \) is Dedekindian. Our proof is independent of all papers devoted to this theme.

According to R. Baer ([Bae1]), the norm \( \mathcal{N}(G) \) of a group \( G \) is the intersection of normalizers of all subgroups of \( G \). Clearly, the subgroup \( \mathcal{N}(G) \) is characteristic in \( G \) and Dedekindian, i.e., \( \mathcal{N}(G) \) is either abelian or \( \mathcal{N}(G) = Q \times E \times A \), where \( Q \) is the ordinary quaternion group, \( \exp(E) \leq 2 \) and all elements of the abelian subgroup \( A \) have odd order ([H, Theorem 12.5.4]) (it is possible that \( \mathcal{N}(G) \) has subgroups that are not normal in \( G \)). Some additional information on the norm is contained in [Bae1, Bae2, Bae3, BHN, Sch, W] and in a number of other papers. For example, it is proved in [Sch] that, for an arbitrary group \( G \), \( G/C_G(\mathcal{N}(G)) \) is abelian and \( \mathcal{N}(G) \leq Z_2(G) \), where \( Z_2(G) \) is the second member of the upper central series of \( G \); this was obtained as a result of intricate computations (for finite groups, see [BJ, Theorem 140.9]). It follows ([Bae3]) that if \( Z(G) = \{1\} \), then \( \mathcal{N}(G) = \{1\} \) (for finite groups this is easily to prove using the same argument as in the proof of Theorem 3, below).

We use the same standard notation as in [Ber].

We offer a new proof of the following remarkable result due to R. Baer.

THEOREM 1 ([Bae2]). If the norm \( \mathcal{N}(G) \) of a 2-group \( G \) is nonabelian, then \( \mathcal{N}(G) = G \).

We do not assume, in Theorem 1, that \( G \) is finite.

2010 Mathematics Subject Classification. 20D15.

Key words and phrases. Norm, ordinary quaternion group, Dedekindian groups, 2-groups of maximal class, 2-groups with nonabelian norm.
All prerequisites are collected in the following

**Lemma 2.** Let $G$ be a finite nonabelian 2-group.

(a) (Dedekind; see [Ber, Theorem 1.20] and [H, Theorem 12.5.4]) If all subgroups of $G$ are normal, then $G = Q \times E$, where $Q \in \{\{1\}, Q_8\}$, $\exp(E) \leq 2$ (this is also true if $G$ is an infinite 2-group).

(b) (Burnside; see [Ber, Theorem 1.2]) If $G$ has a cyclic subgroup of index 2, then $G$ is one of the following groups: dihedral, semidihedral, generalized quaternion or minimal nonabelian of order $> 2^3$ with cyclic center of index 4 (in the last case, $N(G) \leq \Phi(G)$).

(c) (see [Ber, Propositions 10.17 and 1.6]) If $B < G$ is nonabelian of order 8 and $C_G(B) < B$, then $G$ is of maximal class. If $G$ is of maximal class, then it has a cyclic subgroup of index 2 (see (b)).

(d) (see [Ber, Appendix 16]) If $G = B \ast C$ (central product) has order 16, where $B$ is nonabelian of order 8, $C$ is cyclic of order 4, then $G$ has exactly 7 subgroups of order 2 and only one of them is normal in $G$.

(e) If $G$ is of maximal class and order $> 8$ then $N(G)$ is of order 2 unless $G$ is generalized quaternion group with cyclic $N(G)$ of order 4.

(f) [Ber, Theorem 10.28] $G$ is generated by minimal nonabelian subgroups.

If $M < N(G)$ and $M$ is normal in $G$, then $N(G)/M \leq N(G/M)$.

**Proof of Theorem 1.** Assume, to the contrary, that $N(G) < G$.

By Lemma 2(a), there is in $N(G)$ a subgroup $Q \cong Q_8$. By assumption, $Q < G$ so that $|Q| \geq 16$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be distinct cyclic subgroups of $Q$ of order 4.

(i) If $Q \leq H \leq G$, then $Q \leq N(H)$. This is obvious.

(ii) If $Q < H \leq G$ and $|H : Q| = 2$, then $H = Q \times C$ is Dedekindian. Assume that this is false. Then $H$ has a nonnormal cyclic subgroup $L$ of order $\leq 4$ such that $L \not\leq Q$. Since $QL = H$, it follows that $Q$ does not normalize $L$, contrary to the hypothesis. Thus, $H$ is Dedekindian, and our claim follows in view of Lemma 2(a).

(iii) $C_G(Q)$ has no cyclic subgroup of order 4. Assume, on the contrary, that $L = \langle x \rangle \leq C_G(Q)$ is cyclic of order 4. Set $H = Q \ast L$; then $16 \leq |H| \leq 32$. However, $|H| \neq 16$, by (ii). Now let $|H| = 32$; then $H = Q \times L$ and $\langle ax \rangle$ is not b-invariant, a contradiction.

(iv) By (ii), $|G : Q| > 2$.

(v) It follows from hypothesis and (iii) that if $D = \langle d \rangle < G$ is cyclic of order 4, then $Q \cap D = \{1\}$. Assume, on the contrary, that $Q \cap D = \{1\}$. Then, by (iii), $Q$ is not normal in $F = QD$ (otherwise, $F = Q \times D$) so $F/Q_F \cong D_8$. But the norm of $D_8$ coincides with its center, and this is a contradiction since $Q/Q_F \leq N(F/Q_F)$ and $Q/Q_F \not\leq \mathbb{Z}(G/Q_F)$.

(vi) We claim that $\exp(G) = 4$. Assume, on the contrary, that $T < G$ is cyclic of order 8. Since $Q$ normalizes $T$, we get $H = QT \leq G$. Taking in mind
our aim, one may assume that $G = QT$. By (v), one has $Q \cap T \neq \{1\}$ so that $16 \leq |G| \leq 32$. By (iv), $|G| > 16$. Let $|G| = 32$; then $Q \cap T = Z(Q)$. Write $H_1 = AT$ and $H_2 = BT$. Since, by (i), $A \leq \mathcal{N}(H_1)$, it follows that $H_1$ is not of maximal class in view of $A \not\leq \Phi(H_1)$ (Lemma 2(b)). Similarly, $H_2$ is not of maximal class. Then, by Lemma 2(b), $\mathcal{O}_1(T) = \Phi(H_i) \leq Z(H_i)$, $i = 1, 2$. Thus, $\mathcal{O}_1(T)$, a cyclic subgroup of order 4, centralizes $(A, B) = Q$, contrary to (iii).

Now we are ready to complete the proof.

By (v) and (vi), $\mathcal{O}_1(G) = \mathcal{O}_1(Q)$ so that $G/\mathcal{O}_1(G)$, being of exponent 2, is abelian, and we conclude that $G' = \mathcal{O}_1(G)$ has order 2 since $G$ is nonabelian. The quotient group $G/C_G(Q)$ is isomorphic to a subgroup of $D_8 \in \text{Syl}_2(\text{Aut}(Q))$, and $G/C_G(Q)$ contains a four-subgroup $\cong Q/\langle Z(Q) \rangle$. Since $G'/\mathcal{O}_1(Q) \leq C_G(Q)$, we get $G/C_G(Q) \cong Q/\langle Z(Q) \rangle$ since $Q \cap C_G(Q) = Z(Q)$, and we conclude that $G = QC_G(Q)$. Since $\exp(C_G(Q)) = 2$, by (iii), we get $C_G(Q) = Z(Q) \times E$, where $E < C_G(Q)$. In that case, $G = QC_G(Q) = Q(Z(Q) \times E) = QE$, where $Q \cap E = \{1\}$. It follows that $G = Q \times E$ hence $G$ is Dedekindian since $\exp(E) = 2$. □

The following theorem is a partial case of Schenkman’s result [Sch] mentioned above.

**Theorem 3.** Let $G$ be an arbitrary finite group such that $\mathcal{N}(G)$ is nonabelian. Then $P \in \text{Syl}_2(\mathcal{N}(G))$ centralizes all elements of $G$ of odd order and $P \leq Z_2(G)$.

**Proof.** Let $Q_8 \cong Q \leq \mathcal{N}(G)$, let $a \in Q^\#$ and let $x \in G$ be of order $p^k$, where a prime $p > 2$. Set $H = \langle a, x \rangle$. Let $y \in \langle x \rangle$ be of order $p$ and $F = \langle a, y \rangle$. Assume that $F$ is nonabelian. If $o(a) = 2$, then $F$ is a nonabelian group of order $2p$ so its norm equals $\{1\}$, a contradiction since $a \in \mathcal{N}(G)$. If $o(a) = 4$, then $F$ is minimal nonabelian with norm of order 2, a contradiction again. Thus, $F$ is abelian. Thus, $Q$ centralizes all subgroups of $G$ of odd order. Since, by Lemma 2(f), $P \in \text{Syl}_2(\mathcal{N}(G))$ is generated by its minimal nonabelian subgroups all of which are $\cong Q_8$, it follows that $P$ centralizes all subgroups of $G$ of odd order. Note that $P$ is normal in $G$.

Now let $P \leq P_1 \in \text{Syl}_2(G)$. By Theorem 1, the subgroup $P_1$ is Dedekindian since $P \leq \mathcal{N}(P_1)$. Therefore, $Z(P) \leq Z(P_1)$ (Lemma 2(a)). Since, by the above, $Z(P)$ centralizes all elements of $G$ of odd order, we get $Z(P) \leq Z(G)$. Since $P/Z(P) \leq P_1/Z(P)$, $P_1/Z(P)$ is abelian and $P/Z(P)$ centralizes all elements of $G/Z(P)$ of odd order, we get $P/Z(P) \leq Z(G/Z(P))$ so that $P \leq Z_2(G)$, proving the last assertion. □

Let $Q \cong Q_8$ be a subgroup of a 2-group $G$ not necessarily finite. If $Q$ normalizes all cyclic subgroups of $G$ of order $\leq 8$, then $G$ is Dedekindian, as follows immediately from the proof of Theorem 1.
Problems

1. Classify the $p$-groups $G$ satisfying $\mathcal{N}(H) \leq \mathcal{N}(G)$ for all nonabelian $H \leq G$. In particular, classify the $p$-groups $G$ such that $\mathcal{N}(H) = H \cap \mathcal{N}(G)$ for all nonabelian $H \leq G$.

2. Classify the finite groups $H$ such that $\mathcal{N}(Q_8 \times H)$ is nonabelian (if $H$ is a $2$-group of exponent $> 2$, then $\mathcal{N}(Q_8 \times H)$ is abelian, by Theorem 1).

3. Classify the $p$-groups $G$ such that $\mathcal{N}(G)$ is maximal in $G$ (if $G = \langle a, b \mid a^{p^n} = b = 1, a^b = a^{1+p} \rangle$, where $n > 1$ and $n > 2$ provided $p = 2$, then $\mathcal{N}(G) = \langle a^p, b \rangle$ is maximal in $G$).

4. Classify the $p$-groups $G$ satisfying $\mathcal{N}(H) = Z(H)$ for all nonabelian $H \leq G$.

5. Study the finite groups $G$ such that, whenever $H < G$ is either minimal nonabelian or minimal nonnilpotent, then $H \cap \mathcal{N}(G) = \{1\}$.

6. Describe $\mathcal{N}(A \times B)$ and $\mathcal{N}(A \ast B)$ in the terms of $A$ and $B$ (if $A \cong Q_8 \cong B$ and $G = A \ast B$, the central product of order 32, then $\mathcal{N}(G) = Z(G)$ is of order 2 and $A = \mathcal{N}(A) \cong Q_8 \cong \mathcal{N}(B) = B$).

7. Study the pairs of $p$-groups $H < G$ such that $H$ normalizes all $C < G$ with $C \not\leq H$ (for each $H$ this is a separate problem. Note that $H \not\cong D_8$).

Acknowledgements.
I am indebted to Zvonimir Janko drawing my attention to Theorem 1, for discussion of this note and a number of useful remarks. Note that he also produced simultaneously a new proof of Theorem 1 based on entirely other ideas; see [BJ, Theorem 140.9].

References


Y. Berkovich
Department of Mathematics
University of Haifa
Mount Carmel, Haifa 31995
Israel
Received: 3.2.2011.
Revised: 18.7.2011.