ON THE STRUCTURE OF THE AUTOMORPHISM GROUP OF A MINIMAL NONABELIAN $p$-GROUP (METACYCLIC CASE)

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Abstract. In this paper we find the complete structure for the automorphism groups of metacyclic minimal nonabelian $2$-groups. This, together with [6, 7], gives the complete answer to the Question 15 from [5] (respectively Question 20 from [4]) in the case of metacyclic groups. We also correct some inaccuracies and extend the results from [13].

1. Introduction

All groups considered here are finite and the notation used is standard. Finite $p$-groups are an important group class of finite groups. Since the classification of finite simple groups is finally completed, the study of finite $p$-groups becomes more and more active. Many leading group theorists, for example, Berkovich, Glauberman, Janko etc., turn their attention to the study of finite $p$-groups, see [1–4, 9, 10, 12]. Since a finite $p$-group has "too many" normal subgroups and, consequently, there is an extremely large number of nonisomorphic $p$-groups of a given fixed order, the classification of finite $p$-groups in the classical sense is impossible. In [1–3] Berkovich and Janko have developed some techniques for working with minimal non-abelian subgroups of finite $p$-groups. Roughly speaking, they show that some control over the lattice of subgroups in $p$-groups can be gained by considering maximal abelian subgroups together with minimal non-abelian subgroups. In [12] Janko points out that in studying the structure of non-abelian $p$-groups $G$, the minimal non-abelian subgroups of $G$ play an important role since they generate the group $G$. More precisely, if $A$ is a maximal normal abelian subgroup of $G$, then

2010 Mathematics Subject Classification. 20D45, 20D15.
Key words and phrases. Automorphisms, $p$-groups.
minimal non-abelian subgroups of $G$ cover the set $G \setminus A$ (see Proposition 1.6 in [12]). A $p$-group $G$ is said to be \textit{minimal nonabelian} (or brevity, $A_{1}$-group), if $G$ is nonabelian, but all its proper subgroups are abelian. In [5] Berkovich formulated 22 questions concerning $p$-groups. In Question 15 (respectively Question 20 from [4]) he proposed to describe the automorphism groups of $A_{1}$-groups. The following lemma gives the classification of $A_{1}$-groups.

**Lemma 1.1.** (L. Redei) Let $G$ be a minimal nonabelian $p$-group. Then $G = \langle x, y \rangle$ and one of the following holds

1. $x^{p^m} = y^{p^n} = z = 1$, $[x, y] = z$, $[x, z] = [y, z] = 1$, $m, n \in \mathbb{N}$, $m \geq n \geq 1$; where in case $p = 2$ we must have $m > 1$;
2. $x^{p^m} = y^{p^n} = 1$, $[x, y] = x^{p^m - 1}$, $m, n \in \mathbb{N}$, $m \geq 2$, $n \geq 1$;
3. $a^4 = 1$, $a^2 = b^2$, $[a, b] = a^2$, $G \cong Q_8$.

In this paper we find the complete structure for the automorphism groups of metacyclic minimal nonabelian 2-groups. This, together with [6, 7], gives the complete answer to the Question 15 from [5] (respectively Question 20 from [4]) in the case of metacyclic groups. In Section 2 we generalize the results from [13] and we specify a method of finding relations in an automorphism group, that we will use in the next Sections. In the first part of Section 3 we state some results from [13], that we will use in the next part of the note, but also we specify the exact statements. Unfortunately we must point out that in Section 3 of [13] in Case A the expression "$C_{K}(G')$" should be replaced by "$\Omega_{m-1}(K)$." In the end of Section 3 we state the relations in the automorphism group of a split metacyclic 2-group. In this way we remove some inaccuracies from Theorem 3.7 in [8] (see Example 1 in [13]). In Section 4 we find the complete structure of the automorphism group of a metacyclic minimal nonabelian 2-group. These relations were not considered in [8].

If $L$ is a subgroup of a group $G$, then $C_{\text{Aut} G}(L)$ denotes the group of those automorphisms of $G$ that centralize $L$ and $N_{\text{Aut} G}(L)$ denotes the group of those automorphisms of $G$ that normalize $L$. If $M$ and $N$ are normal subgroups of a group $G$, then $\text{Aut}_{N}(G) = C_{\text{Aut}(G)}(G/N)$ denotes the group of all automorphisms of $G$ normalizing $N$ and centralizing $G/N$. Also $\text{Aut}_{N}^{M}(G)$ denotes $\text{Aut}_{N}(G) \cap C_{\text{Aut}(G)}(M)$. If $L$ is a subgroup of a $p$-group $G$ and $l \in \mathbb{N}$ then we set $\Omega_{l}(L) = \langle g \in L \mid g^{p^{l}} = 1 \rangle$ and $\bar{\Omega}_{l}(L) = \langle g^{p^{l}} \mid g \in L \rangle$.

In [15] the authors investigated the automorphism group of a semidirect product $G = H \rtimes K$. They defined the following subgroups

\begin{align*}
A &= \{ \theta \in \text{Aut} G \mid [K, \theta] = 1 \text{ and } H^{\theta} = H \}, \\
B &= \{ \theta \in \text{Aut} G \mid [H, \theta] = 1 \text{ and } [K, \theta] \subseteq H \}, \\
C &= \{ \theta \in \text{Aut} G \mid [K, \theta] = 1 \text{ and } [H, \theta] \subseteq K \}, \\
D &= \{ \theta \in \text{Aut} G \mid [H, \theta] = 1 \text{ and } K^{\theta} = K \}.
\end{align*}

By definition, we have $BD = B \rtimes D \subseteq C_{\text{Aut} G}(K)$ and $AC = C \rtimes A \subseteq C_{\text{Aut} G}(H)$. 
2. Crossed homomorphisms and automorphisms

We call an ordered triple \((Q,N,\theta)\) data if \(N\) is an abelian group, \(Q\) is a group and \(\theta : Q \to \text{Aut} N\) is a homomorphism. If \(\theta\) is a homomorphism of \(Q\) into \(\text{Aut} N\), then \(Q\) acts on \(N\) when we define, for each \(x \in Q\) and \(a \in N\), \(a^x\) is the image of \(a\) under \(x^\theta\). If \(N\) is a normal subgroup of \(G\), then the action of \(G/N\) on \(Z(N)\) is given by \(a^N = a^{gN} = a^g\). Given data \((Q,N,\theta)\) a crossed homomorphism is a function \(\lambda : Q \to N\) such that \((xy)^\lambda = (x^\lambda y)^\lambda\) for all \(x, y \in Q\). We denote the set of such crossed homomorphisms by \(Z^1(Q,N)\).

It forms a group under the operation \(q^{\lambda_1+\lambda_2} = q^{\lambda_1}q^{\lambda_2}\); if \(\theta\) is trivial, then \(Z^1(Q,N) = \text{Hom}(Q,N)\).

We recall a known result ([11], Satz 1.17.1) needed in the sequel:

**Lemma 2.1.** Let \(N\) be a normal subgroup of \(G\). Then there is a natural isomorphism from \(Z^1(G/N,Z(N))\) to \(\text{Aut}_N^Z(G)\) sending each crossed homomorphism \(f : G/N \to Z(N)\) to the automorphism \(\varphi_f : x \mapsto x(xN)^f\) of \(G\).

Lemmas 2.2–2.3 are more general versions of Lemma 2.5 and Theorem 2.6 (see also [13]).

**Lemma 2.2.** Let \(N\) be an normal subgroup of \(G\). Let \(M\) be a normal subgroup of \(G\) such that \(M \leq Z(G)\). Assume that that \(L = \{\lambda \in Z^1(G/N,Z(N)) \mid (G/N)^\lambda \subseteq M\}\) and \(A = N_{\text{Aut} G}(M) \cap N_{\text{Aut} G}(N)\). Then

1. \(A \subseteq \text{Aut}(G)\) and \(L \leq Z^1(G/N,Z(N))\).

2. If \(\alpha \in A\) and \(\lambda \in L\) then the function \(\mu : gN \mapsto ((g^{\alpha^{-1}}N)^\lambda)^\alpha\) is a crossed homomorphism and \(\mu \in L\).

**Proof.** The first part of (1) is obvious.

(2) Assume that \(\alpha \in A\) and \(\lambda \in L\). First let \(Nq_1 = Nq_2\), then \(q_2 = q_1h\) for some \(h \in N\). Then

\[
(g_2N)^\mu = ((g_2^{\alpha^{-1}}N)^\lambda)^\alpha = (((g_1h)^{\alpha^{-1}}N)^\lambda)^\alpha = ((g_1^{\alpha^{-1}}N)^\lambda)^\alpha = (g_1N)^\mu
\]

since \(N\) is normalized by \(\alpha\). So \(\mu\) is well defined.

Let \(g_1N, g_2N \in G/N\). We have

\[
(g_1N \cdot g_2N)^\mu = (g_1g_2N)^\alpha = (((g_1g_2)^{\alpha^{-1}}N)^\lambda)^\alpha = (((g_1^{\alpha^{-1}}N)^\lambda g_2^{\alpha^{-1}}(g_2^{\alpha^{-1}}N)^\lambda))^{\alpha} = (((g_1^{\alpha^{-1}}N)^\lambda)^\alpha g_2((g_2^{\alpha^{-1}}N)^\lambda))^{\alpha} = ((g_1N)^\alpha g_2N \cdot (g_2N)^\mu).
\]

It is evident that \(\mu \in L\) since \((G/N)^\mu \subseteq M\). □

**Lemma 2.3.** Let \(G, N, M, L\) and \(A\) be as in Lemma 2.2. Assume that \(E := \{\varphi \in \text{Aut}_N^Z(G) \mid [G, \varphi] \subseteq M\}\). Then
(1) $E \leq \text{Aut } G$ and there is a natural isomorphism from $L$ to $E$ sending each crossed homomorphism $f : G/N \rightarrow M$ to the automorphism $\varphi_f : x \mapsto x(xN)^f$ of $G$.

(2) If $\alpha \in A$ and $\varphi \in E$ is determined by the crossed homomorphism $\lambda \in L$, then $\alpha^{-1}\lambda\alpha$ is determined by the crossed homomorphism $\mu \in L$ defined by $\mu : gN \mapsto ((g^{\alpha^{-1}}N)^{\lambda})^\alpha$.

(3) $A$ normalizes $E$ and $AE \leq \text{Aut } G$.

**Proof.** (1) It is evident that $E \leq \text{Aut } G$. By definitions of $M, L, E$ and Lemma 2.1 we get the second part of the statement.

(2)-(3) Assume that $\alpha \in A$ and $\beta \in E$. By (1) there exists $\lambda \in Z^1(G/N, Z(N))$ such that $h^\beta = h(hN)^\lambda$ ($h \in G$) and $(hN)^\lambda \in M$ for all $h \in G$. If $h \in G$ then

$$h^{\alpha^{-1}\beta\alpha} = (h^{\alpha^{-1}})^\beta = (h^{\alpha^{-1}}(h^{\alpha^{-1}}N)^\lambda)^\alpha = h((h^{\alpha^{-1}}N)^{\lambda})^\alpha$$

and $((h^{\alpha^{-1}}N)^{\lambda})^\alpha \in M$. Hence by Lemmas 2.1 and 2.2 $\alpha^{-1}\beta\alpha \in E$, so $A$ normalizes $E$. Now it is clear that $AE \leq \text{Aut } G$.

For the sake of completeness we recall some results from [13]. We will use them in this note.

**Lemma 2.4 ([13]).** Let $N$ be an normal subgroup of $G$ such that $G/N$ is cyclic of order $n$. Assume that $g$ is an element of $G$ with $G = \langle N, g \rangle$.

(1) If $a \in Z(N)$ and $a^{g^{n^{-1}+\cdots+g+1}} = 1$, then the function $\lambda : G/N \rightarrow Z(N)$, defined by $g^iN)^\lambda = a^{g^{i-1}+\cdots+g+1}$ ($i \in \mathbb{N}$) and $N^\lambda = 1$, is a crossed homomorphism.

(2) If $\lambda \in Z^1(G/N, Z(N))$ then there exists $a \in Z(N)$ such that $a^{g^{n^{-1}+\cdots+g+1}} = 1$, $(g^iN)^\lambda = a^{g^{i-1}+\cdots+g+1}$ ($i \in \mathbb{N}$) and $N^\lambda = 1$.

**Lemma 2.5 ([13]).** Let $G, N, g$ be as in Lemma 2.4. Let $M$ be a normal subgroup of $G$ such that $M \leq Z(N)$ and for all $a \in M$ $a^{g^{n^{-1}+\cdots+g+1}} = 1$. Assume that $L = \{\lambda \in Z^1(G/N, Z(N)) \mid (G/N)^\lambda \leq M\}$ and $A = N_{\text{Aut } G}(N) \cap N_{\text{Aut } G}(M)$. Then

(1) $A \leq \text{Aut } G$ and $L \leq Z^1(G/N, Z(N))$; moreover $L \cong M$.

(2) If $\alpha \in A$ and $\lambda \in L$ then the function $\mu : G/N \rightarrow Z(N)$ defined by $\mu : hN \mapsto ((h^{\alpha^{-1}}N)^{\lambda})^{\alpha}$ is a crossed homomorphism and $\mu \in L$.

**Theorem 2.6 ([13]).** Let $G, N, L, M, g$ and $A$ be as in Lemma 2.5. Assume that $E := \{\varphi \in \text{Aut } N^G \mid |G, \varphi| \leq M\}$. Then $E \leq \text{Aut } G$, $L \cong E \cong M$, $A$ normalizes $E$, $AE \leq \text{Aut } G$ and $A \cap E \cong \{g^{-1}g^\varphi \mid \varphi \in A \cap E\}$.

We will need the following lemma:
Lemma 2.7. Let \( G \) be a group, \( g, h, z \in G \) and \([h, g] = z\), \([g, z] = 1 = [h, z]\). Assume that \( i, j \in \mathbb{N} \) and \( \alpha \in \text{Aut} \, G \). Then

1. \( h^{g^{-1} + \ldots + g + 1} = h^i z^{\beta(i-1)} \);
2. if \( g^\alpha = g, h^\alpha = h^i, z^\alpha = z \), then \( (h^{g^{-1} + \ldots + g + 1})^\alpha = h^{ij} z^{\beta(i-1)} \);
3. if \( g^\alpha = g, h^\alpha = h^i, z^\alpha = z^j \), then \( (h^{g^{-1} + \ldots + g + 1})^\alpha = h^{ij} z^{\beta(i-1)} \);
4. if \( g^\alpha = g^i, h^\alpha = h, z^\alpha = z \), then \( (h^{g^{-1} + \ldots + g + 1})^\alpha = h^i z^{\beta(i-1)} \);
5. if \( g^\alpha = g^i, h^\alpha = h, z^\alpha = z \), then \( (h^{g^{-1} + \ldots + g + 1})^\alpha = h^i z^{\beta(i-1)} \).

By Lemmas 2.3, 2.4 and 2.7 we get

Lemma 2.8. Let \( G, N, M, E, g \) be as in Theorem 2.6 and \( i, j \in \mathbb{N}, i = j^{-1} \mod n \). Assume that \( \lambda \in \mathbb{Z}^1(G/N, Z(N)) \), \((gN)^\lambda = h \) for some \( h \in M \) and \( \beta \in E \) is an automorphism determined by \( \lambda \). Assume also that \( \alpha \in \text{Aut} \, G, [h, g] = z \) and \([g, z] = 1\). Then

1. if \( g^\alpha = g^i, h^\alpha = h, z^\alpha = z \), then \( ((g^\alpha^{-1} N)^\lambda)^\alpha = h^i z^{\beta(i-1)} \);
   in particular if \( z = 1 \), then \( \beta^\alpha = \beta^i \);
2. if \( g^\alpha = g^i, h^\alpha = h, z^\alpha = z \), then \( ((g^\alpha^{-1} N)^\lambda)^\alpha = h^i z^{\beta(i-1)} \);
   in particular if \( z = 1 \), then \( \beta^\alpha = \beta^i \);
3. if \( g^\alpha = g, h^\alpha = h^i \), then \( ((g^\alpha^{-1} N)^\lambda)^\alpha = h^i \) and \( \beta^\alpha = \beta^i \).

3. A split metacyclic 2-group

Let \( G = H \times K \) be a split metacyclic 2-group, where \( H = \langle x \rangle \) and \( K = \langle y \rangle \) and let \( A, B, C \) and \( D \) be the subgroups of \( \text{Aut} \, G \) defined in the introduction. In this section we refer to the appropriate cases of the split metacyclic 2-groups from [8], but occasionally we repeat some known results for readers’ convenience. In fact we consider only Case A.

Let \( G = H \times K = \langle x, y \mid x^{2^m} = y^{2^n} = 1, x^y = x^{1+2^{m-1}} \rangle \), where \( m \geq 3, n \geq 1, 1 \leq r \leq \min\{m-2, n\} \).

It is convenient to consider \( G \) in the following three subcases (see [8])

(I) \( m \leq n \), (II) \( n \leq m - r < m \), (III) \( m - r < n < m \).

Moreover there exist two special cases. They are case (II), when \( m = 2r \), \( n = r = m - r \geq 2 \) and \( G = \langle x, y \mid x^{2^r} = y^2 = 1, x^y = x^{1+2^r} \rangle \) and case (III), when \( r = n > m - n \geq 2 \) and \( G = \langle x, y \mid x^{2^m} = y^2 = 1, x^y = x^{1+2^m-n} \rangle \).

These are referred to as exceptional cases. We will also need the following number theoretic result (see [8,13]), which is easily established by induction.

Lemma 3.1. Let \( m, n \) and \( r \) be positive integers.

1. For all \( m \geq 2, n \geq 1 \), \((1 + 2^m)^{2^n} \equiv 1 + 2^{m+n} \) (mod \( 2^{2m+n-1} \))
   and \((1 + 2^m)^{2^n-1} \equiv 1 + 2^{m+n-1} \) (mod \( 2^{m+n} \)).
(2) For \( n \geq 2, r \geq 1 \) and \( m = n + r \), let \( S = 1 + u + \cdots + u^{2r-1} \), where \( u \equiv 1 \pmod{2^n} \). Then \( S \equiv 2^r + 2^{m-1} \pmod{2^n} \) if \( u \not\equiv 1 \pmod{2^{n+1}} \) and \( S \equiv 2^r \pmod{2^n} \) if \( u \equiv 1 \pmod{2^{n+1}} \).

Using Lemma 3.1 the following lemmas are easily established.

**Lemma 3.2.**

(1) \( \Omega_H(K) = \langle x^{2^r} \rangle \),
(2) \( \Omega_K(H) = \langle y^{2^r} \rangle \),
(3) \( \Omega' = [H, K] = \langle x^{2^{m-r}} \rangle \),
(4) \( G \) is nil \( 2 \leq r \leq m \).

**Lemma 3.3.** \( \Omega_{m-r}(K), [H, \Omega_{m-r}(K)] \) are given in the three cases as follows:

(I) \( \Omega_{m-r}(K) = \langle y^{2^{n-m+r}} \rangle \leq Z(G) \) if \( [H, \Omega_{m-r}(K)] = 1 \);
(II) \( \Omega_{m-r}(K) = \langle y \rangle = C_K(G'), [H, \Omega_{m-r}(K)] = \langle x^{2^{m-r}} \rangle = G' \leq Z(G) \);
(III) \( \Omega_{m-r}(K) = \langle y^{2^{n-m+r}} \rangle \leq C_K(G'), [H, \Omega_{m-r}(K)] = \langle x^{2^r} \rangle \leq Z(G) \).

As in [14] when \( p \) was odd or by considering matrices of maps from [8] one could find the effect of an automorphism \( \varphi \) on the generators of \( G \).

**Lemma 3.4.** Let \( G, x, y \) be as above.

(1) Assume that \( n \neq r \). Then a map \( \varphi : G \to G \) is an automorphism if and only if \( x^{-1}x^\varphi \in \Omega_1(H)\Omega_{m-r}(K), \ y^\varphi y^{-1} \in \Omega_n(H)C_K(H) \);
(2) Assume that \( n = r \). Then a map \( \varphi : G \to G \) is an automorphism if and only if either \( x^{-1}x^\varphi \in \Omega_1(H)\Omega_{m-r}(K), y^\varphi y^{-1} \in \Omega_n(H) \) or \( x^{-1}x^\varphi \in \Omega_1(H)\Omega_{m-r}(K) \), \( y^\varphi y^{-1} \in \Omega_n(H)g^{2^{m-r}} \).

By Theorem 2.6 and the definitions of \( A, B \) and \( D \) we get the following lemma.

**Lemma 3.5.** Let \( G, A, B, D \) be as above. Then

(1) \( B \cong \text{Aut}_H^G(G) \),
(2) \( AD = A \times D \) normalizes \( B \),
(3) \( B \cap D = 1 \).

For the proofs of Theorem 3.6 and Lemma 3.7 see [13].

**Theorem 3.6.** Let \( G \) be as above.

(1) \( \text{Aut} G = C_{\text{Aut} G}^1(G)C_{\text{Aut} G}^0(K) \) if and only if \( r \neq n \);
(2) \( C_{\text{Aut} G}^1(K) = BD \);
(3) \( C_{\text{Aut} G}^1(K) = AC \) if and only if \( n \leq m \).

We set \( M := [H, \Omega_{m-r}(K)]\Omega_{m-r}(K), N := G'K \) and

\[ E := \{ \varphi \in \text{Aut}_N^G(G) \mid [H, \varphi] \subseteq M \} \subseteq \text{Aut}_N^G(G) \).

**Lemma 3.7.** Let \( G, M \) be as above and \( n \neq r \).

(1) \( M \) is abelian and normal in \( G \).
(2) If \( a \in M \) then \( a^{2^{m-r}+\cdots+x+1} = 1 \).
Lemma 3.8. Let $G, A, D, E$ be as above and $n \neq r$. Then
(1) $E \leq \text{Aut } G$;
(2) $E \cong M$;
(3) $AD = A \times D$ normalizes $E$;
(4) $E \cap A \cong [H, \Omega_{m-r}(K)]$;
(5) $C_{\text{Aut } G}(K) = AE$;
(6) $D \cong \text{Aut}_{C_{\text{Aut } H}}(K)$.

Proof. In the proof of Lemma 3.9 in [13] we put $\Omega_{m-r}(K)$ instead of $C_{K}(G')$.

We define $c \in \text{Aut } G$ by setting $x^c = xy$, when $m - r \geq n \neq r$, and $x^c = x^{2^{m-r}+r}$, when $m - r < n \neq r$, $y^c = y$. We also set $F := (c) \leq E$.

Theorem 3.9. Let $G, E, A, F$ be as above and $n \neq r$. Then
(1) $F \cong \Omega_{m-r}(K)$, $AF = AE$ and $A \cap F = 1$;
(2) $\text{Aut } G = BDAF$ and $|\text{Aut } G| = |B||D||A||F|$.

Proof. In the proof of Theorem 3.10 in [13] we put $\Omega_{m-r}(K)$ instead of $C_{K}(G')$.

By Theorem 3.9 and Lemma 3.4 it is obvious that

Theorem 3.10. Let $G, A, B, D, F, T$ be as above. Then
(1) $A \cong \text{Aut } H \cong C_2 \times C_{2^{m-2}}$ and $B \cong \Omega_{n}(H) \cong C_{2^{\text{min}(m,n)}}$;
(2) $D \cong C_{K}(H) \cong C_{2^{m-r}}$ except if $n > 1 = r$ when $D \cong \text{Aut } K \cong C_2 \times C_{2^{m-2}}$;
(3) If $n \neq r$, then $F \cong \Omega_{m-r}(K) \cong C_{2^{\text{min}(m-r,n)}}$;
(4) Assume that $n = r$. Then $T \cong \Omega_{m-r}(K) \cong C_{2^{\text{min}(m-r,n)}}$ except if $r = 2$ when $T \cong C_2 \times C_2$.

We define automorphisms of $G$ on generators as follows

$x^{a_1} = x^{-1}$, $x^{a_2} = x^5$, $y^{a_1} = y^{a_2} = y$;

$x^b = x$, $y^b = \begin{cases} xy, & n \geq m \\ x^{2^{m-n}} y, & n < m \end{cases}$;

$x^c = \begin{cases} xy, & m - r \geq n \\ xy^{2^{m-r}+r} y, & m - r < n \end{cases}$, $y^c = y$.

Now we assume that $n \neq r$ and $r \geq 2$. In this case we define

$x^d = x$, $y^d = y^{1+2^r}$.

By Theorem 3.6, 3.9 and Lemma 3.8 it is clear that $\text{Aut } G = FABD$ and each automorphism $\varphi$ of $G$ can be presented uniquely as $\varphi = \alpha \beta \gamma \delta$, where $\alpha \in F, \beta \in A, \gamma \in B, \delta \in D$. It is clear that $A = \langle a_1, a_2 \rangle$, $B = \langle b \rangle$, $D = \langle d \rangle$ and $AD$ is abelian. It is evident that $G = HK = KH$, so if $g \in G$, then $g = kh$ for some $k \in K, h \in H$. In the proof of Lemma 3.11 (2) we will use this reverse notation of elements of $G$. 
We define $i, j, k, s, t, u, w, z$ are such that

$i = 0$ in (I), $5^i = 1 + 2^{m-r}$ mod $2^m$ in (II), $5^i = 1 + 2^m$ mod $2^m$ in (III),

$j = 0$ in (I), $5^j = 1 - 2^{m-r+1}$ mod $2^m$ in (II),

$5^i = 1 - 2^{n+1}$ mod $2^m$ in (III),

$k = 1 + 2r + 2^{m-1}$ in (I), $k = 1 + 2^r$ in (II)&(III),

$u = 1 - 2^{m+s}$ in (I), $u = 1 - 2^{m-n}$ in (II), $u = 1 - 2^s$ in (III),

$5^i = (1 - 2^{n-1}) u^{-1}$ mod $2^n$ in (I),

$5^i = (1 - 2^{m-r-n-1}) u^{-1}$ mod $2^m$ in (II),

$5^i = (1 - 2^{m-1}) u^{-1}$ mod $2^m$ in (III),

$s = u^{-1}$ mod $2^n$ in (I), $s = u^{-1}$ mod $2^m$ in (II)&(III),

$(1 + 2^r) w = u$ mod $2^n$,

$z = -2^{m-n+r} + 2^{n-1}$ in (I), $z = -2^{m-n} + 2^{m-r+1}$ in (II),

$z = -2^r + 2^{n-1}$ in (III).

**Lemma 3.11.** Let $a_1, a_2, b, c, d$ be as above. Assume that $n \neq r$ and $r \geq 2$.

Then

1. $e^{a_1} = c^{-1} a_1^2$, $e^{a_2} = c^5 a_2^2$, $e^d = c^{1+2r}$;
2. $b^{a_1} = b^{-1}$, $b^{a_2} = b^5$, $b^{d^{-1}} = b^k$;
3. $e^b = c^a b^d w^u$.

**Proof.** (1) Let $N = G'K$ and $M = [H, \Omega_{m-r}(K)]\Omega_{m-r}(K)$. Then $a_1, a_2, d \in N_{Aut(G)}(N) \cap N_{Aut(G)}(M)$, $c \in Aut\Omega N(G)$ and $h := x^{-1} x^2 \in M$. By Lemmas 2.1, 2.4, 3.7 and 3.3 we get $g^e = g(gN)^h$ ($g \in G$), $(x^i N)^h = h^{x^{-1} + \ldots + x^{i+1}}$ ($i \in \mathbb{N}$). By Lemma 2.8 (3) we get the last relation. Now we use Lemma 2.8 (1) to get the first two relations: in (I) we have $[h, x] = [y^{2^{m-r}}, x] = 1$; in (II) since $[h, x] = [y, x] = x^{-2^{m-n}}$, we obtain

$$((x^{a_1-1} N)^h a_1 = y^{-1} x^{2^{m-r} (2^m-1) (2^{m-1}-1)} = y^{-1} x^{2^{m-r}},$$

$$((x^{a_2} N)^h a_2^{-1} = y^5 x^{2^{m-r+1}};$$

in (III) since $[h, x] = [y^{2^{m-r}}, x] = x^{-2^r}$, by Lemma 2.8 (1) we obtain

$$((x^{a_1-1} N)^h a_1 = x^2 y^{2^{m-r}},$$

$$(x^{a_2} N)^h a_2^{-1} = x^{-2^{n+1}} y^5 x^{2^{m-r}}.$$}

(2) Note that $x^b = x$ and $y^b = y x^{1+2^{m-r}}$ in (I), $y^b = y x^{2^{m-n}}$ in (II),

$y^b = y x^{2^{m-n} + 2^{m-r}}$ in (III). Let $Q = \langle x \rangle$. Then $a_1, a_2, d \in N_{Aut(G)}(Q)$, $b \in Aut\Omega Q(G)$ and $h := y^{-1} y^b \in Q$. By Lemmas 2.1, 2.4, 3.7 and 3.3 we get $g^h = g(gQ)^h$ ($g \in G$), $(y^Q)^h = h^{y^{-1} + \ldots + y^{i+1}}$ ($i \in \mathbb{N}$). By Lemma 2.8 (3)
we obtain the first two relations. Now we use Lemma 2.8 (2) to get the last relation: in (I) since \([h, y] = [x^{1+2m-n}, y] = x^{2m-r(1+2m-r)}\) we obtain
\[
((y^dN)^\lambda)^{-1} = (x^{1+2m-r})^{1+2r} x^{2m-r(1+2m-r)2r^{-1}(2r+1)} = x^{(1+2m-r)(1+2r+2m-r)};
\]
in (II) we get \([h, y] = [x^{2m-n}, y] = 1\); in (III) since \([h, y] = [x^{2m-n+22m-n-r}, y]\) we obtain
\[
((y^dN)^\lambda)^{-1} = (x^{2m-n+22m-n-r})^{1+2r} x^{2m-r(2m-n+22m-n-r)(2r+1)2r^{-1}} = (x^{2m-n+22m-n-r})^{1+2r}.
\]

(3) The direct computations with the help of Lemma 3.1 give the relation.

\[\square\]

**Theorem 3.12.** Let \(G\) be as above and \(m \geq 3\), \(n \geq 1\), \(1 \leq r \leq \min\{m - 2, n\}\), \(n \neq r\) and \(r \geq 2\). Then \(\text{Aut} G\) can be given by the following presentation, where the relations with commuting generators are omitted:
\[
\text{Aut} G = \langle a_1, a_2, b, c, d \mid a_1^2 = a_2^{2m-2} = b_{2n}^{2m-n} = c_{2m-n}^2 = d_{2m-n}^2 = 1, c^{a_1} = c^{-1}a_1, c^{a_2} = c^{5}a_2, c^{d} = c^{1+2r}, b^{a_1} = b^{-1}, b^{a_2} = b^5, b^{d} = b^k, c^{b} = c^{a_2}b^d d^e \rangle.
\]

4. Metacyclic minimal nonabelian 2-groups

In this section we will deal with groups \(G = \langle x, y \mid x^{2m} = y^{2n} = 1, x^y = x^{1+2m-1} \rangle\); where \(m, n \in \mathbb{N}\), \(m \geq 2, n \geq 1\). So \(G = H \rtimes K\) is a split metacyclic 2-group, where \(H = \langle x \rangle\) and \(K = \langle y \rangle\).

First assume that \(n \geq m \geq 3\). We define automorphisms of \(G\) on generators as follows
\[
x^{a_1} = x^{-1}, \quad x^{a_2} = x^5, \quad y^{a_1} = y^{a_2} = y;
\]
\[
x^b = x, \quad y^b = \begin{cases} x^y, & m \geq n \\ x^{2^{m-n}} y, & n < m \end{cases};
\]
\[
x^c = \begin{cases} x^y, & m > n \\ x^{2^{m-n+1}} y, & m \leq n \end{cases}, \quad y^c = y;
\]
\[
x^{d_1} = x^{d_2} = x, \quad y^{d_1} = y^{-1}, \quad y^{d_2} = y^5.
\]

By Theorems 3.6, 3.9 and Lemma 3.8 it is clear that \(\text{Aut} G = FABD\) and each automorphism \(\varphi\) of \(G\) can be presented uniquely as \(\varphi = \alpha \beta \gamma \delta\), where \(\alpha \in F, \beta \in A, \gamma \in B, \delta \in D\). It is clear that \(A = \langle a_1, a_2 \rangle, \quad B = \langle b \rangle, \quad D = \langle d_1, d_2 \rangle\) and \(AD\) is abelian. It is evident that \(G = HK = KH\), so if \(g \in G\), then \(g = kh\) for some \(k \in K, h \in H\). In the proof of Lemma 4.1(2) we will use also this reverse notation of elements of \(G\).
Lemma 4.1. Let \( a_1, a_2, b, c, d_1, d_2 \) be as above. Assume that \( m \geq 3, n \geq 3 \).

Then

1. \( c^{a_1} = c^{-1} a_2 \), \( c^{a_2} = c^5, c^{d_1} = c^{-1}, c^{d_2} = c^5 \), where \( i = 0 \) when \( m > n \) and \( i = 2m-3 \) when \( m \leq n \);
2. \( b^{a_1} = b^{-1}, b^{a_2} = b^5, b^{d_1} = b^{-1}, b^{d_2} = b^5 \);
3. if \( n-m \geq 1 \), then \( c^b = c^s a_2 b^{-2^{m-n+1}} d_2 w \), where \( s, t, w \) are such that
   \( s = 5^t = (1 - 2^{m-n+1})^{-1} \mod 2m, 5^w = 1 - 2^{m-n+1} \mod 2n \);
4. if \( m = n, \) then \( c^b = c^{-1} a_1 a_2^{2m-3} b^{-2+2^{m-n-1}} d_1 \);
5. if \( m-n > 1, \) then \( c^b = c^s a_2 b^{-2^{m-n}} d_2 \), where \( s, t, w \) are such that
   \( s = 5^t = (1 - 2^{m-n})^{-1} \mod 2m, 5^w = 1 - 2^{m-n} \mod 2n \);
6. if \( m = n+1, \) then \( c^b = c^{-1} a_1 a_2^{2m-3} b^{-2+2^{m-2}} d_1 \).

Proof. (1) Let \( N = G' K \) and \( M = [H, \Omega_{m-r}(K)] \Omega_{m-r}(K) \). Then \( a_k, d_k \in N_{Aut G}(N) \cap N_{Aut G}(M) \) \((k = 1, 2, c \in Aut N(G) \) and \( h := x^{-1} x^5 \in M \). By Lemmas 2.1, 2.4, 3.7 and 3.3 we get \( g^5 = g(N)^{b_1} \langle g \rangle \), \((x^4) N \rangle \) = \( h^2 x^2 \cdots x \) (i \( \in \mathbb{N} \)). For the first two relations see the proof of Lemma 3.11 (1) with \( r = 1 \). By Lemma 2.8 (3) we obtain the last two relations.

(2) Note that \( x^b = x \) and \( y^b = y x^1 + 2^{m-1} \) when \( m \geq n, \) \( y^b = y x^{2^{m-n}} \) when \( m > n \). Let \( Q = \langle x \rangle \). Then \( a_k, d_k \in N_{Aut G}(Q) \) \((k = 1, 2, c \in Aut Q(G) \) and \( y^{-1} y^b \in Q \). By Lemmas 2.1, 2.4, 3.7 and 3.3 we get \( g^b = g(N)^{b_1} \langle g \rangle \), \((y^q) N \rangle \) = \( h^q x^2 \cdots x \) (i \( \in \mathbb{N} \)). By Lemma 2.8 (3) we obtain the first two relations. Now we will use Lemma 2.8 (2) to get the last two relations: if \( m > n \) then \[ y^{2^{m-n}} y \] = \( 1 \), so we get the last two relations; if \( m \leq n, \) then \[ x^{1+2^{m-n}} y \] = \( x^{2^{m-1}} \) and we get \[ (y^{d_1} N)^{b_1} \] = \( x^{1+2^{m-n}} x^{2^{m-n} - 1} x^{2^{m-n} - 1} \) = \( x^{-1} \) and \[ (y^{d_2} N)^{b_2} \] = \( x^{1+2^{m-1}} x^{2^{m-1} - 10} = (x^{1+2^{m-1}})^5 \).

(3)-(6) The direct computations with the help of Lemma 3.1 give the relations.

In the next theorems the relations with commuting generators are omitted.

Theorem 4.2. Let \( G \) be as above and \( m, n \geq 3 \). Then \( Aut G \) can be given by the following presentation: 
\[ Aut G = \langle a_1, a_2, b, c, d_1, d_2 | a_1^2 = a_2^{2m-2} = b^{2^{min(m,n)}} = c^{2^{min(n,m)}} = d^{2^{m-n}} = 1, c^{d_1} = c^{-1} a_2, c^{d_2} = c^{-1}, c^{d_2} = c^5, b^{d_1} = b^{-1}, b^{d_2} = b^5, b^{d_1} = b^{-1}, b^{d_2} = b^5, c^{d_1} = c^5, c^{d_2} = c \rangle, \] where \( i \) is given in Lemma 4.1 and \( \alpha \) is the appropriate relation in (3)-(4) of Lemma 4.1.

If \( m = 2 \) and \( n = 1 \), then \( G \cong Aut G \) is dihedral of order 8.
Now assume that $m > n = 2$. We define automorphisms of $G$ on generators as follows
\[ x^{a_1} = x^{-1}, \quad x^{a_2} = x^{5}, \quad y^{a_1} = y^{a_2} = y; \quad x^b = x, \quad y^b = x^{2m-2}y; \]
\[ x^c = xy, \quad y^c = y; \quad x^d = x, \quad y^d = y^{-1}. \]

**Theorem 4.3.** Let $G$ be as above and $m > n = 2$. Then $\text{Aut} G$ can be given by the following presentation:

1. If $m > 3$, then $\text{Aut} G = \langle a_1, a_2, b, c, d | a_1^2 = a_2^{2m-2} = b^4 = c^4 = d^2 = 1, c^{a_1} = c^{-1}a_2^{2m-3}, c^d = c^{-1}, b^{a_1} = b^{-1}, b^d = b^{-1}, b^c = ba_1^2 \rangle$, where $5^m = 1 - 2^{1-m}$.
2. If $m = 3$, then $\text{Aut} G = \langle a_1, a_2, b, c, d | a_1^2 = a_2^2 = b^4 = c^4 = d^2 = 1, c^{a_1} = c^{-1}a_2, c^d = c^{-1}, b^{a_1} = b^{-1}, b^d = b^{-1}, c^b = c^{-1}a_1a_2d \rangle$.

Now assume that $m \geq 3$, $n = 1$. We define automorphisms of $G$ on generators as follows
\[ x^{a_1} = x^{-1}, \quad x^{a_2} = x^{5}, \quad y^{a_1} = y^{a_2} = y; \]
\[ x^b = x, \quad y^b = x^{2m-1}y; \quad x^c = xy, \quad y^c = y. \]

**Theorem 4.4.** Let $G$ be as above and $m \geq 3$, $n = 1$. Then $\text{Aut} G$ can be given by the following presentation:
\[ \text{Aut} G = \langle a_1, a_2, b, c | a_1^2 = a_2^{2m-2} = b^2 = c^2 = 1, c^{a_1} = ca_2^{2m-3}, c^b = ca_2^{2m-3} \rangle. \]

Now assume that $m = n = 2$. We define automorphisms of $G$ on generators as follows
\[ x^a = x^{-1}, \quad y^a = y; \quad x^b = x, \quad y^b = xy; \]
\[ x^c = xy^2, \quad y^c = y; \quad x^d = x, \quad y^d = y^{-1}. \]

**Theorem 4.5.** Let $G$ be as above and $m = n = 2$. Then $\text{Aut} G$ can be given by the following presentation:
\[ \text{Aut} G = \langle a, b, c, d | a^2 = b^4 = c^2 = d^2 = 1, b^a = b^{-1}, b^c = bd \rangle. \]

**References**


