General rotational surfaces in Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map

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Abstract. In this paper, we study general rotational surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map. We consider general rotational surfaces in $\mathbb{E}^4$ whose meridian curves lie in two-dimensional planes. We firstly obtain all general rotational surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind. Then we classify minimal rotational surfaces of $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the second kind.

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1. Introduction

In late 1970’s, B. Y. Chen introduced the notion of finite type immersion into a Euclidean space. Since then many works have been written in this field (see [5, 6], etc.). A submanifold $M$ of a Euclidean space $\mathbb{E}^m$ is said to be of finite type if its position vector $x$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, that is, $x = x_0 + x_1 + \cdots + x_k$, where $x_0$ is a constant map, $x_1, \ldots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then $M$ is said to be of $k$-type. In [7], Chen and Piccinni similarly extended this definition to differentiable maps, in particular, to Gauss maps of submanifolds. They made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map, and for hypersurfaces they proved that a compact hypersurface $M$ of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if $M$ is a hypersphere in $\mathbb{E}^{n+1}$. Also, many geometers studied submanifolds with finite type Gauss map ([2, 3, 4, 7, 20] etc.).

If a submanifold $M$ of a Euclidean space has 1-type Gauss map $\nu$, then $\Delta \nu = \lambda (\nu + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$. However, the Laplacian of the Gauss map of several surfaces such as helicoid, catenoid and right cones in $\mathbb{E}^3$, and also some hypersurfaces take the form

$$\Delta \nu = f(\nu + C)$$  \hspace{1cm} (1)

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for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold of a Euclidean space is said to have pointwise 1-type Gauss map if it satisfies (1). A submanifold with pointwise 1-type Gauss map is said to be of the first kind if $C$ is the zero vector. Otherwise, it is said to be of the second kind. A pointwise 1-type Gauss map is called proper if the function $f$ defined by (1) is non-constant. A non-proper pointwise 1-type Gauss map is just usual 1-type Gauss map.

**Remark 1.** For an $n$-dimensional plane $M$ in a Euclidean space, the Gauss map $\nu$ is constant and $\Delta \nu = 0$. For $f = 0$ if we write $\Delta \nu = 0 \cdot \nu$, then $M$ has pointwise 1-type Gauss map of the first kind. If we choose $C = -\nu$ for any nonzero smooth function $f$, then (1) holds. In this case, $M$ has pointwise 1-type Gauss map of the second kind. Therefore we say that an $n$-dimensional plane $M$ in a Euclidean space is a trivial submanifold with pointwise 1-type Gauss map of the first kind or the second kind.

Surfaces and some hypersurfaces in Euclidean spaces with pointwise 1-type Gauss map were recently studied in [1, 8, 9, 10, 12, 13, 14, 15, 17]. In [14], the characterizations of surfaces in the Euclidean space $E^4$ with pointwise 1-type Gauss map were given. Also, in [1], simple rotational surfaces in $E^4$ whose meridian curves lie in 3-spaces were considered, and the meridian curve of the flat rotational surfaces with pointwise 1-type Gauss map of the second kind was described.

In this paper, we study general rotational surfaces in $E^4$ with meridian curves lying in two-dimensional planes and pointwise 1-type Gauss map. We firstly prove that there exists no non-planar minimal general rotational surface with pointwise 1-type Gauss map of the first kind. Then we obtain all general rotational surfaces with proper pointwise 1-type Gauss map of the first kind which includes the results given in [19]. We also classify minimal general rotational surfaces of $E^4$ with proper pointwise 1-type Gauss map of the second kind.

2. Preliminaries

Let $M$ be an oriented $n$-dimensional submanifold in an $(n+2)$-dimensional Euclidean space $E^{n+2}$. We choose an oriented local orthonormal frame $\{e_1, \ldots, e_{n+2}\}$ on $M$ such that $e_1, \ldots, e_n$ are tangent to $M$ and $e_{n+1}, e_{n+2}$ are normal to $M$. We use the following convention on the range of indices: $1 \leq i, j, k, \ldots \leq n$, $n + 1 \leq r, s, t, \ldots \leq n+2$.

Let $\tilde{\nabla}$ be the Levi-Civita connection of $E^{n+2}$ and $\nabla$ the induced connection on $M$. Denote by $\{w^1, \ldots, w^{n+2}\}$ the dual frame and by $\{w^A_B\}, A, B = 1, \ldots, n+2$, the connection forms associated to $\{e_1, \ldots, e_{n+2}\}$. Then we have

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} w^j_i (e_k) e_j + \sum_{r=n+1}^{n+2} h^r_{ik} e_r, \quad \tilde{\nabla}_{e_k} e_s = -A_s (e_k) + \sum_{r=n+1}^{n+2} w^r_s (e_k) e_r$$

and

$$D_{e_k} e_s = \sum_{r=n+1}^{n+2} w^r_s (e_k) e_r,$$
where \( D \) is the normal connection, \( h_{ij}^r \) the coefficients of the second fundamental form \( h \), and \( A_r \) the Weingarten map in the direction \( e_r \).

The mean curvature vector \( H \) and the squared length \( \| h \|^2 \) of the second fundamental form \( h \) are defined, respectively, by

\[
H = \frac{1}{n} \sum_{r,i} h_{ij}^r e_r \quad \text{and} \quad \| h \|^2 = \sum_{r,i,j} h_{ij}^r h_{ji}^r .
\]

A submanifold \( M \) is said to have parallel mean curvature vector if the mean curvature vector satisfies \( DH = 0 \) identically.

The Codazzi equation of \( M \) in \( \mathbb{E}^{n+2} \) is given by

\[
h_{ij,k}^r = h_{jk,i}^r ,
\]

\[
h_{jk,i}^r = e_i(h_{jk}^r) + \sum_{s=n+1}^{n+2} h_{jk}^s w_s^r(e_i) - \sum_{\ell=1}^{n} (w_{j}^\ell(e_i) h_{\ell k}^r + w_{k}^\ell(e_i) h_{j \ell}^r) .
\]

Also, from the Ricci equation of \( M \) in \( \mathbb{E}^{n+2} \), we have

\[
R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_j}](e_j), e_k \rangle = \sum_{i=1}^{n} (h_{ik}^r h_{ij}^s - h_{ij}^r h_{ik}^s) ,
\]

where \( R^D \) is the normal curvature tensor.

Let \( M \) be an oriented \( n \)-dimensional submanifold of a Euclidean space \( \mathbb{E}^m \). The map \( \nu : M \to G(m-n, m) \) defined by \( \nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_m)(p) \) is called the Gauss map of \( M \) that is a smooth map which carries a point \( p \in M \) into the oriented \((m-n)\)-plane in \( \mathbb{E}^m \) which is obtained from the parallel translation of the normal space of \( M \) at \( p \) in \( \mathbb{E}^m \), where \( G(m-n, m) \) denotes the Grassmannian manifold consisting of all oriented \((m-n)\)-planes through the origin of \( \mathbb{E}^m \). Since \( G(m-n, m) \) is canonically embedded in \( \bigwedge^{m-n} \mathbb{E}^m = \mathbb{E}^{N} \), \( N = (m-n) \), then the notion of the type of the Gauss map is naturally defined.

### 2.1. General rotational surfaces

In [16], Moore introduced general rotational surfaces in the Euclidean space \( \mathbb{E}^4 \). A rotational surface in \( \mathbb{E}^4 \) is a surface left invariant by a rotation in \( \mathbb{E}^4 \) which is defined as a linear transformation of positive determinant preserving distance and leaving one point fixed.

In [11], Cole studied the general theory of rotations in \( \mathbb{E}^4 \). In \( \mathbb{E}^4 \), two planes which have no line in common are called completely (or absolutely) perpendicular to each other. A rotation in general leaves two completely perpendicular planes invariant not fixed point for point, but only as planes. A rotation which leaves one of the invariant planes fixed point for point and converts the other invariant plane to itself is called a simple rotation. In general, every general rotation (also called double rotation) of \( \mathbb{E}^4 \) can be reduced to a succession of two simple rotations whose fixed planes are completely perpendicular to each other (for details see [11]). By a suitable isometry of \( \mathbb{E}^4 \), two completely perpendicular planes at a point in \( \mathbb{E}^4 \) can be transformed to completely perpendicular \( xy \) - and \( zw \) -planes at the origin of \( \mathbb{E}^4 \).
Let $\beta(s) = (x(s), y(s), z(s), w(s))$ be a regular smooth curve in $\mathbb{E}^4$ on an open interval $I$ in $\mathbb{R}$, and let $a$ and $b$ be some real numbers. Then, considering the equations of the general rotation given in [11], a general rotational surface $M$ in $\mathbb{E}^4$ with the meridian curve $\beta$ is given by

$$X(s, t) = \left( x(s) \cos at - y(s) \sin at, x(s) \sin at + y(s) \cos at, 
\begin{array}{l}
z(s) \cos bt - w(s) \sin bt, \\
z(s) \sin bt + w(s) \cos bt
\end{array}\right),$$  \hspace{1cm} (4)

where $a$ and $b$ are the rates of rotation in fixed planes of the rotation, [16]. If $a$ or $b$ is zero, then a surface $M$ defined by (4) is a simple rotational surface as the rotation subgroup which produces $M$ is a simple rotation.

Let $M$ be a general rotational surface in $\mathbb{E}^4$ whose meridians lie in 2-planes. Then these planes of meridians are perpendicular to the two fixed planes of the rotation that generates the surface $M$. If $M$ with planar meridians is parametrized by (4), then the planes of meridians are perpendicular to the invariant $xy$- and $zw$-planes of the rotation which generates the surface $M$. Therefore, without loss of generality, we can choose a meridian curve $\beta$ of $M$ in the $xz$-plane as $\beta(s) = (x(s), 0, z(s), 0)$, and thus a general rotational surface $M$ in $\mathbb{E}^4$ whose meridians lie in 2-planes is given by the parametrization

$$F(s, t) = (x(s) \cos at, x(s) \sin at, z(s) \cos bt, z(s) \sin bt)$$  \hspace{1cm} (5)

with the rates of rotation $a$ and $b$, where $s \in I \subset \mathbb{R}$, $t \in (0, 2\pi)$. Throughout this work we suppose that $ab \neq 0$. Since $\beta$ is a regular smooth curve, parametrization (5) is an immersion if and only if $a^2x^2(s) + b^2z^2(s) > 0$ on $I$.

Moreover, a rotational surface in $\mathbb{E}^4$ defined by (5) for $a = b = 1$, $x(s) = u(s) \cos s$ and $z(s) = u(s) \sin s$ is also known as a Vranceanu rotational surface [18], where $u$ is a differentiable function.

From now on, since $\beta$ is a plane curve, without loss of generality, we consider $\beta$ of the form $\beta(s) = (x(s), z(s))$ on some open interval $I$.

Suppose that $s$ is the arc length parameter of $\beta$. Then, $x'^2 + z'^2 = 1$, and the curvature function $\kappa$ of $\beta$ is given by $\kappa(s) = z''(s)x' - x''(s)z'$. If

Let $M$ be a general rotational surface in $\mathbb{E}^4$ defined by (5). We consider the following orthonormal moving frame field $\{e_1, e_2, e_3, e_4\}$ on $M$ such that $e_1, e_2$ are tangent to $M$, and $e_3, e_4$ are normal to $M$:

$$e_1 = \frac{\partial}{\partial s}, \hspace{0.5cm} e_2 = \frac{1}{\sqrt{a^2x^2 + b^2z^2}} \frac{\partial}{\partial t},$$  \hspace{1cm} (6)

$$e_3 = (-z' \cos at, -z' \sin at, x' \cos bt, x' \sin bt),$$  \hspace{1cm} (7)

$$e_4 = \frac{1}{\sqrt{a^2x^2 + b^2z^2}} (-bz \sin at, bz \cos at, ax \sin bt, -ax \cos bt).$$  \hspace{1cm} (8)

By a direct computation we have components of the second fundamental form and
the connection forms as
\[
\begin{align*}
    h_{11}^3 &= \kappa, & h_{22}^3 &= \frac{a^2xz' - b^2zx'}{a^2x^2 + b^2z^2}, & h_{12}^3 &= 0, \\
    h_{12}^4 &= \frac{ab(zx' - x'z)}{a^2x^2 + b^2z^2}, & h_{22}^4 &= h_{12}^4 = 0, \\
    w_1^2(e_1) &= 0, & w_1^2(e_2) &= \frac{a^2xx' + b^2zz'}{a^2x^2 + b^2z^2}, \\
    w_1^3(e_1) &= 0, & w_1^3(e_2) &= \frac{ab(xx' + zz')}{a^2x^2 + b^2z^2}.
\end{align*}
\]
Thus, the shape operators of \( M \) are of the form
\[
A_3 = \begin{pmatrix} h_{11}^3 & 0 \\ 0 & h_{22}^3 \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & h_{12}^4 \\ h_{12}^4 & 0 \end{pmatrix},
\]
from which we obtain the mean curvature vector and the normal curvature of \( M \) as follows:
\[
H = \frac{1}{2}(h_{11}^3 + h_{22}^3)e_3, \quad R^D(e_1, e_2; e_3, e_4) = h_{12}^4(h_{22}^3 - h_{11}^3).
\]
On the other hand, from Codazzi equation (2) we have
\[
\begin{align*}
    e_1(h_{22}^3) &= w_1^2(e_2)(h_{22}^3 - \kappa) + h_{12}^4w_1^3(e_2), \\
    e_1(h_{12}^4) &= 2w_1^3(e_2)h_{12}^4 - \kappa w_1^3(e_2).
\end{align*}
\]

**Remark 2.** If the meridian curve \( \beta \) of \( M \) is a line \( z = c_0x \) passing through the origin, and the rates of rotation \( a \) and \( b \) hold \( a^2 = b^2 \), then the rotational surface \( M \) is given by \( F(x, t) = (x \cos t, x \sin t, c_0x \cos t, \varepsilon c_0x \sin t) \), \( x > 0 \), \( \varepsilon = a/b = \pm 1 \). It can be easily shown that \( M \) is an open part of a plane in \( \mathbb{E}^4 \).

### 3. General rotational surfaces with pointwise 1-type Gauss map of the first kind

In this section, we obtain all general rotational surfaces defined by (5) with pointwise 1-type Gauss map of the first kind.

The Laplacian of the Gauss map \( \nu \) for an \( n \)-dimensional submanifold \( M \) in the Euclidean space \( \mathbb{E}^{n+2} \) was given by

**Lemma 1** (See [14]). Let \( M \) be an \( n \)-dimensional submanifold of Euclidean space \( \mathbb{E}^{n+2} \). Then, the Laplacian of the Gauss map \( \nu = e_{n+1} \wedge e_{n+2} \) is given by

\[
\Delta \nu = \| h \|^2 \nu + 2 \sum_{j<k} R^D(e_j, e_k; e_{n+1}, e_{n+2})e_j \wedge e_k + n \sum_{j=1}^n w_{n+2}^j(e_j) \wedge H + \nabla(\text{tr} A_{n+1}) \wedge e_{n+2} - \nabla(\text{tr} A_{n+2}) \wedge e_{n+1},
\]

\[\text{(18)}\]
where $\|h\|^2$ is the squared length of the second fundamental form, $R^D$ the normal curvature tensor, and $\nabla(\text{tr}A_r)$ the gradient of $\text{tr}A_r$.

In [14], the following results were given for the characterization of surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind.

**Theorem 1** (See [14]). An oriented minimal surface $M$ in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if $M$ has flat normal bundle.

**Theorem 2** (See [14]). An oriented non-minimal surface $M$ in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if $M$ has parallel mean curvature vector in $\mathbb{E}^4$.

We will classify rotational surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind by using the above theorems.

**Theorem 3.** Let $M$ be a general rotational surface in $\mathbb{E}^4$ defined by (5) for the rates of rotation $a$ and $b$. Then, $M$ is minimal, and its normal bundle is flat if and only if $M$ is an open part of a plane.

**Proof.** Let $M$ be a general rotational surface given by (5). Then, we have an orthonormal moving frame $\{e_1,e_2,e_3,e_4\}$ on $M$ in $\mathbb{E}^4$ given by (6)-(8), and the shape operators $A_3$ and $A_4$ are given by (13). If $M$ is minimal, and its normal bundle is flat, then (14) and (15) imply, respectively,

$$\kappa + h_{12}^3 = 0,$$

$$h_{12}^4(h_{22}^3 - \kappa) = 0,$$

as $h_{11}^3 = \kappa$, where $\kappa$ is the curvature of the meridian curve of $M$. By using these equations we get $h_{12}^3 = 0$ which implies either $\kappa = 0$ or $h_{22}^3 = 0$.

**Case 1.** $\kappa = 0$. Then the meridian curve of $M$ is a line. We may put

$$x(s) = x_1 s + x_2, \quad z(s) = z_1 s + z_2$$

for some constants $x_1$, $x_2$, $z_1$, $z_2$ with $x_1^2 + z_1^2 = 1$. From (19) we also have $h_{22}^3 = 0$. By using the second equation in (9) and (21) we obtain

$$h_{22}^3 = \frac{(a^2-b^2)x_1z_1 s + (a^2x_2z_1 - b^2x_1z_2)}{a^2(x_1 s + x_2)^2 + b^2(z_1 s + z_2)^2} = 0$$

which yields

$$(a^2 - b^2)x_1z_1 = 0,$$

$$a^2x_2z_1 - b^2x_1z_2 = 0.$$
Now, assume that \( x_1 z_1 \neq 0 \) and \( a^2 - b^2 = 0 \). Then, (23) implies \( x_2 z_1 = x_1 z_2 \) from which and (21) we get \( x_1 z = z_1 x \), i.e., line (21) is passing through the origin. In view of Remark 2, \( M \) is an open part of a plane.

**Case 2.** \( h_{12}^1 = 0 \). From the first equation in (10) we have \( x' z - x z' = 0 \), i.e., \( z = c_0 x \), where \( c_0 \) is a constant. Hence, \( \beta \) is an open part of a line passing through the origin. Therefore \( M \) is an open part of a plane because of Remark 2.

The converse of the proof is trivial. \( \square \)

From Theorem 1 and Theorem 3 we state

**Theorem 4.** There exists no non-planar minimal general rotational surface in \( \mathbb{E}^4 \) defined by (5) with pointwise 1-type Gauss map of the first kind.

In [20], Yoon studied flat Vranceanu rotational surfaces in \( \mathbb{E}^4 \) with pointwise 1-type Gauss map of the first kind. He proved that a flat Vranceanu rotational surface \( M \) in \( \mathbb{E}^4 \) has pointwise 1-type Gauss map of the first kind if and only if \( M \) is a Clifford torus in \( \mathbb{E}^4 \), that is, the product of two plane circles with the same radius.

Now we investigate non-minimal general rotational surfaces in \( \mathbb{E}^4 \) with parallel mean curvature vector to obtain surfaces in \( \mathbb{E}^4 \) with proper pointwise 1-type Gauss map of the first kind. For this reason we prove

**Theorem 5.** A non-minimal general rotational surface \( M \) in \( \mathbb{E}^4 \) defined by (5) has parallel mean curvature vector if and only if it is an open part of the surface defined by

\[
F(s, t) = \left( r_0 \cos\left( \frac{s}{r_0} \right) \cos at, r_0 \cos\left( \frac{s}{r_0} \right) \sin bt, r_0 \sin\left( \frac{s}{r_0} \right) \cos at, r_0 \sin\left( \frac{s}{r_0} \right) \sin bt \right) \quad (24)
\]

which is minimal in \( S^3(r_0) \subset \mathbb{E}^4 \).

**Proof.** Let \( M \) be a non-minimal general rotational surface in \( \mathbb{E}^4 \) defined by (5). Let \( \{e_1, e_2, e_3, e_4\} \) be an orthonormal moving frame on \( M \) in \( \mathbb{E}^4 \) given by (6)-(8). From (13) we have \( H = \frac{1}{2} (h_{11}^3 + h_{22}^3) e_3 \). Suppose that the mean curvature vector \( H \) is parallel, i.e., \( DH = 0 \). By considering (12), we obtain that

\[
D_{e_4} H = - \frac{a b (h_{11}^3 + h_{22}^3)(xx' + zz')}{2(a^2 x^2 + b^2 z^2)} e_4 = 0.
\]

Since \( M \) is non-minimal, this equation yields \( xx' + zz' = 0 \), i.e., \( x^2 + z^2 = r_0^2 \), where \( r_0 \) is a positive real number. Hence, the meridian curve \( \beta \) is an open part of a circle which is parametrized by

\[
x(s) = r_0 \cos \frac{s}{r_0}, \quad z(s) = r_0 \sin \frac{s}{r_0}.
\]

Therefore, \( M \) is an open part of the surface given by (24).

The converse follows from a direct calculation. \( \square \)

By Theorem 2 and Theorem 5 we have
Corollary 1. A non-minimal general rotational surface \( M \) in \( \mathbb{E}^4 \) defined by (5) has pointwise 1-type Gauss map of the first kind if and only if it is an open part of the surface given by (24).

By computation we have
\[
\|h\|^2 = \text{tr}(A_3)^2 + \text{tr}(A_4)^2 = \frac{2}{r_0^2} \left( 1 + \frac{a^2 b^2}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \right)
\]
for the rotational surface (24).

By combining the results obtained in this section we state a classification theorem:

**Theorem 6.** Let \( M \) be a general rotational surface in \( \mathbb{E}^4 \) defined by (5). Then \( M \) has pointwise 1-type Gauss map of the first kind if and only if \( M \) is an open part of a plane or a surface given by (24). Moreover, the Gauss map \( \nu = e_3 \wedge e_4 \) of the rotational surface (24) satisfies (1) for the function
\[
f = \frac{2}{r_0^2} \left( 1 + \frac{a^2 b^2}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \right).
\]

**Corollary 2.** The only general rotational surface \( M \) in \( \mathbb{E}^4 \) defined by (5) with proper pointwise 1-type Gauss map of the first kind is the surface given by (24) for \( a^2 \neq b^2 \).

In particular, if the rates of rotation \( a \) and \( b \) in (5) meet \( a^2 = b^2 \), then the rotational surface (24) is a Clifford torus in \( \mathbb{E}^4 \) which has (global) 1-type Gauss map of the first kind studied in [19, 20].

**4. Minimal general rotational surfaces with pointwise 1-type Gauss map of the second kind**

In [16], Moore proved that a general rotational surface \( M \) defined by (5) for \( a = b = 1 \) is minimal if and only if its meridian curve is an open part of the hyperbola
\[
c_1(z^2 - x^2) + 2xz + c_2 = 0,
\]
where \( c_1 \) and \( c_2 \) are some real numbers. A direct calculation shows that this result still holds if \( a^2 = b^2 \).

In [14], a characterization of minimal surfaces in \( \mathbb{E}^4 \) with pointwise 1-type Gauss map of the second kind was given as follows:

**Theorem 7** (See [14]). A non-planar minimal oriented surface \( M \) in the Euclidean space \( \mathbb{E}^4 \) has pointwise 1-type Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame \( \{e_1, e_2, e_3, e_4\} \) on \( M \), the shape operators of \( M \) are given by
\[
A_3 = \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & \varepsilon \rho \\ \varepsilon \rho & 0 \end{pmatrix},
\]
where \( \varepsilon = \pm 1 \) and \( \rho \) is a smooth non-zero function on \( M \).
By using Theorem 7 we classify non-planar minimal general rotational surfaces in $E^4$ defined by (5) with pointwise 1-type Gauss map of the second kind.

**Theorem 8.** Let $M$ be a non-planar general rotational surface in $E^4$ defined by (5) for the rates of rotation $a$ and $b$. Then,

1. if $a^2 = b^2$, then the minimal surface $M$ whose meridian curve is given by (25) has proper pointwise 1-type Gauss map of the second kind.

2. if $a^2 \neq b^2$, then $M$ is minimal and its Gauss map is of pointwise 1-type of the second kind if and only if the meridian curve of $M$ is given by

$$z = cx^{\mp b/a}, \quad x > 0$$

(27)

for some real number $c \neq 0$.

**Proof.** Let $M$ be a non-planar general rotational surface in $E^4$ defined by (5). Then we have an orthonormal moving frame $\{ e_1, e_2, e_3, e_4 \}$ on $M$ in $E^4$ given by (6)-(8), and the shape operators $A_3$ and $A_4$ are given by (13). For $a^2 = b^2$, assume that $M$ is minimal. Thus we have $h_{11}^1 + h_{22}^1 = 0$ which gives the differential equation

$$x^2 z'' - z' x'' + \frac{xz' - zz'}{x^2 + z^2} = 0$$

that has a general solution given by (25). Also, from the second equation in (9) and the first equation in (10) we have $(h_{22}^1)^2 = (h_{11}^1)^2$. If we put $\rho = h_{11}^3$, then $h_{22}^2 = -\rho$ and $h_{11}^3 = \varepsilon \rho$, where $\varepsilon = \pm 1$. Thus, the shape operators $A_3$ and $A_4$ of $M$ are of the form (26). A direct calculation (or see the proof of Theorem 7) shows that the function $f$ satisfying (1) is given by $f = 8\rho^2 = 8\kappa^2$ as $\rho = h_{11}^3 = \kappa$ from (9). Since $\kappa$ is not constant for the hyperbola given by (25), $f$ is not a constant function. As a result $M$ has proper pointwise 1-type Gauss map of the second kind by Theorem 7. This gives case 1 of the theorem.

Now, for $a^2 \neq b^2$ assume that a non-planar general rotational surface $M$ in $E^4$ defined by (5) is minimal and its Gauss map $\nu = e_3 \wedge e_4$ is of pointwise 1-type of the second kind. Then, Theorem 7 implies that the shape operators $A_3$ and $A_4$ of $M$ are of the form (26). Hence we have $h_{11}^1 + h_{22}^1 = 0$ and $h_{12}^4 = \varepsilon h_{11}^3 = -\varepsilon h_{22}^3$, where $\varepsilon = \pm 1$.

From the second equation in (9) and the first equation in (10) it is seen that $h_{12}^3 = -\varepsilon h_{22}^3$ implies the differential equation $axz' = -\varepsilon bzx'$ as $a^2 \neq b^2$, and its solution gives (27).

Conversely, suppose that the meridian curve of the rotational surface $M$ is given by (27). We will show that the shape operators $A_3$ and $A_4$ of $M$ are of the form (26).

From (27) if we write $z = cx^{-eb/a}$, then we have $axz' = -\varepsilon bzx'$ from which, the second equation in (9) and the first equation in (10) it is seen that $h_{12}^3 = -\varepsilon h_{22}^3$. Now, let us show that the minimality condition holds, i.e., $h_{11}^3 + h_{22}^3 = 0$ or equivalently, $h_{11}^3 - \varepsilon h_{12}^3 = 0$. Using the second equation in (9) and the first equation in (10), the equation $h_{11}^3 - \varepsilon h_{12}^3 = 0$ produces the differential equation

$$x^2 z'' - z' x'' + \frac{\varepsilon ab(xz' - zz')}{a^2 x^2 + b^2 z^2} = 0$$
which is expressed as
\[
\frac{d}{ds} \left( \tan^{-1} \left( \frac{z'}{x'} \right) \right) + \varepsilon \frac{d}{ds} \left( \tan^{-1} \left( \frac{bz}{ax} \right) \right) = 0
\] (28)
because of \(x'^2 + z'^2 = 1\). Since \(\tan^{-1}\) is an odd function, it is easily seen that the equation \(axz' = -\varepsilon bz x'\) which produces (27) satisfies (28). That is, the minimality condition holds.

If we put \(\rho = h_{11}^3\), then \(h_{22}^3 = -\rho\) and \(h_{12}^4 = \varepsilon \rho\). Thus, the shape operators \(A_3\) and \(A_4\) are of the form (26). Therefore, \(M\) is minimal and its Gauss map is of pointwise 1-type of the second kind by Theorem 7. By a direct calculation it is easy to show that the Gauss map is of proper pointwise 1-type of the second kind. This completes the proof of case 2. \(\square\)

Here, using (25) and (27) we give two examples of a general rotational surface in \(E^4\) which are minimal and have proper pointwise 1-type Gauss map of the second kind.

**Example 1.** For \(c_1 = 0\) and \(c_2 = -1\) in (25) we have the hyperbola \(2xz = 1\) or equivalently \(x^2 - z^2 = 1\). Let \(x = \cosh u,\ z = \sinh u\) be the parametrization of the right-hand branch of the hyperbola \(x^2 - z^2 = 1\). Then, the general rotational surface \(M\) defined by
\[
F(u, t) = (\cosh u \cos t, \cosh u \sin t, \sinh u \cos t, \sinh u \sin t)
\]
is minimal in \(E^4\) with proper pointwise 1-type Gauss map of the second kind. Moreover, following the proof of Theorem 7, the Gauss map \(\nu = e_3 \wedge e_4\) satisfies (1) for the function \(f = 8 \text{sech}^3(2u)\) and for the constant vector \(C = -\frac{1}{2} e_1 \wedge e_2 - \frac{1}{2} e_2 \wedge e_3\) for some suitable orthonormal frame \(\{e_1, e_2, e_3, e_4\}\) on \(M\).

**Example 2.** If we choose \(a = 1, b = 2\) and \(z = x^2\) from (27), then the general rotational surface \(M\) defined by
\[
F(x, t) = (x \cos t, x \sin t, x^2 \cos 2t, x^2 \sin 2t), \ x > 0
\]
is minimal in \(E^4\) with proper pointwise 1-type Gauss map of the second kind. Also, the Gauss map \(\nu = e_3 \wedge e_4\) satisfies (1) for the function \(f = \frac{32}{(1+4x^2)^2}\) and for the constant vector \(C = \frac{1}{2} e_1 \wedge e_2 - \frac{1}{2} e_2 \wedge e_3\) for some suitable orthonormal frame \(\{e_1, e_2, e_3, e_4\}\) on \(M\).

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**References**

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