On the degree of approximation of continuous functions by means of Fourier series

∗Baorong Wei1 and Dansheng Yu1,†

1 Department of Mathematics, Hangzhou Normal University, Xiasha Economic Development Area, Hangzhou, Zhejiang 310036, P. R. China

Received March 15, 2009; accepted May 29, 2011

Abstract. We generalize some results on the degree of approximation of continuous functions by means of Fourier series, which were obtained by Chandra ([1, 2]) and Leindler ([4]). Some applications of the main results are given.

AMS subject classifications: 42A24, 41A25

Key words: A-transformation, degree of approximation, Fourier series

1. Introduction

Let \( f(x) \) be a \( 2\pi \)-periodic continuous function. Denote by \( S_n(f, x) \) the \( n \)-th partial sum of its Fourier series, \( \omega(\delta) := \omega(f, \delta) \) the modulus of continuity of \( f \). Let \( A := (a_{nk})_{(k, n = 0, 1, \cdots)} \) be a lower triangular infinite matrix of real numbers, that is, \( a_{nk} = 0 \) for all \( k > n \). The \( A \)-transform of \( \{S_n(f, x)\} \) is given by

\[
T_n(f) := T_n(f, x) := \sum_{k=0}^{n} a_{nk} S_k(f, x), \quad n = 0, 1, \cdots.
\]

The following theorems can be found in [1], [2]:

Theorem A. Let \((a_{nk})\) satisfy the following conditions:

\[
a_{nk} \geq 0 \quad \text{and} \quad \sum_{k=0}^{n} a_{nk} = 1, \quad (1)
\]

\[
a_{nk} \leq a_{n,k+1}, \quad k = 0, 1, \cdots, n-1; \quad n = 0, 1, \cdots. \quad (2)
\]

Suppose \( \omega(t) \) is such that

\[
\int_{u}^{\pi} t^{-2} \omega(t) dt = O(H(u)), \quad (u \to 0+), \quad (3)
\]

∗This work was supported by the NSF of China (10901044), Gianjiang Rencai Program of Zhejiang Province (2010R10101), SRF for ROCS, SEM and Program for Excellent Young Teachers in Hangzhou Normal University

†Corresponding author. Email addresses: weibr1234568126.com (B. R. Wei), danshengyu@yahoo.com.cn (D. S. Yu)

http://www.mathos.hr/mcc ©2012 Department of Mathematics, University of Osijek
where $H(u) \geq 0$ and
\[
\int_0^t H(u) du = O(tH(t)), \quad (t \to 0^+).
\] (4)

Then
\[
\| T_n(f) - f \| = O(a_{nn}H(a_{nn})),
\] (5)

where $\| \cdot \|$ denotes the supnorm.

**Theorem B.** Let (1), (2) and (3) hold. Then
\[
\| T_n(f) - f \| = O(\omega(\pi/n) + a_{nn}H(\pi/n)).
\] (6)

If, in addition, $\omega(t)$ satisfies (4), then
\[
\| T_n(f) - f \| = O(a_{nn}H(\pi/n)).
\] (7)

**Theorem C.** Let us assume that (1) and
\[
a_{nk} \geq a_{n,k+1}, \; k = 0, 1, \ldots, n - 1; n = 0, 1, \ldots
\] (8)

hold. Then
\[
\| T_n(f) - f \| = O\left(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1} \omega(\pi/k) \sum_{r=0}^{k+1} a_{nr}\right).
\] (9)

**Theorem D.** Let (1), (3), (4) and (8) hold. Then
\[
\| T_n(f) - f \| = O\left(a_{n0}H(a_{n0})\right).
\] (10)

Recently, Leindler [4] has showed that the monotonic condition in (2) and (8) can be essentially relaxed. To state his results, we need some notions.

For a fixed $n$, $\alpha_n := \{a_{nk}\}_{k=0}^\infty$ of nonnegative numbers tending to zero is called **rest bounded variation**, or briefly $\alpha_n \in RBVS$, if there is a constant $K(\alpha_n)$ only depending on $\alpha_n$ such that
\[
\sum_{k=m}^\infty |\Delta a_{nk}| := \sum_{k=m}^\infty |a_{nk} - a_{n,k+1}| \leq K(\alpha_n)a_{nm}
\] (11)

for all natural numbers $m$.

For a fixed $n$, $\alpha_n = \{a_{nk}\}_{k=0}^\infty$ of nonnegative numbers tending to zero is called **head bounded variation**, or briefly $\alpha_n \in HBVS$, if there is a constant $K(\alpha_n)$ only depending on $\alpha_n$ such that
\[
\sum_{k=0}^{m-1} |\Delta a_{nk}| \leq K(\alpha_n)a_{n,m}
\] (12)
for all natural numbers \( m \), or only for all \( m \leq N \) if the sequence \( \alpha_n \) has only finite nonzero terms, and the last nonzero term is \( a_{nN} \).

Leindler’s main result in [4] can be read as follows:

**Theorem E.** The statements of Theorem A, B, C and D hold with (12) in place of (2), and with (11) in place of (8), respectively; naturally maintaining all the other assumptions.

It should be noted that in the previous theorems of Chandra and Leindler, a sequence of sequences \( \alpha_n := \{a_{nk}\}_{k=0}^\infty \) has appeared. Thus, it is natural to assume that \( \{K(\alpha_n)\}_{n=0}^\infty \) is bounded, that is, there is an absolute constant \( K \) such that

\[
0 \leq K(\alpha_n) \leq K \quad \text{for} \quad n = 1, 2, \ldots .
\]

In the present paper, we further generalize Theorem E by establishing the following:

**Theorem 1.** Let (1) hold. Suppose that \( \omega(t) \) satisfies (3). Then

\[
\|T_n(f) - f\| = O\left(\omega(\pi/n) + \sum_{k=0}^n |\Delta a_{nk}| H(\pi/n)\right),
\]

(13)

If, in addition, \( \omega(t) \) satisfies (4), then

\[
\|T_n(f) - f\| = O\left(\sum_{k=0}^n |\Delta a_{nk}| H\left(\sum_{k=0}^n |\Delta a_{nk}|\right)\right),
\]

(14)

\[
\|T_n(f) - f\| = O\left(\sum_{k=0}^n |\Delta a_{nk}| H(\pi/n)\right).
\]

(15)

**Theorem 2.** Let (1) hold. Then

\[
\|T_n(f) - f\| = O\left(\omega(\pi/n) + \sum_{k=1}^n k^{-1}\omega(\pi/k) \sum_{r=0}^{k+1} a_{nr} + \sum_{k=1}^n \omega(\pi/k) \sum_{r=k}^n |\Delta a_{nr}|\right).
\]

(16)

As an application of our results, we will show that Theorem 1 and Theorem 2 imply all the results of Theorem E, thus Theorem A-Theorem D. Also, we will give some generalizations of Theorem A-Theorem E by applying Theorem 1 and Theorem 2 to a more general class of sequences than RBVS.

**2. Proofs of theorems**

We need some Lemmas.

**Lemma 1** (see [1]). If (3)and (4) hold, then

\[
\int_0^{\pi/n} \omega(t)dt = O(n^{-2}H(\pi/n)).
\]

(17)

**Lemma 2** (see [1]). If (3)and (4) hold, then

\[
\int_0^r t^{-1}\omega(t)dt = O(rH(r)), \quad r \to 0 + .
\]

(18)
Lemma 3. For any lower triangular infinite matrix \((a_{nk})\), \(k, n = 0, 1, 2, \cdots\) of nonnegative numbers, it holds uniformly in \(0 < t \leq \pi\), that
\[
\sum_{k=0}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t = O \left( \sum_{r=0}^{\tau} a_{nr} + \frac{1}{\tau} \sum_{r=\tau}^{n} |\Delta a_{nr}| \right),
\]
(19)
where \(\tau\) denotes the integer part of \(\pi/t\). It also holds that
\[
\sum_{k=0}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t = O \left( \frac{1}{t} \sum_{r=0}^{n} |\Delta a_{nr}| \right),
\]
(20)

Proof. Since \((a_{nk})\) is a lower triangular infinite matrix, that is, \(a_{nk} = 0\) for \(k > n\), then
\[
a_{nm} \leq \sum_{k=m}^{n} |\Delta a_{nk}|
\]
(21)
for \(m = 0, 1, 2, \cdots, n\). It is elementary to deduce that for arbitrary \(\lambda_n \geq 0\) and for \(n \geq m \geq 0\),
\[
\left| \sum_{k=m}^{n} \lambda_k \sin \left( k + \frac{1}{2} \right) t \sin \frac{t}{2} \right| \leq \frac{1}{2} \left( \lambda_m + \sum_{k=m}^{n-1} |\Delta \lambda_k| + \lambda_n \right)
\]
(22)
holds.

By (21) and (22), assuming that \(n \geq \tau\), we have
\[
\left| \sum_{k=0}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t \right| \leq \sum_{r=0}^{\tau} a_{nr} + \left| \sum_{r=\tau}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t \right|
\]
\[
\leq \sum_{r=0}^{\tau} a_{nr} + O \left( \frac{1}{\tau} \left( a_{n\tau} + \sum_{r=\tau}^{n-1} |\Delta a_{nk}| + a_{nn} \right) \right)
\]
\[
= O \left( \sum_{r=0}^{\tau} a_{nr} + \frac{1}{\tau} \sum_{r=\tau}^{n} |\Delta a_{nr}| \right),
\]
which completes (19).

Similarly, we have
\[
\left| \sum_{k=0}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t \right| = O \left( \frac{1}{\tau} \left( a_{n0} + \sum_{r=0}^{n-1} |\Delta a_{nk}| + a_{nn} \right) \right)
\]
\[
= O \left( \frac{1}{\tau} \sum_{r=0}^{n} |\Delta a_{nr}| \right),
\]
hence, (20) is finished. \(\Box\)
Proof of Theorem 1. We will follow the ideas of Chandra ([1,2]) and Leindler ([4]).

Write

\[ \Phi_x(t) := \frac{1}{2} (f(x + t) + f(x - t) - 2f(x)). \]

Then

\[ T_n(f, x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \left( \Phi_x(t) \left( 2 \sin \frac{t}{2} \right) - \sum_{k=0}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t \right) dt. \tag{23} \]

Proof of (13). By (23), we have

\[ \| T_n(f) - f \| \leq \frac{2}{\pi} \left( \int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right) := I_1 + I_2. \tag{24} \]

By (1) and the inequality \(|\sin t| \leq t\), we have for \(0 \leq t \leq \pi/n\),

\[ \left| \sum_{k=0}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t \right| = O(nt). \]

Therefore,

\[ I_1 = O(n) \int_0^{\pi/n} \omega(t) dt = O(\omega(\pi/n)). \tag{25} \]

By (20) and (3),

\[ I_2 = O \left( \sum_{k=0}^{n} |\Delta a_{nk}| \right) \int_{\pi/n}^{\pi} t^{-2} \omega(t) dt = O \left( \sum_{k=0}^{n} |\Delta a_{nk}| H(\pi/n) \right). \tag{26} \]

We complete (13) by combining (24)-(26).

Proof of (14). By (23) again, and

\[ \sum_{k=0}^{n} |\Delta a_{nk}| \leq 2 \sum_{k=0}^{n} a_{nk} = 2 < \pi, \]

we get

\[ \| T_n(f) - f \| \leq \frac{2}{\pi} \left( \int_0^{\sum_{k=0}^{n} |\Delta a_{nk}|} + \int_{\sum_{k=0}^{n} |\Delta a_{nk}|}^{\pi} \right) := J_1 + J_2. \tag{27} \]

By (1), we have

\[ \left| \sum_{k=0}^{n} a_{nk} \sin \left( k + \frac{1}{2} \right) t \right| \leq 1. \]
Hence, by (18), we have
\[
J_1 = O(1) \int_0^n t^{-1} \omega(t) dt = O \left( \sum_{k=0}^n |\Delta a_{nk}| H \left( \sum_{k=0}^n |\Delta a_{nk}| \right) \right).
\] (28)

By (20) and (3), we have
\[
J_2 = O \left( \sum_{k=0}^n |\Delta a_{nk}| \int_0^n t^{-1} \omega(t) dt \right) = O \left( \sum_{k=0}^n |\Delta a_{nk}| H \left( \sum_{k=0}^n |\Delta a_{nk}| \right) \right).
\] (29)

We finish (14) by combining (27)-(29).

**Proof of (15).** Note that \(a_{nk} = 0\) for \(k > n\), we deduce that
\[
a_{nj} \leq \sum_{k=j}^n |\Delta a_{nk}| \leq \sum_{k=0}^n |\Delta a_{nk}|
\]
for \(j = 0, 1, \ldots, n\), which implies that
\[
1 = \sum_{j=0}^n a_{nj} \leq (n + 1) \sum_{k=0}^n |\Delta a_{nk}|
\]
or in other words,
\[
\sum_{k=0}^n |\Delta a_{nk}| \geq \frac{1}{2n}.
\]

Hence, by (17), we obtain that
\[
I_1 = O \left( \frac{1}{n} H(\pi/n) \right) = O \left( \sum_{k=0}^n |\Delta a_{nk}| H(\pi/n) \right).
\] (30)

Altogether by (24), (26) and (30), (15) is proved. \(\square\)

**Proof of Theorem 2.** By (19) and the monotonicity of \(\omega(t)\), we deduce that (see (24) for \(I_2\))
\[
I_2 = O(1) \int_{\pi/n}^{\pi} t^{-1} \omega(t) \left( \sum_{r=0}^\tau a_{nr} + \frac{1}{\tau} \sum_{r=\tau}^n |\Delta a_{nr}| \right) dt
\]
\[
= O(1) \sum_{k=1}^{n-k} \int_{\pi/(k+1)}^{\pi/k} t^{-1} \omega(t) \left( \sum_{r=0}^\tau a_{nr} + \frac{1}{\tau} \sum_{r=\tau}^n |\Delta a_{nr}| \right) dt
\]
\[
= O \left( \sum_{k=1}^n k^{-1} \omega(\pi/k) \sum_{r=0}^k a_{nr} + \sum_{r=0}^n \omega(\pi/k) \sum_{r=k}^n |\Delta a_{nr}| \right).
\] (31)

Altogether by (24), (25) and (31), we obtain (16). \(\square\)
3. Applications of Theorems

Application 1. We remark that Theorem 1 implies Theorem E, and thus Theorem A–Theorem D. In fact, if \( \{a_{nk}\} \in HBVS \), then
\[
\sum_{k=0}^{n} |\Delta a_{nk}| = \sum_{k=0}^{n-1} |\Delta a_{nk}| + a_{nn} \leq (K(\alpha_n) + 1) a_{nn}.
\]
Thus, (14), (13) and (15) imply (5), (6) and (7), respectively.

If \( \{a_{nk}\} \in RBVS \), then
\[
\sum_{k=0}^{n} |\Delta a_{nk}| \leq \sum_{k=0}^{n-1} |\Delta a_{nk}| + a_{nn} \leq 2 \sum_{k=0}^{n-1} |\Delta a_{nk}| + a_{nn} \leq 2 (K(\alpha_n) + 1) a_{nn},
\]
(32)
hence, (14) implies (10). Also, we derive from (15) and (13) that
\[
\|T_n(f) - f\| = O(\omega(\pi/n) + a_{n0} H(\pi/n)),
\]
and
\[
\|T_n(f) - f\| = O(\omega(\pi/n) + a_{n0} H(\pi/n)),
\]
which are new results not stated in Theorem A-Theorem E.

Finally, we prove that (16) implies (9) if \( \{a_{nk}\} \in RBVS \). In fact, since \( \{a_{nk}\} \in RBVS \), then, similarly to (32), we get
\[
\sum_{r=k}^{n} |\Delta a_{nr}| \leq (2K(\alpha_n) + 1) a_{nk},
\]
(33)
By using the definition of \( RBVS \), we have
\[
a_{nk} \leq \sum_{r=j}^{k-1} |\Delta a_{nr}| + a_{nj} \leq (2K(\alpha_n) + 1) a_{nj}
\]
for \( j = \lfloor k/2 \rfloor + 1, \ldots, k \), which implies that
\[
a_{nk} = O \left( \frac{1}{k} \sum_{r=\lfloor k/2 \rfloor + 1}^{k} |a_{nr}| \right) = O \left( \frac{1}{k} \sum_{r=0}^{k+1} |a_{nr}| \right),
\]
(34)
By (33) and (34), we have
\[
\sum_{r=k}^{n} |\Delta a_{nr}| = O \left( \frac{1}{k} \sum_{r=0}^{k+1} |a_{nr}| \right),
\]
which shows that (16) implies (9).

**Application 2.** We can apply theorems to some A–transform with \( \{a_{nk}\}_{k=0}^{\infty} \) may have lacunary terms for \( 0 \leq k \leq n \), which is impossible for \( \{a_{nk}\}_{k=0}^{\infty} \in HBVS \) or \( \{a_{nk}\}_{k=0}^{\infty} \in RBVS \).

**Application 3.** Very recently, Leindler [5] has extended the definition of \( RBVS \) to the so-called \( \gamma RBVS \). In our case, we can state the definition of \( \gamma RBVS \) as follows:

For a fixed \( n \), let \( \gamma_n := \{\gamma_{nk}\}_{k=0}^{\infty} \) be a nonnegative sequence. If a null-sequence \( \alpha_n := \{a_{nk}\}_{k=0}^{\infty} \) of real numbers has the property

\[
\sum_{k=m}^{\infty} |\Delta a_{nk}| \leq K(\alpha_n) \gamma_{nm}
\]

for every positive integer \( m \), then we call the sequence \( \alpha_n := \{a_{nk}\}_{k=0}^{\infty} \) a \( \gamma RBVS \), briefly denoted by \( \alpha_n \in \gamma RBVS \).

If \( \gamma_n = \alpha_n \), then \( \gamma RBVS \equiv RBVS \).

Similarly, we can introduce a new kind of sequences \( \gamma HBVS \) as follows:

For a fixed \( n \), let \( \gamma_n := \{\gamma_{nk}\}_{k=0}^{\infty} \) be a nonnegative sequence. If a null-sequence \( \alpha_n := \{a_{nk}\}_{k=0}^{\infty} \) of real numbers has the property

\[
\sum_{k=0}^{m-1} |\Delta a_{nk}| \leq K(\alpha_n) \gamma_{nm}
\]

for every positive integer \( m \), then we call the sequence \( \alpha_n := \{a_{nk}\}_{k=0}^{\infty} \) a \( \gamma HBVS \), briefly denoted by \( \alpha_n \in \gamma HBVS \).

By a discussion similar to Application 1, Theorem 1 and Theorem 2, we have the following generalizations of Theorem E:

**Theorem 3.** Let \( (a_{nk}) \) satisfy (1). Suppose that \( \omega(t) \) satisfies (3), then

(i) If \( \{a_{nk}\}_{k=0}^{\infty} \in \gamma HBVS \), then

\[
\|T_n(f) - f\| = O\left( \omega(\pi/n) + \gamma_{n0} H(\pi/n) \right).
\]

If, in addition, \( \omega(t) \) satisfies (4), then

\[
\|T_n(f) - f\| = O\left( \gamma_{n0} H(\gamma_{n0}) \right),
\]

and

\[
\|T_n(f) - f\| = O\left( \gamma_{n0} H(\gamma_{n0}) \right).
\]

(ii) If \( \{a_{nk}\}_{k=0}^{\infty} \in \gamma RBVS \), then

\[
\|T_n(f) - f\| = O\left( \omega(\pi/n) + \gamma_{n0} H(\pi/n) \right).
\]

If, in addition, \( \omega(t) \) satisfies (4), then

\[
\|T_n(f) - f\| = O\left( \gamma_{n0} H(\gamma_{n0}) \right),
\]

\[
\|T_n(f) - f\| = O\left( \gamma_{n0} H(\gamma_{n0}) \right).
\]
Theorem 4. If \( (a_{nk}) \) satisfies (1) and \( \{a_{nk}\}_{k=0}^{\infty} \in \gamma RBVS \), then

\[
\|T_n(f) - f\| = O\left(\frac{\omega(\pi/n)}{n} + \sum_{k=1}^{n} k^{-1} \omega(\pi/k) \sum_{r=0}^{k-1} a_{nr} + \sum_{k=1}^{n} \omega(\pi/k) \gamma_{nk}\right).
\]

Acknowledgement

The authors would like to thank the referees for their helpful suggestions.

References


