Some common fixed point theorems in fuzzy metric spaces

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Abstract. *Some common fixed point theorems in complete fuzzy metric spaces (in sense of Song [17] and Vasuki-Veeramani [19]) are proved which generalize earlier results due to Vasuki [18], Chugh and Kumar [3] and others. We also introduce the concept of R-weak commutativity of type (P) in fuzzy metric spaces. Some related results and illustrative examples are also discussed.*

Key words: *fuzzy metric space, fixed point, R-weakly commuting mappings*

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1. Introduction

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [20] which laid the foundation of fuzzy mathematics. Consequently the last three decades were very productive for fuzzy mathematics and the recent literature has observed the fuzzification in almost every direction of mathematics such as arithmetic, topology, graph theory, probability theory, logic etc. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. No wonder that fuzzy fixed point theory has become an area of interest for specialists in fixed point theory, or fuzzy mathematics has offered new possibilities for fixed point theorists.

Deng [4], Erceg [5], Kaleva and Seikkala [11] and Kramosil and Michalek [12] have introduced the concept of fuzzy metric spaces in various ways. George and Veeramani [8] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [12] and defined Hausdorff topology of metric spaces which is later proved to be metrizable. They also showed that every metric induces a fuzzy metric

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and by now there exists considerable literature on this topic. To mention a few, we cite [2,4,5,8-12,17,19].

Recently, Chugh and Kumar [3] proved a Pant [13, Theorem 1] type theorem for two pairs of R-weakly commuting mappings satisfying a Boyd and Wong [1] type contraction condition which in turn, generalizes a fixed point theorem of Vasuki [18, Theorem 2].

In fact Chugh and Kumar [3] proved the following.

Theorem A. *Let A, B, S and T be mappings from a complete fuzzy metric space $(X, M, \star)$ into itself satisfying $A(X) \subset T(X), B(X) \subset S(X)$ and $M(Ax, By, t) \geq r(M(Sx, Ty, t))$ for all $x, y \in X$, where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(s) > s$ for each $0 < s < 1$. Suppose that one of A, B, S and T is continuous, pairs (A, S) and (B, T) are R-weakly commuting on X. Then A, B, S and T have a unique common fixed point in X.*

Note that *Theorem A* for a pair of R-weakly commuting mappings was proved by Vasuki [18] provided one of the mapping is continuous.

The purpose of this paper is to improve the main theorem of Vasuki [18] along with *Theorem A* (due to Chugh and Kumar [3]) besides adopting R-weak commutativity of type (A_f), type (Ag) to fuzzy setting and to introduce R-weak commutativity of type (P) which are to be used to prove our results in this paper.

Our improvement in this paper is four-fold:

(i) to relax the continuity requirement of maps completely,

(ii) to minimize the commutativity requirement of the maps to the point of coincidence,

(iii) to weaken the completeness requirement of the space to four alternative conditions,

(iv) to employ a more general contraction condition in proving our results.

2. Preliminaries

In what follows we collect relevant definitions, results and examples for our future use.

Definition 2.1.(cf. [20]) *A fuzzy set A in X is a function with domain X and values in [0, 1].*

Definition 2.2.(cf. [16]) *A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $\{[0, 1], \star\}$ is an abelian topological monoid with unit 1 such that $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.*

Definition 2.3.(cf. [12]) *The triplet $(X, M, \star)$ is a fuzzy metric space if X is an arbitrary set, $\star$ is a continuous t-norm, M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:*

(i) $M(x, y, 0) = 0$,

(ii) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$,

(iii) $M(x, y, t) = M(y, x, t) \neq 0$ *for* $t \neq 0$,

(iv) $M(x,y,t) \star M(y,z,s) \leq M(x,z,t+s)$,

(v) $M(x,y,.) : [0,\infty) \to [0,1]$ *is left continuous for all* $x,y,z \in X$ *and* $s,t > 0$.

Example 2.1.(cf. [8]) *Every metric space induces a fuzzy metric space. Let* (X,d) *be a metric space. Define* $a \star b = ab$ *and* $M(x,y,t) = \dfrac{kt^n}{kt^n + md(x,y)}$, $k,m,n,t \in \Re^+$. *Then* $(X,M,\star)$ *is a fuzzy metric space. If we put* $k = m = n = 1$, *we get*

$$M(x,y,t) = \frac{t}{t + d(x,y)}.$$

The fuzzy metric induced by a metric d *is referred to as a* standard fuzzy metric.

Definition 2.4.(cf. [10]) *A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X,M,\star)$ *is convergent to* $x \in X$ *if*

$$\lim_{n\to\infty} M(x_n,x,t) = 1 \quad \text{for each } t > 0.$$

Recently, Song [17] and Vasuki and Veeramani [19] again critically reviewed the existing definitions of Cauchy sequence in a fuzzy metric space. Vasuki and Veeramani [19] suggested that the definition of Cauchy sequence due to Grabiec [10] is weaker than that contained in [17, 19] and called it a G-Cauchy sequence.

Definition 2.5.(cf. [10]) *A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X,M,\star)$ *is called Cauchy if* $\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$ *for every* $t > 0$ *and each* $p > 0$. $(X,M,\star)$ *is complete if every Cauchy sequence in* X *converges in* X.

Definition 2.6. *A pair of self-mappings* (f,g) *of a fuzzy metric space* $(X,M,\star)$ *is said to be*

(i) weakly commuting (cf.[18]) if $M(fgx,gfx,t) \geq M(fx,gx,t)$,

(ii) R-weakly commuting (cf.[18]) if there exists some $R > 0$ *such that*

$$M(fgx,gfx,t) \geq M(fx,gx,t/R),$$

(iii) R-weakly commuting mappings of type (A_f) *if there exists some* $R > 0$ *such that* $M(fgx,ggx,t) \geq M(fx,gx,t/R)$,

(iv) R-weakly commuting mappings of type (Ag) *if there exists some* $R > 0$ *such that* $M(gfx,ffx,t) \geq M(fx,gx,t/R)$,

(v) R-weakly commuting mappings of type (P) *if there exists some* $R > 0$ *such that* $M(ffx,ggx,t) \geq M(fx,gx,t/R)$,
for all $x \in X$ *and* $t > 0$.

Notice that *Definition 2.6 (iii)* and *Definition 2.6 (iv)* are inspired by Pathak et al. [15] whereas *Definition 2.6 (v)* seems to be unreported.

Example 2.2 (cf. [18]) *Let* $X = \Re$, *the set of real numbers. Define* $a \star b = ab$ *and*

$$M(x,y,t) = \begin{cases} \left(e^{\frac{|x-y|}{t}}\right)^{-1}, & \text{for all } x,y \in X \text{ and } t > 0 \\ 0, & \text{for all } x,y \in X \text{ and } t = 0. \end{cases}$$

Then it is well known (cf. [18]) that $(X, M, \star)$ is a fuzzy metric space. Define $fx = 2x - 1$ and $gx = x^2$. Then by a straightforward calculation, one can show that

$$M(fgx, gfx, t) = \left(e^{\frac{2|x-1|^2}{t}}\right)^{-1} = M(fx, gx, t/2)$$

which shows that the pair (f, g) is R-weakly commuting for R=2. Note that the pair (f, g) is not weakly commuting due to a strict increasing property of the exponential function.

However, various kinds of above mentioned 'R-weak commutativity' notions are independent of one another and none implies the other. The earlier example can be utilized to demonstrate this inter-independence.

To demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (A_f) notice that

$$\begin{aligned} M(fgx, ggx, t) &= \left(e^{\frac{|x^4-2x^2+1|}{t}}\right)^{-1} = \left(e^{\frac{R(x-1)^2}{t}\frac{(x+1)^2}{R}}\right)^{-1} \\ &< \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, t/R) \text{ when } x > 1 \end{aligned}$$

which shows that 'R-weak commutativity' does not imply 'R-weak commutativity' of type (A_f).

Secondly, in order to demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (P) note that

$$\begin{aligned} M(ffx, ggx, t) &= \left(e^{\frac{|x^4-4x+3|}{t}}\right)^{-1} = \left(e^{\frac{R(x-1)^2}{t}\frac{(x^2+2x+3)}{R}}\right)^{-1} \\ &< \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, t/R) \text{ for } x > 1. \end{aligned}$$

Finally, for a change the pair (f, g) is R-weakly commuting of type (Ag) as

$$\begin{aligned} M(gfx, ffx, t) &= \left(e^{\frac{|(2x-1)^2-4x+3|}{t}}\right)^{-1} = \left(e^{\frac{4|x-1|^2}{t}}\right)^{-1} \\ &= M(fx, gx, t/4) \end{aligned}$$

which shows that (f, g) is R-weakly commuting of type (Ag) for R=4. This situation may also be utilized to interpret that an R-weakly commuting pair of type (Ag) need not be R-weakly commuting pair of type (A_f) or type (P). It is not difficult to find examples to establish the independence of one of these definitions from the others which shows that there exist situations to suit a definition but not the others.

3. Results

Now, let $(X, M, \star)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X satisfying the following conditions:

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{3.1}$$

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\}) \tag{3.2}$$

for all $x, y \in X$, where $\phi : [0, 1] \to [0, 1]$ is a continuous function such that $\phi(s) > s$ for each $0 < s < 1$. Then for any arbitrary point $x_0 \in X$, by (3.1), we choose a point $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this point x_1, there exists a point $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \text{ for } n = 0, 1, 2 \ldots \tag{3.3}$$

Firstly, we prove the following lemma.

Lemma 3.1. *Let A, B, S and T be self-mappings of a fuzzy metric space $(X, M, \star)$ satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X.*

Proof. For $t > 0$,

$$\begin{aligned} M(y_{2n}, y_{2n+1}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \phi(\min\{M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}) \\ &= \phi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)\}) \\ &> \begin{cases} M(y_{2n-1}, y_{2n}, t), & \text{if } M(y_{2n-1}, y_{2n}, t) < M(y_{2n}, y_{2n+1}, t) \\ M(y_{2n}, y_{2n+1}, t), & \text{if } M(y_{2n-1}, y_{2n}, t) \geq M(y_{2n}, y_{2n+1}, t), \end{cases} \end{aligned} \tag{3.4}$$

as $\phi(s) > s$ for $0 < s < 1$. Thus $\{M(y_{2n}, y_{2n+1}, t), n \geq 0\}$ is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $l \leq 1$. We assert that $l = 1$. If not, $l < 1$ which on letting $n \to \infty$ in (3.4) one gets $l \geq \phi(l) > l$ a contradiction yielding thereby $l = 1$. Therefore for every $n \in N$, using analogous arguments one can show that $\{M(y_{2n+1}, y_{2n+2}, t), n \geq 0\}$ is a sequence of positive real numbers in [0, 1] which tends to a limit $l = 1$. Therefore for every $n \in N$

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t) \text{ and } \lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1.$$

Now for any positive integer p

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) \star \ldots \star M(y_{n+p-1}, y_{n+p}, t/p).$$

Since $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$ for $t > 0$, it follows that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 \star 1 \star \ldots \star 1 = 1$$

which shows that $\{y_n\}$ is a Cauchy sequence in X. □

Now we prove our main result as follows:

Theorem 3.1. *Let A, B, S and T be four self-mappings of a fuzzy metric space $(X, M, \star)$ satisfying the condition*

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\})$$

for all $x, y \in X$ and $t > 0$ where $\phi : [0, 1] \to [0, 1]$ is a continuous function with $\phi(s) > s$ whenever $0 < s < 1$. If $A(X) \subset T(X)$ and $B(X) \subset S(X)$ and one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of X, then *(i) A and S have a point of coincidence,*

(ii) B and T have a point of coincidence.

Moreover, if the pairs (A,S) and (B,T) are coincidentally commuting, then A,B,S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Then following arguments of Fisher [7], one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}.$$

Then due to *Lemma 3.1*, $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that $S(X)$ is a complete subspace of X, then the subsequence $y_{2n+1} = Sx_{2n+2}$ must get a limit in $S(X)$. Call it to be u and $v \in S^{-1}u$. Then $Sv = u$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of $\{y_{2n}\}$ being a subsequence of the convergent sequence $\{y_n\}$. On setting $x = v$ and $y = x_{2n+1}$ in (3.2) one gets (for $t > 0$)

$$\begin{aligned} M(Av, y_{2n+1}, t) &= M(Av, Bx_{2n+1}, t) \\ &\geq \phi(\min\{M(u, y_{2n}, t), M(Av, u, t), M(y_{2n+1}, y_{2n}, t)\}) \end{aligned}$$

which on letting $n \to \infty$ reduces to

$$M(Av, u, t) \geq \phi(M(Av, u, t)) > M(Av, u, t)$$

a contradiction. Therefore $Av = u = Sv$, which shows that the pair (A,S) has a point of coincidence.

As $A(X) \subset T(X)$, $Av = u$ implies that $u \in T(X)$. Let $w \in T^{-1}u$, then $Tw = u$. Now using (3.2) again

$$\begin{aligned} M(y_{2n}, Bw, t) &= M(Ax_{2n}, Bw, t) \\ &\geq \phi(\min\{M(y_{2n-1}, Tw, t), M(y_{2n-1}, y_{2n}, t), M(u, Bw, t)\}) \end{aligned}$$

which on letting $n \to \infty$ reduces to

$$M(u, Bw, t) \geq \phi(M(u, Bw, t)) > M(u, Bw, t)$$

a contradiction. Therefore $u = Bw$. Thus we have shown $u = Av = Sv = Bw = Tw$ which amounts to say that both pairs have point of coincidence. If one assumes $T(X)$ to be complete, then an analogous argument establishes this claim.

The remaining two cases pertain essentially to the previous cases. Indeed if $A(X)$ is complete, then $u \in A(X) \subset T(X)$ and if $B(X)$ is complete, then $u \in B(X) \subset S(X)$. Thus (i) and (ii) are completely established.

Since the pairs (A,S) and (B,T) are coincidentally commuting at v and w respectively, then

$$Au = A(Sv) = S(Av) = Su \text{ and}$$

$$Bu = B(Tw) = T(Bw) = Tu.$$

If $Au \neq u$, then for $t > 0$

$$\begin{aligned} M(Au, u, t) &= M(Au, Bw, t) \\ &\geq \phi(\min\{M(Su, Tw, t), M(Su, Au, t), M(Bw, Tw, t)\} \\ &= \phi(\min\{M(Au, u, t), 1, 1\} \\ &= \phi(M(Au, u, t)) > M(Au, u, t) \end{aligned}$$

a contradiction. Therefore $Au = u$. Similarly, one can show that $Bu = u$. Thus u is a common fixed point of A, B, S and T. The uniqueness of a common fixed point follows easily. Also u remains the unique common fixed point of both pairs separately. This completes the proof. □

Remark 3.1. *If*

$$\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t)\} = M(Sx, Ty, t)$$

one obtains an improved version of Theorem A *(due to Chugh and Kumar [3]) as* Theorem 3.1 *is proved without any continuity requirement besides confining the commutativity to points of coincidence alone.*

Remark 3.2. *If*

$$\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t)\} = M(Sx, Ty, t),$$

by setting $A = B$ and $S = T$, one obtains a substantially improved version of [18, Theorem 2] due to Vasuki as our result is proved under tight commutativity condition without any continuity requirement.

Theorem 3.2. Theorem 3.1 *remains true if a 'coincidentally commuting' property is replaced by any one (retaining the rest of the hypotheses) of the following:*

(i) R-weakly commuting property,

(ii) R-weakly commuting property of type (A_f),

(iii) R-weakly commuting property of type (Ag),

(iv) R-weakly commuting property of type (P),

(v) weakly commuting property.

Proof. Since all the conditions of *Theorem 3.1* are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pair (A, S), then using R-weak commutativity one gets

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R) = 1$$

which amounts to say that $ASx = SAx$. Thus the pair (A, S) is coincidentally commuting. Similarly (B, T) commutes at all of its coincidence points. Now applying *Theorem 3.1*, one concludes that A, B, S and T have a unique common fixed point.

In case (A, S) is an R-weakly commuting pair of type (A_f), then

$$M(ASx, S^2x, t) \geq M(Ax, Sx, t/R) = 1$$

which amounts to say that $ASx = S^2x$. Now

$$\begin{aligned} M(ASx, SAx, t) &\geq M\Big(ASx, S^2x, \frac{t}{2}\Big) \star M\Big(S^2x, SAx, \frac{t}{2}\Big) \\ &= 1 \star 1 = 1 \end{aligned}$$

yielding thereby $ASx = SAx$. Similarly, if pair is R-weakly commuting mappings of type (Ag) or type (P) or weakly commuting, then (A, S) also commutes at their points of coincidence. Similarly, one can show that the pair (B,T) is also coincidentally commuting. Now in view of *Theorem 3.1*, in all four cases A, B, S and T have a unique common fixed point. This completes the proof. □

As an application of *Theorem 3.1*, we prove a common fixed point theorem for four finite families of mappings which runs as follows:

Theorem 3.3. *Let* $\{A_1, A_2, \dots, A_m\}, \{B_1, B_2, \dots, B_n\}, \{S_1, S_2, \dots, S_p\}$ *and* $\{T_1, T_2, \dots, T_q\}$ *be four finite families of self-mappings of a fuzzy metric space* $(X, M, \star)$ *such that* $A = A_1A_2 \dots A_m$, $B = B_1B_2 \dots B_n$, $S = S_1S_2 \dots S_p$ *and* $T = T_1T_2 \dots T_q$ *satisfy condition (3.2) with* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$. *If one of* $A(X), B(X), S(X)$, *or* $T(X)$ *is a complete subspace of* X, *then*

(i) A *and* S *have a point of coincidence,*

(ii) B *and* T *have a point of coincidence.*

Moreover, if $A_iA_j = A_jA_i$, $B_kB_l = B_lB_k$, $S_rS_s = S_sS_r$, $T_tT_u = T_uT_t$, $A_iS_r = S_rA_i$ *and* $B_kT_t = T_tB_k$ *for all* $i, j \in I_1 = \{1, 2, \dots, m\}$, $k, l \in I_2 = \{1, 2, \dots, n\}$, $r, s \in I_3 = \{1, 2, \dots, p\}$ *and* $t, u \in I_4 = \{1, 2, \dots, q\}$, *then (for all* $i \in I_1, k \in I_2, r \in I_3$ *and* $t \in I_4$*)* A_i, S_r, B_k *and* T_t *have a common fixed point.*

Proof. The conclusions (i) and (ii) are immediate as A, S, B and T satisfy all the conditions of *Theorem 3.1*. Now appealing to component wise commutativity of various pairs, one can immediately prove that $AS = SA$ and $BT = TB$ and hence, obviously both pairs (A,S) and (B,T) are coincidentally commuting. Note that all the conditions of *Theorem 3.1* (for mappings A, S, B and T) are satisfied ensuring the existence of a unique common fixed point, say z. Now one needs to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned}
A(A_iz) &= ((A_1A_2 \dots A_m)A_i)z = (A_1A_2 \dots A_{m-1})((A_mA_i)z) \\
&= (A_1 \dots A_{m-1})(A_iA_mz) = (A_1 \dots A_{m-2})(A_{m-1}A_i(A_mz)) \\
&= (A_1 \dots A_{m-2})(A_iA_{m-1}(A_mz)) = \dots \\
&= A_1A_i(A_2A_3A_4 \dots A_mz) = A_iA_1(A_2A_3 \dots A_mz) = A_i(Az) = A_iz.
\end{aligned}$$

Similarly, one can show that

$$A(S_rz) = S_r(Az) = S_rz, \quad S(S_rz) = S_r(Sz) = S_rz,$$

$$S(A_iz) = A_i(Sz) = A_iz, \quad B(B_kz) = B_k(Bz) = B_kz,$$

$$B(T_tz) = T_t(Bz) = T_tz, \quad T(T_tz) = T_t(Tz) = T_tz$$

$$\text{and} \quad T(B_kz) = B_k(Tz) = B_kz,$$

which show that (for all i, r, k and t) A_iz and S_rz are other fixed points of the pair (A,S) whereas B_kz and T_tz are other fixed points of the pair (B,T). Now appealing to the uniqueness of common fixed points of both pairs separately, we get

$$z = A_iz = S_rz = B_kz = T_tz,$$

which shows that z is a common fixed point of A_i, S_r, B_k and T_t for all i, r, k and t. □

By setting $A = A_1 = A_2 = \ldots = A_m, B = B_1 = B_2 = \ldots = B_n, S = S_1 = S_2 = \ldots = S_p$ and $T = T_1 = T_2 = \ldots = T_q$, one deduces the following for certain iterates of maps, which runs as follows:

Corollary 3.1. *Let A, B, S and T be four self-mappings of a fuzzy metric space $(X, M, \star)$ such that A^m, B^n, S^p and T^q satisfy the conditions (3.1) and (3.2). If one of $A^m(X), B^n(X), S^p(X)$ or $T^q(X)$ is a complete subspace of X, then A, B, S and T have a unique common fixed point provided (A, S) and (B, T) commute.*

The following example furnishes an instance where *Corollary 3.1* is applicable but *Theorem A* (due to Chugh and Kumar [3]) cannot be used due to the absence of continuity requirement.

Example 3.1. *Consider $X = [0, 1]$ equipped with the natural metric $d(x, y) = |x - y|$. Now for $t \in [0, \infty)$ define*

$$M(x, y, t) = \begin{cases} 0, & \text{if } t = 0 \text{ and } x, y \in X \\ \frac{t}{t+|x-y|}, & \text{if } t > 0 \text{ and } x, y \in X. \end{cases}$$

Clearly $(X, M, \star)$ is a fuzzy metric on X where $\star$ is defined as $a \star b = ab$. This fuzzy metric space is also sometimes referred to as induced fuzzy metric space. It is well known (cf. [8]) that this fuzzy metric space is complete if and only if X is complete.

Define A, B, S and T on [0, 1] as

$$Ax = \begin{cases} 1, & \text{if } x \in [0, 1] \cap Q \\ 0, & \text{if } x \notin [0, 1] \cap Q, \end{cases} \qquad Bx = \begin{cases} 1, & \text{if } x \in [0, 1] \cap Q \\ \frac{1}{2}, & \text{if } x \notin [0, 1] \cap Q, \end{cases}$$

$$Sx = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1, \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{4}, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

Then $A^2(X) = \{1\} \subset \{\frac{1}{4}, 1\} = T^2(X)$ and $B^2(X) = \{1\} \subset \{\frac{1}{2}, 1\} = S^2(X)$. Define $\phi : [0, 1] \to [0, 1]$ as $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(s) = \sqrt{s}$ for $0 < s < 1$. Then

$$\begin{aligned} 1 &= M(A^2x, B^2y, t) \\ &\ge \phi(\min\{M(S^2x, T^2y, t), M(S^2x, A^2x, t), M(B^2y, T^2y, t)\}) \end{aligned}$$

for all $t > 0$. Also the various componentwise commutativity ensures the commutativity of both pairs (A, S) and (B, T). Thus all the conditions of Corollary 3.1 *are satisfied and 1 is the common fixed point of A, B, S and T.*

Here one needs to note that Theorem A *(due to Chugh and Kumar [3]) cannot be used in the context of this example because if we take $x, y \notin Q$, then*

$$\frac{t}{t + \frac{1}{2}} = M(Ax, By, t) \ge \phi\left(\min\left\{\frac{t}{t + \frac{1}{4}}, \frac{t}{t + \frac{1}{2}}, \frac{t}{t + \frac{1}{4}}\right\}\right)$$

which is not always true for $t > 0$ (e.g. $t = 0.5$). On the other hand, all the four mappings are discontinuous, which is not in lieu of requirement of Theorem A *(due to Chugh and Kumar [3]).*

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