Sensitivity analysis of the proportionate change of a subset of outputs or/and inputs in DEA

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Abstract. Sensitivity analysis in data envelopment analysis (DEA) is studied for the case of the proportionate change of a subset of outputs or/and of a subset of inputs of an efficient decision making unit (DMU) according to the Charnes–Cooper–Rhodes (CCR) ratio model. Sufficient conditions for an efficient DMU to preserve its efficiency under the proportionate change of a subset of inputs or/and of outputs are obtained. An illustrative numerical example is provided.

Key words: data envelopment analysis, efficiency, proportionate change of a subset of outputs or/and of inputs, sensitivity analysis, linear programming

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1. Introduction

Sensitivity analysis in Data Envelopment Analysis (DEA) for the Charnes–Cooper–Rhodes (CCR) ratio model (see Charnes and Cooper [1], Cooper et al. [5]) for the cases (a) with the proportionate change (increase) of all inputs and (b) with the proportionate change (decrease) of all outputs were studied by Charnes and Neralić [3]. In (a) sufficient conditions for an efficient DMU to preserve its efficiency after the proportionate change (increase) of all inputs were obtained and in (b) sufficient conditions for an efficient DMU after the proportionate change (decrease) of all outputs were given. The case of the simultaneous proportionate change (increase) of all inputs and proportionate change (decrease) of all outputs of an efficient DMU

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preserving its efficiency for the CCR ratio model was studied by Charnes and Neralić [4] and sufficient conditions for efficiency preservation of efficient DMU were established. Similar results for the additive model were obtained by Neralić [7].

Sufficient conditions for an efficient DMU to preserve efficiency for the case of the proportionate change (increase) of all inputs or/and the proportionate change (decrease) of all outputs for the CCR ratio model and for the Additive model with different coefficients of proportionality for all inputs or/and for all outputs were obtained by Neralić and Sexton [8].

The aim of this paper is firstly to study the case of the simultaneous proportionate change (decrease) of a subset of outputs or/and the proportionate change (increase) of a subset of inputs of an efficient DMU preserving its efficiency for the CCR ratio model. Secondly, the case of the proportionate change (decrease) of a subset of outputs or/and the proportionate change (increase) of a subset of inputs for the CCR ratio model with different coefficients of proportionality for outputs and/or for inputs will be studied too. Sufficient conditions for an efficient DMU according to the CCR ratio model to preserve efficiency after the simultaneous proportionate change (decrease) of a subset of outputs or/and of the proportionate change (increase) of a subset of inputs are given for the first case. (Similar results can be obtained for the second case.) In that way a measure of stability of efficiency for an efficient DMU is obtained.

The paper is organized as follows. Results in sensitivity analysis for the proportionate change (increase) of a subset of inputs or/and the proportionate change (decrease) of outputs for two considered cases are contained in Section 2. Section 3 gives an illustrative example. The last Section contains some conclusions and suggestions for further research.

2. Sensitivity analysis of a subset of outputs or/and of a subset of inputs

2.1. Let us suppose that there are \( n \) Decision Making Units (DMUs) with \( m \) inputs and \( s \) outputs. Let \( x_{ij} \) be the observed amount of \( i \)th type of input of the \( j \)th DMU (\( x_{ij} > 0, \ i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \)) and let \( y_{rj} \) be the observed amount of output of the \( r \)th type for the \( j \)th DMU (\( y_{rj} > 0, r = 1, 2, \ldots, s, j = 1, 2, \ldots, n \)). Let \( Y_j, X_j \) be the observed vectors of outputs and inputs of the DMU \( j \), respectively, \( j = 1, 2, \ldots, n \). Let \( e \) be the column vector of ones and let \( T \) as a superscript denote the transpose. In order to see if the DMU \( j_0 = \text{DMU}_0 \) is efficient according to the CCR ratio model the following linear programming problem should be solved:

\[
\begin{align*}
\min & \quad 0\lambda_1 + \cdots + 0\lambda_0 + \cdots + 0\lambda_n - \varepsilon e^T s^+ - \varepsilon e^T s^- + \theta \\
\text{subject to} & \\
Y_1\lambda_1 & + \cdots + Y_0\lambda_0 + \cdots + Y_n\lambda_n = s^+ = Y_0 \\
-X_1\lambda_1 & - \cdots - X_0\lambda_0 - \cdots - X_n\lambda_n = s^- + X_0\theta = 0 \\
\lambda_1, \ldots, \lambda_n, s^+, s^- & \geq 0,
\end{align*}
\]  
(1)

with \( Y_0 = Y_{j_0}, X_0 = X_{j_0}, \lambda_0 = \lambda_{j_0} \) and \( \theta \) unconstrained. The symbol \( \varepsilon \) represents the infinitesimal we use to generate the non-Archimedean ordered extension field.
In this extension field $\varepsilon$ is less than every positive number in our base field, but greater than zero. DMU$_0$ is DEA efficient if and only if for the optimal solution $(\lambda^*, s^{++}, s^{--}, \theta^*)$ of the linear programming problem (1) both of the following are satisfied (for details see Charnes and Cooper [1]):

$$\begin{align*}
\theta^* &= 1 \\
\theta^{++} &= \theta^{--} = 0, \quad \text{in all alternative optima.}
\end{align*}$$

(2)

Linear programming problem (1), with discussion about its use, can be found in Cooper et al [5], pp 73-75. As it is pointed out in that book on p. 73, representing the non-Archimedean infinitesimal $\varepsilon > 0$ by a small real number could get erroneous results. Because of that, a two-phase procedure for solving (1) is suggested (see Cooper et al [5], p. 43-45, 50-51). In Phase I we solve the LP problem

$$\min \theta$$

(3)

subject to the constraints in (1). Using the optimal value $\theta^* = \min \theta$ of the LP problem (3) from Phase I, in Phase II we solve the LP problem

$$\min (-e^T s^+ - e^T s^-)$$

(4)

subject to the constraints in (1) with the substitution $\theta = \theta^*$. DMU$_0$ is called CCR efficient if an optimal value of (3) and (4) satisfies $\theta^* = 1$ and $s^{++} = 0, s^{--} = 0$ respectively (see Cooper et al [5], p. 45).

2.2. We are interested in the proportionate change of a subset of outputs or/and of a subset of inputs of an efficient DMU$_0$ preserving its efficiency. An increase of any output cannot worsen an already achieved efficiency rating. Upward variations of outputs are not possible in the efficiency rating for an efficient DMU$_0$. Hence, without loss of generality, we can restrict attention to the proportionate decrease of the subset of the first $\bar{s}$ $(\bar{s} < s)$ outputs which can be written as

$$\hat{y}_r = \hat{\alpha} y_{r0}, \quad 0 < \hat{\alpha} \leq 1, \quad r = 1, 2, \ldots, \bar{s},$$

(5)

with the last $s - \bar{s}$ outputs fixed

$$\hat{y}_r = y_{r0}, \quad r = \bar{s} + 1, \bar{s} + 2, \ldots, s.$$  

(6)

Similarly, a decrease of any input cannot worsen an already achieved efficiency rating. Downward variations of inputs are not possible in the efficiency rating for an efficient DMU$_0$. Hence we can restrict attention to upward variations of inputs of an efficient DMU$_0$. Without loss of generality, let us consider the proportionate increase of the subset of the first $\overline{m}$ $(\overline{m} < m)$ inputs, with the last $m - \overline{m}$ inputs fixed. It can be written as

$$\hat{x}_{i0} = \hat{\beta} x_{i0}, \quad \hat{\beta} \geq 1, \quad i = 1, 2, \ldots, \overline{m},$$

(7)

and

$$\hat{x}_{i0} = x_{i0}, \quad i = \overline{m} + 1, \overline{m} + 2, \ldots, m.$$  

(8)
Let us introduce the following substitution
\[ \hat{\alpha} = 1 - \alpha, \quad 0 \leq \alpha < 1. \] (9)

Using (9) in (5) we have
\[ \hat{y}_r = y_r - \alpha_r > 0, \quad r = 1, 2, \ldots, \bar{s}, \] (10)
with
\[ \alpha_r = \alpha y_r, \quad \alpha_r \geq 0, \quad r = 1, 2, \ldots, \bar{s}. \] (11)

Because of (6) we have
\[ \hat{\alpha} = 1, \quad \alpha = 0, \quad \alpha_r = 0, \quad r = \bar{s} + 1, \bar{s} + 2, \ldots, s. \] (12)

Let us also introduce the substitution
\[ \hat{\beta} = 1 + \beta, \quad \beta \geq 0. \] (13)

Using (13) in (7) we have
\[ \hat{x}_i = x_i + \beta_i, \quad i = 1, 2, \ldots, m, \] (14)
with
\[ \beta_i = \beta x_i, \quad \beta_i \geq 0, \quad i = 1, 2, \ldots, m. \] (15)

Because of (8) we have
\[ \hat{\beta} = 1, \quad \beta = 0, \quad \beta_i = 0, \quad i = m + 1, m + 2, \ldots, m. \] (16)

It means that the proportionate change of a subset of outputs (5), with the other outputs fixed (6), can be considered as the additive change (10), with \( \alpha_r \) in (11) and (12). Similarly, the proportionate change of a subset of inputs (7), with the other inputs fixed (8), can be considered as the additive change (14) with \( \beta_i \) in (15) and (16). Because of that we will consider additive changes (10) of outputs together with (11) - (12) or/and additive changes (14) of inputs together with (15) - (16).

2.3. We will also consider the proportionate decrease of a subset of the first \( \bar{s} \) \((\bar{s} < s)\) outputs with different coefficients of proportionality which can be written as
\[ \hat{y}_r = \hat{\alpha}_r y_r, \quad 0 < \hat{\alpha}_r \leq 1, \quad r = 1, 2, \ldots, \bar{s}, \] (17)
with the last \( s - \bar{s} \) outputs fixed
\[ \hat{y}_r = y_r, \quad r = \bar{s} + 1, \bar{s} + 2, \ldots, s. \] (18)

Similarly, we will consider the proportionate increase of the subset of the first \( \bar{m} \) \((\bar{m} < m)\) inputs with different coefficients of proportionality and the last \( m - \bar{m} \) inputs fixed. It can be written as
\[ \hat{x}_i = \hat{\beta}_i x_i, \quad \hat{\beta}_i \geq 1, \quad i = 1, 2, \ldots, \bar{m}, \] (19)
and
\[ \hat{x}_{i0} = x_{i0}, \quad i = \overline{m+1, m+2, \ldots, m}. \] (20)

Let us introduce the following notation
\[ \hat{\alpha}_r = 1 - \bar{\alpha}_r, \quad 0 \leq \bar{\alpha}_r < 1. \] (21)

Using (21) in (17) we have
\[ \hat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad r = 1, 2, \ldots, \bar{s}, \] (22)
with
\[ \alpha_r = \bar{\alpha}_r y_{r0}, \quad \alpha_r \geq 0, \quad r = 1, 2, \ldots, \bar{s}. \] (23)

Because of (18) we have
\[ \hat{\alpha}_r = 1, \quad \bar{\alpha}_r = 0, \quad \alpha_r = 0, \quad r = \bar{s} + 1, \bar{s} + 2, \ldots, s. \] (24)

Similarly, let us introduce notation
\[ \hat{\beta}_i = 1 + \bar{\beta}_i, \quad \bar{\beta}_i \geq 0. \] (25)

Using (25) in (19) we have
\[ \hat{x}_{i0} = x_{i0} + \beta_i, \quad i = 1, 2, \ldots, \overline{m}, \] (26)
with
\[ \beta_i = \bar{\beta}_i x_{i0}, \quad \beta_i \geq 0, \quad i = 1, 2, \ldots, \overline{m}. \] (27)

Because of (20) we have
\[ \hat{\beta}_i = 1, \quad \bar{\beta}_i = 0, \quad \beta_i = 0, \quad i = \overline{m+1, m+2, \ldots, m}. \] (28)

So, the proportionate change of a subset of outputs with different coefficients of proportionality (17), with the other outputs fixed (18), can be considered as the additive change (22), with \( \alpha_r \) in (23) and (24). Similarly, the proportionate change of a subset of inputs with different coefficients of proportionality (19), with the other inputs fixed (20), can be considered as the additive change (26) with \( \beta_i \) in (27) and (28). Because of that we will consider additive changes (22) of outputs together with (23) - (24) or/and additive changes (26) of inputs together with (27) - (28).

2.4. For an efficient DMU \( 0 \) because of (2) vectors \( [ Y_0 - X_0 ]^T \) and \( [ 0 \quad X_0 ]^T \) must occur in some optimal basis, which means that there is a basic optimal solution to (1) with \( \lambda_S^* = 1 \) and \( \theta^* = 1 \). Similarly to Charnes and Neralić [2], simultaneous changes (10) - (12), and changes (14) - (16), are then accompanied by alterations in the inverse \( B^{-1} \) of the optimal basis matrix
\[ B = \begin{bmatrix} Y_B - I_B^T & 0 & 0 \\ -X_B & -I_B & 0 \end{bmatrix}, \] (29)
which corresponds to the optimal solution \((\lambda^*, s^{+*}, s^{-*}, \theta^*)\) of (1) with \(\lambda_0^* = 1\) and \(\theta^* = 1\). Let
\[
B^{-1} = \begin{bmatrix} b_{ij}^{-1} \end{bmatrix}, \quad i, j = 1, 2, \ldots, s + m,
\]
be the inverse of the optimal basis matrix \(B\) in (29). Let \(P_j, j = 1, 2, \ldots, n + s + m + 1\) be the columns of the matrix and let \(P_0\) be the right-hand side of the linear programming problem (1). We will use the following notation:
\[
\Gamma_j = B^{-1}P_j, \quad j = 0, 1, \ldots, n + s + m + 1, \\
\omega^T = c^T_0B^{-1}, \\
z_j = c^T_0B^{-1}P_j \\
= \omega^TP_j, \quad j = 0, 1, \ldots, n + s + m + 1.
\]

The simultaneous change of outputs (10) together with (11) - (12) and of inputs (14) together with (15) - (16) leads to the following change of the optimal basis matrix \(B\)
\[
\hat{B} = B + \Delta B
\]
(30)
with
\[
\Delta B = \begin{pmatrix}
0 & \cdots & 0 & -\alpha_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -\alpha_s & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -\beta_1 & 0 & \cdots & \beta_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -\beta_{s-1} & 0 & \cdots & \beta_{s-1} \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
and the following change of the right-hand side vector
\[
\hat{P}_0 = P_0 + [-\alpha_1 - \alpha_2 \ldots - \alpha_s 0 \ldots 0]^T,
\]
(32)
where indices \(k\) and \(s + m\) correspond to the optimal basic variables \(\lambda_0^* = 1\) and
θ∗ = 1, respectively. Using matrices

\[
U = U_{(s+m)\times 2} = \begin{bmatrix}
\alpha_1 & \alpha_1 \\
\vdots & \vdots \\
\alpha_s & \alpha_s \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
\beta_1 & 0 \\
\vdots & \vdots \\
\beta_m & 0 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0
\end{bmatrix}
\]  \hspace{1cm} (33)

and

\[
V^T = V_{2\times(s+m)}^T = \begin{bmatrix}
0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1
\end{bmatrix}
\]  \hspace{1cm} (34)

we can write the perturbation matrix (31) as

\[
\triangle B = UV^T.
\]  \hspace{1cm} (35)

As in Charnes and Neralić [2], because of (30) and (35) we can use the Sherman-Morrison-Woodbury formula (see, for example, Golub and Van Loan [6], p. 3) to get the following perturbed basis matrix inverse

\[
(\hat{B})^{-1} = (B + UV^T)^{-1} = B^{-1} - B^{-1}U(I + V^TB^{-1}U)^{-1}V^TB^{-1}.
\]  \hspace{1cm} (36)

Using the abbreviation

\[
D = U(I + V^TB^{-1}U)^{-1}V^T
\]  \hspace{1cm} (37)

we can write (36) as

\[
(\hat{B})^{-1} = B^{-1} - B^{-1}DB^{-1}
\]

\[
= B^{-1}(I - DB^{-1})
\]

\[
= (I - B^{-1}D)B^{-1}.
\]  \hspace{1cm} (38)

Also, using (33), (34) we can get \(V^TB^{-1}U\) and

\[
M = I + V^TB^{-1}U
\]  \hspace{1cm} (39)
with

\[
\det M = 1 - \sum_{t=1}^{g} b_{k,t}^{-1}\alpha_t + \sum_{t=1}^{\overline{n}} (-b_{k,s+t}^{-1} + b_{s+m,s+t}^{-1})\beta_t + \\
+ \left( \sum_{t=1}^{g} b_{s+m,t}^{-1}\alpha_t \right) \left( \sum_{t=1}^{\overline{n}} b_{k,s+t}^{-1}\beta_t \right) - \left( \sum_{t=1}^{g} b_{k,t}^{-1}\alpha_t \right) \left( \sum_{t=1}^{\overline{n}} b_{s+m,s+t}^{-1}\beta_t \right).
\]

(40)

It is easy to get inverse \( M^{-1} \) of the matrix \( M \) and matrix \( \Gamma D = U M^{-1} V^T \) in the form

\[
D = \begin{bmatrix}
0 & \ldots & 0 & d_{1k} & 0 & \ldots & 0 & d_{1,s+m} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & d_{k,k} & 0 & \ldots & 0 & d_{k,s+m} \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & d_{s+1,k} & 0 & \ldots & 0 & d_{s+1,s+m} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & d_{s+m,k} & 0 & \ldots & 0 & d_{s+m,s+m} \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix},
\]

(41)

with

\[
d_{t,k} = -\frac{1}{\det M} \left( 1 + \sum_{t=1}^{\overline{n}} b_{s+m,s+t}^{-1}\beta_t \right)\alpha_t, \quad t = 1, 2, \ldots, \overline{s},
\]

(42)

\[
d_{s+t,k} = \frac{1}{\det M} \left( -1 + \sum_{t=1}^{g} b_{s+m,t}^{-1}\alpha_t \right)\beta_t, \quad t = 1, 2, \ldots, \overline{m},
\]

(43)

\[
d_{t,s+m} = \frac{1}{\det M} \left( \sum_{t=1}^{\overline{n}} b_{k,s+t}^{-1}\beta_t \right)\alpha_t, \quad t = 1, 2, \ldots, \overline{s},
\]

(44)

\[
d_{s+t,s+m} = \frac{1}{\det M} \left( 1 - \sum_{t=1}^{g} b_{k,t}^{-1}\alpha_t \right)\beta_t, \quad t = 1, 2, \ldots, \overline{m}.
\]

(45)

Now we can prove the following

**Theorem 1.** Let us suppose that DMU_0 is efficient. Conditions

\[
\omega^T \Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables},
\]

(46)

are sufficient for DMU_0 to continue to be efficient after the simultaneous proportional changes of a subset of outputs (5) - (6) and of a subset of inputs (7) - (8).

The proof is omitted because it is similar to the proof of Theorem 1 in Charnes and Neralić [2].
Remark 1. In conditions (46) there is det \( M \) in the denominator of elements of matrix \( D \). In order to get the system of inequalities in \( \alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_m \) we have to multiply (46) by \( \text{det} M \). So, if we suppose that \( \text{det} M > 0 \), multiplying (46) by \( \text{det} M > 0 \) will keep the sign \( \geq \) in (46). If we suppose that \( \text{det} M < 0 \), because of multiplying (46) by \( \text{det} M < 0 \) the sign in (46) has to be changed to \( \leq \). Besides, constraints (10) - (12) and (14) - (16), should be added to conditions (46). The solution set of the corresponding system of inequalities \( S_{\bar{0}} \) will be a set of points \( (\alpha_1, \ldots, \alpha_s, 0, \ldots, 0, \beta_1, \ldots, \beta_m, 0, \ldots, 0) \) in \( \mathbb{R}^{s \times m} \). Because of (11) and (15) we can get the corresponding system of inequalities in \( \alpha_\bar{i} \) and the solution set \( S_{\bar{0}}^* \). For all points \( (\bar{\alpha}, \bar{\beta}) \) in the solution set \( S_{\bar{0}}^* \) after the changes of outputs according to (5) - (6) and the changes of inputs according to (7) - (8) efficiency of DMU \( 0 \) will be preserved. The solution set \( S_{\bar{0}}^* \) gives an area \( A_{\bar{0}}^* \) in the plane with the coordinate system \( \bar{\alpha}\hat{O}\bar{\beta} \).

Remark 2. Using points \( (\bar{\alpha}, \bar{\beta}) \) from the set \( S_{\bar{0}}^* \) in (5), (7) with (6), (8) we can get the corresponding region of efficiency \( R_{\bar{0}} \) around DMU \( 0 \). The size of the region of efficiency around efficient point, within which perturbations (5) - (6) and (7) - (8) keep it efficient, is an important property of the (empirical) efficient production function at this point. It is a measure of stability of efficiency at that point. If for efficient DMU \( 1 \) and DMU \( 0 \) holds \( R_1 > R_2 \) it means that DMU \( 1 \) is more stable than DMU \( 2 \) in preserving efficiency at the simultaneous proportionate changes (5) - (6) and (7) - (8).

For the case with \( \text{det} M > 0 \) (see also Theorem 2 in Charnes and Neralić [4]) it is easy to get from Theorem 1 the following

Corollary 1. Let us suppose that DMU \( 0 \) is efficient and let

\[
\text{det} \ M = 1 - a_1(1 - \bar{\alpha}) + (-b_1 + b_2)(\bar{\beta} - 1) + (a_2b_1 - a_1b_2)(1 - \bar{\alpha})(\bar{\beta} - 1) > 0, \quad (47)
\]

with

\[
a_1 = \sum_{t=1}^{\bar{s}} b_{1t}^{-1} y_{t0}, \quad a_2 = \sum_{t=1}^{\bar{s}} b_{s+m, t}^{-1} y_{t0}, \quad b_1 = \sum_{t=1}^{\bar{m}} b_{s+m, t}^{-1} x_{t0}, \quad b_2 = \sum_{t=1}^{\bar{m}} b_{s+m, s+t}^{-1} x_{t0}. \quad (48)
\]

Let

\[
a_3 = \sum_{t=1}^{\bar{s}} \omega_t y_{t0}, \quad b_3 = \sum_{t=1}^{\bar{m}} \omega_{s+m} x_{t0}, \quad (49)
\]

\[
d_j = -a_3 \Gamma_{kj} + a_1 \bar{c}_j, \quad e_j = -b_3 (\Gamma_{kj} - \Gamma_{s+m,j}) - (-b_1 + b_2) \hat{c}_j, \quad (50)
\]

\[
f_j = (a_2b_3 - a_3b_2) \Gamma_{kj} + (a_3b_1 - a_1b_3) \Gamma_{s+m,j} - (a_2b_1 - a_1b_2) \hat{c}_j, \quad (51)
\]

\[j = 1, 2, \ldots, n + s + m + 1, \]

with \( \bar{c}_j = z_j - c_j \). Then the conditions

\[
d_j(1 - \bar{\alpha}) + e_j(\bar{\beta} - 1) + f_j(1 - \bar{\alpha})(\bar{\beta} - 1) \geq \bar{c}_j, \quad (52)
\]

\[j \text{ an index of nonbasic variables}, \]

with \( \bar{c}_j = z_j - c_j \). Then the conditions

\[
d_j(1 - \bar{\alpha}) + e_j(\bar{\beta} - 1) + f_j(1 - \bar{\alpha})(\bar{\beta} - 1) \geq \bar{c}_j, \quad (52)
\]

\[j \text{ an index of nonbasic variables}, \]
are sufficient for DMU$_0$ to preserve efficiency after the simultaneous proportionate changes of a subset of outputs (5) - (6) and of a subset of inputs (7) - (8).

For fixed inputs, we can consider the proportionate change (decrease) of a subset of outputs (5) with the other outputs fixed (6). In that case it is easy to get the matrix $D_1$ from the matrix $D$ with elements in (42) - (45) taking $\beta_t = 0$, $t = 1, 2, \ldots, \pi$. With the matrix $D_1$ instead of matrix $D$ from Theorem 1 we have the following

\textbf{Corollary 2. Conditions}

\[ \omega^T D_1 \Gamma_j \geq z_j - c_j, \quad \text{\textit{j an index of nonbasic variables}}, \quad (53) \]

are sufficient for DMU$_0$ to be efficient after the proportionate changes of a subset of outputs (5) with the other outputs and all inputs fixed.

For fixed outputs, we can consider the proportionate change (increase) of a subset of inputs (7) with the other inputs fixed (8). In that case it is easy to get the matrix $D_2$ from the matrix $D$ with elements in (42) - (45) taking $\alpha_t = 0$, $t = 1, 2, \ldots, \pi$. With the matrix $D_2$ instead of matrix $D$ from Theorem 1 we have the following

\textbf{Corollary 3. Conditions}

\[ \omega^T D_2 \Gamma_j \geq z_j - c_j, \quad \text{\textit{j an index of nonbasic variables}}, \quad (54) \]

are sufficient for DMU$_0$ to be efficient after the proportionate change of a subset of inputs (7) with other inputs and all outputs fixed.

\textbf{Remark 3.} In the case of the proportionate change (decrease) of a subset of outputs with different coefficients of proportionality (17) - (18) or/and the proportionate change (increase) of inputs with different coefficients of proportionality (19) - (20) of an efficient DMU$_0$ similar results on its efficiency preservation as in Subsection 2.4. (Theorem 1, Corollaries 1, 2 and 3) can be obtained.

3. Illustrative example

\textbf{3.1.} We will consider the following example taken from Rhodes [9] (see also Charnes and Neralić [3]) with five DMUs, one output, two inputs and data in Table 1.

\begin{table}[h]
\centering
\begin{tabular}{l c c c c c}
\hline
\multicolumn{2}{c}{Output/Input} & DMU$_j$ & 1 & 2 & 3 \\
\hline
$y_{1j}$ & 2 & 4 & 2 & 3 & 2 \\
$x_{1j}$ & 4 & 12 & 8 & 6 & 2 \\
$x_{2j}$ & 6 & 8 & 2 & 6 & 8 \\
\hline
\end{tabular}
\caption{Data for the example}
\end{table}

We are interested in the efficiency of DMU$_4$, with $X_0 = [6 \ 6]^T$ and $Y_0 = [3]$. In order to see if DMU$_4$ is efficient, the following linear programming problem should be solved:

\[
\begin{align*}
\text{min} \quad & 0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 0\lambda_4 + 0\lambda_5 - \varepsilon s_1^+ - \varepsilon s_1^- - \varepsilon s_2^- + \theta \\
\text{subject to} \quad & 2\lambda_1 + 4\lambda_2 + 2\lambda_3 + 3\lambda_4 + 2\lambda_5 - s_1^+ = 3 \\
& -4\lambda_1 - 12\lambda_2 - 8\lambda_3 - 6\lambda_4 - 2\lambda_5 - s_1^- + 6\theta = 0 \\
& -6\lambda_1 - 8\lambda_2 - 2\lambda_3 - 6\lambda_4 - 8\lambda_5 - s_2^- + 6\theta = 0 \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4, s_1^+, s_1^-, s_2^- \geq 0.
\end{align*}
\]

According to (3), in Phase I the following LP problem should be solved:

\[
\begin{align*}
\text{min} \quad & \theta \\
\text{subject to the constraints of (55). The optimal solution of the LP problem (56) is} \\
& \lambda_4^* = 1, \theta^* = 1, \lambda_2^* = \lambda_3^* = \lambda_5^* = 0, s_1^{+*} = s_1^{-*} = s_2^{-*} = 0. \text{ Using } \theta^* = 1 \text{ from} \\
& \text{the optimal solution of the LP problem (56) in Phase II we solve the LP problem} \\
& \begin{align*}
\text{min} (-s_1^- - s_1^+ - s_2^-) \\
\text{subject to the constraints of (55) with the substitution } \theta = \theta^* = 1. \text{ Optimal solution} \\
& \text{of (57) is } \lambda_4^* = 1, \lambda_2^* = \lambda_3^* = \lambda_5^* = 0, s_1^{+*} = s_1^{-*} = s_2^{-*} = 0, \text{ which means that} \\
& \text{DMU}_4 \text{ is efficient.}
\end{align*}
\end{align*}
\]

Using these results it is easy to reconstruct the optimum tableau for the problem (55) which is given in Table 2 below. Namely, it is slightly changed optimum tableau for the LP (56), where \( z_6^* - c_6^* = -(1/3), z_7^* - c_7^* = -(1/9) \) and \( z_8^* - c_8^* = -(1/18) \) are changed to \( z_6 - c_6 = -(1/3) + \varepsilon, z_7 - c_7 = -(1/9) + \varepsilon \) and \( z_8 - c_8 = -(1/18) + \varepsilon \).

The optimal solution of problem (55) is \( \lambda_4^* = \lambda_2^* = 1, \theta^* = 1, \lambda_3^* = \lambda_5^* = 0, s_1^{+*} = s_1^{-*} = s_2^{-*} = 0 \) with optimal basic variables \( \lambda_3^* = 0, \lambda_4^* = 1, \theta^* = 1 \). The optimal basis matrix is

\[
B = \begin{bmatrix}
2 & 3 & 0 \\
-8 & -6 & 6 \\
-2 & -6 & 6 \\
\end{bmatrix},
\]

with the inverse

\[
B^{-1} = \begin{bmatrix}
0 & -\frac{1}{7} & -\frac{1}{9} \\
\frac{1}{7} & 0 & -\frac{1}{9} \\
\frac{1}{9} & \frac{1}{7} & 0 \\
\end{bmatrix},
\]

and the corresponding optimum tableau in Table 2. (Because of degeneracy, some other optimal solutions can be obtained, as, for example, the one with basic variables \( \lambda_4^* = 1, \lambda_3^* = 0, \theta^* = 1 \).)

<table>
<thead>
<tr>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \theta )</th>
<th>( z_3 - c_3 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
<th>( \Gamma_4 )</th>
<th>( \Gamma_5 )</th>
<th>( \Gamma_6 )</th>
<th>( \Gamma_7 )</th>
<th>( \Gamma_8 )</th>
<th>( \Gamma_9 )</th>
<th>( \Gamma_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{3} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{1}{3} )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( \frac{1}{2} )</td>
<td>( 0 )</td>
<td>( \frac{1}{2} )</td>
<td>( 0 )</td>
<td>( \frac{1}{2} )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>( \frac{3}{2} )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( \frac{1}{2} )</td>
<td>( -\frac{1}{3} + \varepsilon )</td>
<td>( -\frac{1}{18} + \varepsilon )</td>
<td>( -\frac{1}{18} + \varepsilon )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Optimum tableau
3.2. Let us consider the simultaneous proportionate change (decrease) of output

$$\hat{y}_{10} = 3\hat{\alpha}, \quad 0 < \hat{\alpha} \leq 1,$$

and proportionate change (increase) of input 1

$$\hat{x}_{10} = 6\hat{\beta}, \quad \hat{\beta} \geq 1$$

with input 2 fixed

$$\hat{x}_{20} = 6$$

of DMU 4 preserving its efficiency. So, we consider proportionate change of output and of a subset of inputs of DMU 4.

Using substitutions (9), with (10) - (12), (13) with (14) - (16) it is easy to get

$$\hat{y}_{10} = 3 - \alpha_1 > 0, \quad \alpha_1 = 3\alpha, \quad 0 \leq \alpha_1 < 3.$$ (62)

and

$$\hat{x}_{10} = 6 + \beta_1, \quad \beta_1 = 6\beta, \quad \beta_1 \geq 0,$$

$$\hat{x}_{20} = 6.$$ (64)

Because of $$s = 1, \ \bar{s} = 1, \ m = 2, \ \bar{m} = 1, \ k = 2, \ s + m = 3,$$ the optimal basis perturbation matrix is

$$\Delta B = \begin{bmatrix} 0 & -\alpha_1 & 0 \\ 0 & -\beta_1 & \beta_1 \\ 0 & 0 & 0 \end{bmatrix},$$ (65)

and the change of the right-hand side vector is

$$\hat{P}_0 = P_0 + [-\alpha_1 \ 0 \ 0]^T.$$ (66)

Using matrices

$$U = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \beta_1 & 0 \\ 0 & 0 \end{bmatrix},$$ (67)

and

$$V^T = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$ (68)

and (65) we can write the perturbed optimal basis matrix as

$$\hat{B} = B + \Delta B = B + UV^T.$$ (69)

It is easy to get

$$M = I + V^T B^{-1} U = \begin{bmatrix} 1 - (1/18)\beta_1 & 0 \\ -(-1/18)(6\alpha_1 + \beta_1) & 1 - (1/3)\alpha_1 \end{bmatrix},$$

$$\det M = 1 - (1/3)\alpha_1 - (1/18)\beta_1 + (1/18)\alpha_1\beta_1,$$ (70)

$$M^{-1} = \frac{1}{\det M} \begin{bmatrix} 1 - (1/3)\alpha_1 & 0 \\ (-1/18)(6\alpha_1 + \beta_1) & 1 + (1/18)(-\beta_1) \end{bmatrix}.$$
Proportionate change of a subset of data in DEA

$$D = UM^{-1}V^T$$

$$= \frac{1}{\det M} \begin{bmatrix}
0 & -[1 + (1/18)\beta_1] & [(1/9)\beta_1] & [1 - (1/3)\alpha_1] \\
0 & [1 + (1/3)\alpha_1] & [\beta_1] & [1 - (1/3)\alpha_1] \\
0 & 0 & 0 & 0
\end{bmatrix},$$

(71)

$$\omega^T = c^T B^{-1} = [0 \ 1] B^{-1} = [1/3 \ 1/18 \ 1/9],$$

(72)

$$\omega^T D = \frac{1}{\det M} \begin{bmatrix}
0 & -[1 - (1/3)\alpha_1 - (1/18)\beta_1 + (1/54)\alpha_1 \beta_1]
\end{bmatrix}.$$  

(73)

Let us suppose that

$$\det M = 1 - (1/3)\alpha_1 - (1/18)\beta_1 + (1/54)\alpha_1 \beta_1 > 0.$$  

(74)

Using (73) with (74) and elements of Table 2 in (46) it is easy to get sufficient conditions for DMU4 to preserve its efficiency as the system of inequalities in \(\alpha_1, \beta_1\) in the form

$$5\alpha_1 + \beta_1 \leq 3, \ 0 \leq \alpha_1 < 3, \ \beta_1 \geq 0.$$  

(75)

The other constraints, including (74), are redundant. Using (62), (63) in (75) we can get the corresponding system of inequalities in \(\alpha, \beta\)

$$5\alpha + 2\beta \leq 1, \ 0 \leq \alpha < 1, \ \beta \geq 0.$$  

(76)

Using substitutions (9), (13) it is easy to get from (76) the corresponding system of inequalities in \(\hat{\alpha}, \hat{\beta}\)

$$5\hat{\alpha} - 2\hat{\beta} \geq 2, \ 0 < \hat{\alpha} \leq 1, \ \hat{\beta} \geq 1.$$  

(77)

Let us point out that in order to get the system of inequalities (77) we could use conditions (52) from Corollary 1.

It is easy to see that the solution set \(S^*_4\) of the system of inequalities (77) is the triangle \(\hat{AB}C\) in the plane with the coordinate system \(\hat{\alpha}\hat{O}\hat{\beta}\), with \(\hat{A}(1, 1.5), \ \hat{B}(0.8, 1)\) and \(\hat{C}(1, 1)\). For every point \((\hat{\alpha}, \hat{\beta})\) which belongs to the triangle \(\hat{AB}C\) the efficiency of DMU4 will be preserved after the simultaneous proportionate change (59) of output with the coefficient \(\hat{\alpha}\) and proportionate change (60) of input 1 with the coefficient \(\hat{\beta}\) and input 2 fixed (61). The point \(\hat{C}(1, 1)\) means that there are no changes of output and of input 1, the point \(\hat{A}(1, 1.5)\) means the maximal proportionate increase of input 1 of DMU4 for 50% preserving its efficiency and the point \(\hat{B}(0.8, 1)\) means the maximal proportionate decrease of output of DMU4 for 20% preserving efficiency of DMU4. Let us point out that these maximal changes cannot be done simultaneously.

Using the solution set \(S^*_4\) we can get the corresponding region of efficiency \(R_4\) around DMU4 as the triangle ABC in the coordinate system \(Ox_1x_2y_1\) (\(x_1 = \) input 1, \(x_2 = \) input 2, \(y_1 = \) output 1) with \(A(9,6,3), B(6,6,2.4), C(6,6,3)\). (Because input 2 is fixed, we can project triangle ABC into the plane \(x_1Oy_1\) and get the triangle \(AB'C'\), with \(A'(9,3), B'(6,2.4)\) and \(C'(6,3)\).) The size of the region of efficiency \(R_4\) is 0.9 (which is the area of the triangle ABC or triangle \(AB'C'\) and it is a measure of stability of efficiency of DMU4.
3.3. Let us consider the case of the proportionate change of a subset of outputs or/and of a subset of inputs with different coefficients of proportionality in the example from Subsection 3.1. For efficient DMU_0 = DMU_4 we can consider the proportionate change (decrease) of output
\[ \hat{y}_{10} = 3\hat{\alpha}_1, \quad 0 < \hat{\alpha}_1 \leq 1, \]  
and the proportionate change (increase) of input 1
\[ \hat{x}_{10} = 6\hat{\beta}_1, \quad \hat{\beta}_1 \geq 1 \]
with input 2 fixed
\[ \hat{x}_{20} = 6 \]
preserving its efficiency. So, we consider the proportionate change of output and of a subset of inputs of DMU_4 with different coefficients of proportionality. But, this is just the case considered in Subsection 3.2, with \( \hat{\alpha} = \hat{\alpha}_1 \) and \( \hat{\beta} = \hat{\beta}_1 \). It means that the results in that case are the same as results obtained in the case in Subsection 3.2, with \( \hat{\alpha} = \hat{\alpha}_1 \) and \( \hat{\beta} = \hat{\beta}_1 \).

4. Summary and conclusions

The proportionate change of a subset of inputs and/or proportionate change of a subset of outputs of an efficient DMU_0 preserving efficiency for the case of the CCR ratio model in DEA is studied in the paper. Sufficient conditions for an efficient DMU_0 to preserve efficiency are established for the case of the proportionate decrease of a subset of outputs with the coefficient of proportionality \( \hat{\alpha} \) or/and the proportionate increase of a subset of inputs with the coefficient of proportionality \( \hat{\beta} \). Similar results could be obtained for the case of the proportionate decrease of a subset of outputs with different coefficients of proportionality \( \hat{\alpha}_r, r = 1, 2, \ldots, s < s \) or/and the proportionate increase of a subset of inputs with different coefficients of proportionality \( \hat{\beta}_i, i = 1, 2, \ldots, m < m \). In the case with \( \hat{\alpha} \) and \( \hat{\beta} \) sufficiency conditions give for each efficient DMU_0 region of efficiency and the area the size of which is a measure of stability of efficiency at the proportionate change of a subset of inputs or/and of a subset of outputs. A numerical example illustrating the results is provided.

Sensitivity and stability analysis for the case of the proportionate change of a subset of inputs or/and of a subset of outputs with two parameters (one for outputs and the other for inputs) preserving efficiency of an efficient DMU_0 according to the BCC or Additive model is an interesting open question. The same holds for the case of proportionate changes with different coefficients of proportionality. Also, the question of efficiency preservation of all efficient DMUs according to the CCR ratio model under the proportionate (or additive) changes of all data is open. An application of the results in the paper using data for a real world problem seems to be a challenge too.
References


