ONE–DIMENSIONAL FLOW OF A COMPRESSIBLE VISCOUS MICROPOLAR FLUID: A LOCAL EXISTENCE THEOREM

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Abstract. An initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodinamically perfect and politropic. A local-in-time existence and uniquenes theorem is proved.

1. Introduction

Theory of a polar or Cosserat continuum ([4], [1], [5], [6]) is based on the assumption that an appropriate dynamical field in a medium is a torzor (e.g. [7]), the reduction elements of which are momentum and intrinsic spin. As a consequence, instead of the symmetry of the stress tenzor, a new conservation law (for the momentum moment) appears. Kinematical and contact fields corresponding to the spin are, respectively, microrotation velocity and couple stress tenzor. We consider here an isotropic, viscous and compressible fluid, that is (in a thermodinamical sense) perfect and politropic. In the setting of the field equations we use the Eulerian description.

Notation:

- $\rho$ – mass density
- $\nu$ – velocity
- $D(\nu)$ – stretching, $D(\nu) = \text{sym} \nabla \nu$
- $p$ – pressure
- $T$ – stress tenzor
- $T_{ax}$ – an axial vector with the Cartesian components $(T_{ax})_i = e_{ijk} T_{kj}/2$, where $e_{ijk}$ is the alternating tenzor
- $\omega$ – microrotation velocity
- $\omega_{skw}$ – a skew tenzor with the Cartesian components $(\omega_{skw})_{ij} = e_{ijk} \omega_k$
- $j$ – microinertia density (a positive scalar field)
- $M$ – couple stress tenzor
- $\theta$ – absolute temperature
- $e$ – internal energy density
- $q$ – heat flux density vector
- $f$ – body force density


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Local forms of the conservation laws for the mass, momentum, momentum moment and energy are, respectively, as follows:

\[ \rho + \rho \text{div} \mathbf{v} = 0, \]  
\[ \rho \mathbf{v} = \text{div} T + \rho f, \]  
\[ \rho j \dot{\omega} = \text{div} M + T_{ax} + \rho m, \]  
\[ \rho \dot{e} = T \cdot \nabla \mathbf{v} + M \cdot \nabla \omega - 2T_{ax} \cdot \omega + \text{div} q + \rho r, \]

where \( \dot{a} \) denotes material derivative of a field \( a \):

\[ \dot{a} = \frac{\partial a}{\partial t} + (\nabla a) \mathbf{v}. \]

The linear constitutive equations for stress tensor, couple stress tensor and heat flux density vector are, respectively, of the forms:

\[ T = -pI + \lambda (\text{div} \mathbf{v}) I + 2\mu D(\mathbf{v}) + \chi (\nabla \mathbf{v} + \omega_{skw}), \]  
\[ M = \alpha (\text{div} \omega) I + \beta (\nabla \omega)^T + \gamma \nabla \omega, \]  
\[ q = k \nabla \theta, \]

where \( \lambda, \mu, \chi, \alpha, \beta, \gamma \) and \( k \) are scalar material coefficients, depending generally on mass density and temperature and satisfying the conditions ([5], [6]):

\[ 3\lambda + 2\mu + \chi \geq 0, \quad 2\mu + \chi \geq 0, \quad \chi \geq 0, \]
\[ 3\alpha + \beta + \gamma \geq 0, \quad |\beta| \leq 0, \quad k \geq 0. \]

Assuming that the fluid is perfect and polytropic, for pressure and internal energy we have the equations:

\[ p = R \rho \theta, \]  
\[ e = c \theta, \]

where \( R \) and \( c \) are positive constants.

Initial-boundary value problems for the system (1.1)–(1.7), (1.10)–(1.11) so far were not considered (for incompressible flow see [10], [17], [18], [19], [21], [22], [23]).

It is well known that even for a classical fluid (when the coefficients \( j, \chi, \alpha, \beta \) and \( \gamma \) are equal zero) a few results are obtained for three-and-two-dimensional problems (see [2], [14] and [8] and references therein); a global existence theorems are proved for isentropic case ([15], [16]) and for one-dimensional flow ([11], [12], [2]; see also [9]).

2. Statement of the problem and the main results

In this paper we consider the system (1.1)–(1.7), (1.10)–(1.11) for one-dimensional flow, assuming that all material coefficients (including \( j \)) are constants.
Let (in a Cartesian coordinate frame) \( v_2 = v_3 = \omega_2 = \omega_3 = 0 \) and let the functions \( \rho, v = v_1, \omega = \omega_1 \) and \( \theta \) depend on \( x = x_1 \) and \( t \) only. Inserting (1.5)–(1.7), (1.10)–(1.11) into (1.2)–(1.4) and taking \( f = m = r = 0 \), we obtain the system:

\[
\begin{align*}
\rho + \rho \frac{\partial v}{\partial x} &= 0, \\
\rho \dot{v} &= -\frac{\partial}{\partial x} (R \rho \theta) + \sigma_1 \frac{\partial^2 v}{\partial x^2}, \\
j \rho \omega &= \sigma_2 \frac{\partial^2 \omega}{\partial x^2} - 2 \chi \omega, \\
\rho \phi &= -R \rho \theta \frac{\partial \phi}{\partial x} + \sigma_1 \left( \frac{\partial v}{\partial x} \right)^2 + \sigma_2 \left( \frac{\partial \omega}{\partial x} \right)^2 + 2 \chi \omega^2 + k \frac{\partial^2 \omega}{\partial x^2},
\end{align*}
\]

where

\[
\begin{align*}
\sigma_1 = \lambda + 2 \mu + \chi, \\
\sigma_2 = \alpha + \beta + \gamma.
\end{align*}
\]

Because of (1.8) and (1.9) it holds \( \sigma_1 \geq 0, \sigma_2 \geq 0 \); we assume

\[
\begin{align*}
\sigma_1, \sigma_2, \chi, k \in \mathbb{R}^+ = ]0, +\infty[.
\end{align*}
\]

We shall consider the system (2.1)–(2.4) in the domain \( ]0, L[ \times \mathbb{R}^+, L \in \mathbb{R}^+ \), under the homogeneous boundary conditions:

\[
\begin{align*}
\nu(0, t) = \nu(L, t) &= 0, \\
\omega(0, t) = \omega(L, t) &= 0, \\
\frac{\partial \theta}{\partial x}(0, t) = \frac{\partial \theta}{\partial x}(L, t) &= 0
\end{align*}
\]

for \( t > 0 \) and non-homogeneous initial conditions:

\[
\begin{align*}
\rho(x, 0) &= \rho_0(x), \\
\nu(x, 0) &= \nu_0(x), \\
\omega(x, 0) &= \omega_0(x), \\
\theta(x, 0) &= \theta_0(x)
\end{align*}
\]

for \( x \in ]0, L[ \). Here \( \rho_0, \nu_0, \omega_0 \) and \( \theta_0 \) are given functions. We assume that the functions \( \rho_0 \) and \( \theta_0 \) are strictly positive and bounded:

\[
m \leq \rho_0 \leq M, \ m \leq \theta_0(x) \leq M \quad \text{for} \ \ x \in ]0, L[, 
\]

where \( m, M \in \mathbb{R}^+ \).

It is convenient to transform our problem to the Lagrangian form. For \( \xi \in ]0, L[ \) let \( t \rightarrow \phi_t(\xi) \) be a solution of the Cauchy problem

\[
\frac{d\phi_t}{dt} = \nu(\phi_t, t), \quad \phi_0(\xi) = \xi.
\]

Because of (2.6) the mapping \( \xi \rightarrow x = \phi_t(\xi) \) is a diffeomorphism \( ]0, L[ \rightarrow ]0, L[ \). To an Eulerian field \( f(x, t) \) on \( ]0, L[ \times \mathbb{R}^+ \) it corresponds a Lagrangian field \( \tilde{f}(\xi, t) = f(\phi_t(\xi), t) \) on the same domain. Taking into account the equality

\[
f = \frac{\partial \tilde{f}}{\partial t} \circ \phi_t^{-1},
\]
one can easily obtain the system of equations for the functions \( \tilde{\rho}, \tilde{v}, \tilde{\omega} \) and \( \tilde{\theta} \). Let

\[
\psi(\xi) = \int_0^\xi \rho_0(\xi) \, d\xi, \quad \eta = \psi(L), \quad \delta = \eta \sigma_1^{-1}(2\chi)^{-\frac{1}{2}} \sigma_2^{\frac{1}{2}},
\]

\[
\zeta_1 = \eta^{-1}(2\chi)^{-\frac{1}{2}} \sigma_2^{\frac{1}{2}}, \quad \zeta_2 = \eta \sigma_1^{-1}, \quad \zeta_3 = \eta \sigma_1^{-1} \sigma_2^{\frac{1}{2}}, \quad \zeta_4 = c \eta^2 \sigma_1^{-2}.
\]

It is useful to introduce the new coordinates

\[
x' = \eta^{-1} \psi(\xi), \quad t' = \delta^{-1} t
\]

and the new functions

\[
\rho'(x', t') = \zeta_1 \tilde{\rho}(\psi^{-1}(\eta x'), \delta t'),
\]

\[
\nu'(x', t') = \zeta_2 \tilde{\nu}(\psi^{-1}(\eta x'), \delta t'),
\]

\[
\omega'(x', t') = \zeta_3 \tilde{\omega}(\psi^{-1}(\eta x'), \delta t'),
\]

\[
\theta'(x', t') = \zeta_4 \tilde{\theta}(\psi^{-1}(\eta x'), \delta t').
\]

Let

\[
K = R c^{-1}, \quad A = j^{-1} \sigma_1^{-1} \sigma_2, \quad D = kc^{-1} \sigma_1^{-1},
\]

\[
\rho'_0(x') = \zeta_1 \rho_0(\psi^{-1}(\eta x')),
\]

\[
\nu'_0(x') = \zeta_2 \nu_0(\psi^{-1}(\eta x')),
\]

\[
\omega'_0(x') = \zeta_3 \omega_0(\psi^{-1}(\eta x')),
\]

\[
\theta'_0(x') = \zeta_4 \theta_0(\psi^{-1}(\eta x')).
\]

Then the functions \( \rho', \nu', \omega' \) and \( \theta' \) satisfy the system that we write omitting for simplicity the primes:

\[
\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \nu}{\partial x} = 0,
\]

\[
\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x} \left( \rho \frac{\partial \nu}{\partial x} \right) - K \frac{\partial}{\partial x} (\rho \theta) = 0,
\]

\[
\rho \frac{\partial \omega}{\partial t} = A \left[ \rho \frac{\partial}{\partial x} \left( \rho \frac{\partial \omega}{\partial x} \right) - \omega \right],
\]

\[
\rho \frac{\partial \theta}{\partial t} = -K \rho^2 \rho \frac{\partial \nu}{\partial x} + \rho^2 \left( \frac{\partial \nu}{\partial x} \right)^2 + \rho^2 \left( \frac{\partial \omega}{\partial x} \right)^2 + \omega^2 + D \rho \frac{\partial}{\partial x} \left( \rho \frac{\partial \theta}{\partial x} \right)
\]

in \( ]0,1[ \times \mathbb{R}^+ \),

\[
\nu(0, t) = \nu(1, t) = 0,
\]

\[
\omega(0, t) = \omega(1, t) = 0,
\]

\[
\frac{\partial \theta}{\partial x}(0, t) = \frac{\partial \theta}{\partial x}(1, t) = 0.
\]
for \( t \in \mathbb{R}_+ \),

\[
\begin{align*}
\rho(x,0) &= \rho_0(x), \\
\nu(x,0) &= \nu_0(x), \\
\omega(x,0) &= \omega_0(x), \\
\theta(x,0) &= \theta_0(x),
\end{align*}
\] (2.21)

for \( x \in ]0, 1[ \). The functions \( \rho_0 \) and \( \theta_0 \) satisfy the conditions

\[
m \leq \rho_0 \leq M, \quad m \leq \theta_0(x) \leq M \quad \text{for} \ x \in ]0, 1[,
\] (2.25)

where \( m, M \in \mathbb{R}_+ \). The problem (2.14)–(2.24) is equivalent to the problem (2.1)–(2.4), (2.6)–(2.12).

**Definition 2.1.** Let \( T \in \mathbb{R}_+ \); a generalised solution of the problem (2.14)–(2.24) in the domain \( Q_T = ]0, 1[ \times ]0, T[ \) is a function

\[
(x, t) \rightarrow (\rho, \nu, \omega, \theta)(x, t), \quad (x, t) \in Q_T,
\] (2.26)

where

\[
\begin{align*}
\rho &\in L^\infty(0, T; H^1([0, 1])) \cap H^1(Q_T), \\
\nu, \omega, \theta &\in L^\infty(0, T; H^1([0, 1])) \cap H^1(Q_T) \cap L^2(0, T; H^2([0, 1])),
\end{align*}
\] (2.27)

that satisfies the equations (2.14)–(2.17) a.e. in \( Q_T \), the conditions (2.18)–(2.24) in the sense of traces and the condition

\[
\inf_{Q_T} \rho > 0.
\] (2.29)

**Remark 2.1.** From embedding and interpolation theorems ([13]) one can conclude that from (2.27) and (2.28) it follows:

\[
\begin{align*}
\rho &\in C([0, T], L^2([0, 1])) \cap L^\infty(0, T; C([0, 1])), \\
\nu, \omega, \theta &\in L^2(0, T; C^{(1)}([0, 1])) \cap C([0, T], H^1([0, 1])),
\end{align*}
\] (2.30)

Specially, the condition (2.29) has a sense.

The purpose of this paper is to prove the following results.

**Theorem 2.1.** For each \( T \in \mathbb{R}_+ \) the problem (2.14)–(2.24) has at most one generalised solution in \( Q_T \).

**Theorem 2.2.** Let the functions \( \rho_0, \theta_0 \in H^1([0, 1]) \) satisfy the conditions (2.25) and let \( \nu_0, \omega_0 \in H^1([0, 1]) \). Then there exists \( T_0 \in \mathbb{R}_+ \) such that the problem (2.14)–(2.24) has a generalised solution in \( Q_0 = Q_{T_0} \), having the property

\[
\theta > 0 \quad \text{in} \ Q_0.
\] (2.33)

The analogous theorems for the classical fluid were proved in [24], [25] and [2]. In our proof we use the Faedo–Galerkin method and follow ideas of the book [2].
3. The proof of Theorem 2.1.

Let \((\rho_i, v_i, \omega_i, \theta_i), i = 1, 2\) be generalised solutions of the problem (2.14)–(2.24). Then the function \((\rho, v, \omega, \theta) = (\rho_1, v_1, \omega_1, \theta_1) - (\rho_2, v_2, \omega_2, \theta_2)\) satisfies the system:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \rho_1 \frac{\partial v_1}{\partial x} + \rho_1 \frac{\partial v_2}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial x} \left( \rho_1 \frac{\partial v_1}{\partial x} + \rho \frac{\partial v_2}{\partial x} \right) - K \frac{\partial}{\partial x} \left( \rho_1 \theta + \rho_2 \theta_2 \right), \\
\frac{\partial \omega}{\partial t} &= A \left[ \frac{\partial}{\partial x} \left( \rho_1 \frac{\partial \omega_1}{\partial x} + \rho \frac{\partial \omega_2}{\partial x} \right) - \frac{\omega_1}{\rho_1} + \frac{\omega_2}{\rho_2} \frac{\rho}{\rho_1 \rho_2} \right], \\
\frac{\partial \theta}{\partial t} &= D \frac{\partial}{\partial x} \left( \rho_1 \frac{\partial \theta_1}{\partial x} + \rho \frac{\partial \theta_2}{\partial x} \right) - K \left( \rho_1 \theta_1 \frac{\partial v_1}{\partial x} + \theta_2 \rho \frac{\partial v_1}{\partial x} + \rho_2 \frac{\partial v_2}{\partial x} \right) \\
&\quad + \rho_1 \frac{\partial v_1}{\partial x} + \rho \frac{\partial v_2}{\partial x} + \rho \frac{\partial v_2}{\partial x} \frac{\rho}{\rho_1 \rho_2} \\
&\quad + \rho_1 \frac{\partial \omega_1}{\partial x} + \rho_2 \frac{\partial \omega_2}{\partial x} + \rho \frac{\partial \omega_2}{\partial x}, \\
\frac{\partial \omega(0, t)}{\partial t} &= \frac{\partial \omega(1, t)}{\partial t} = 0, \\
\frac{\partial \omega(0, t)}{\partial x} &= \frac{\partial \omega(1, t)}{\partial x} = 0, \\
\frac{\partial \theta(0, t)}{\partial x} &= \frac{\partial \theta(1, t)}{\partial x} = 0, \\
\rho(x, 0) &= \nu(x, 0) = \omega(x, 0) = \theta(x, 0) = 0.
\end{align*}
\]

In that what follows we denote by \(C > 0\) a generic constant, not depending on \((\rho, v, \omega, \theta)\) and having possibly different values at different places. We also use the notation

\[
\|f\| = \|f\|_{L^2([0, t])}.
\]

Taking into account properties (2.30)–(2.32), from (3.1) and (3.8) we obtain

\[
\|\rho(t)\|^2 \leq C \int_0^t \left[ \left( 1 + \max_{x \in [0, 1]} \left| \frac{\partial v_2}{\partial x} \right| (\tau) \right) \|\rho(\tau)\|^2 + \left( \|\frac{\partial v}{\partial x}(\tau)\|^2 \right) \right] d\tau
\]

or, because of the Gronwall’s inequality,

\[
\|\rho(t)\|^2 \leq C \int_0^t \left\| \frac{\partial v}{\partial x}(\tau) \right\|^2 d\tau.
\] (3.9)

From (3.2), (3.5) and (3.8) we get

\[
\|v(t)\|^2 + \int_0^t \left( \left| \frac{\partial v}{\partial x}(\tau) \right| \right)^2 d\tau \leq C \int_0^t \left[ \left( 1 + \max_{x \in [0, 1]} \left| \frac{\partial v_2}{\partial x} \right| (\tau) \right) \|\rho(\tau)\| \|\frac{\partial v}{\partial x}(\tau)\| \right. \\
&\quad \left. + \|\theta(\tau)\| \left| \frac{\partial v}{\partial x}(\tau) \right| \right] d\tau
\]
or applying the Young's inequality and (3.9),
\[ \|v(t)\|^2 + \int_0^t \left\| \frac{\partial v}{\partial x} (\tau) \right\|^2 d\tau \leq C \int_0^t \left[ \left( 1 + \max_{x \in [0,1]} \left| \frac{\partial v}{\partial x} (\tau) \right| \right)^2 \left\| \nu(\tau) \right\|^2 \]
\[ + \int_0^t \left\| \frac{\partial v}{\partial x} (\lambda) \right\|^2 d\lambda + \|\theta(\tau)\|^2 \right] d\tau. \]

Using now the Gronwall's inequality, we obtain
\[ \|v(t)\|^2 + \int_0^t \left\| \frac{\partial v}{\partial x} (\tau) \right\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau. \quad (3.10) \]

Analogously, from (3.3), (3.4), (3.6)–(3.10) there follow the inequalities
\[ \|\omega(t)\|^2 + \int_0^t \left\| \frac{\partial \omega}{\partial x} (\tau) \right\|^2 d\tau \leq C \int_0^t \left\| \frac{\partial v}{\partial x} (\tau) \right\|^2 d\tau, \quad (3.11) \]
\[ \|\theta(t)\|^2 + \int_0^t \left\| \frac{\partial \theta}{\partial x} (\tau) \right\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau. \quad (3.12) \]

From (3.9)–(3.12) we conclude that \( \rho = v = \omega = \theta = 0. \)

4. Approximate solutions

A local generalised solution to the problem (2.14)–(2.24) we shall find as a limit of approximate solutions
\[ (\rho^n, v^n, \omega^n, \theta^n), \quad n \in \mathbb{N}, \quad (4.1) \]

where
\[ v^n(x, t) = \sum_{i=1}^n v^n_i(t) \sin(nix), \quad (4.2) \]
\[ \omega^n(x, t) = \sum_{j=1}^n \omega^n_j(t) \sin(njx), \quad (4.3) \]
\[ \theta^n(x, t) = \sum_{k=0}^n \theta^n_k(t) \cos(nkx); \quad (4.4) \]

here \( v^n_i, \omega^n_j, \theta^n_k \) are unknown functions, defined and smooth on an interval \([0, T_n], \)
\( T_n \in \mathbb{R}_+. \) Evidently, the boundary conditions
\[ v^n(0, t) = v^n(1, t) = \omega^n(0, t) = \omega^n(1, t) = \frac{\partial \theta^n}{\partial x}(0, t) = \frac{\partial \theta^n}{\partial x}(1, t) = 0 \quad (4.5) \]
are satisfied. According to Feado–Galerkin method, we take the following approximation conditions:

\[
\frac{\partial \rho^n}{\partial t} + (\rho^n)^2 \frac{\partial \nu^n}{\partial x} = 0, \quad \rho^n(x, 0) = \rho_0(x),
\]

(4.6)

\[
\int_0^1 \left[ \frac{\partial \nu^n}{\partial t} - \frac{\partial}{\partial x} \left( \rho^n \frac{\partial \nu^n}{\partial x} \right) + K \frac{\partial}{\partial x} (\rho^n \theta^n) \right] \sin(\pi ix) \, dx = 0, \quad i = 1, 2, \ldots, n,
\]

(4.7)

\[
\int_0^1 \left[ \frac{\partial \omega^n}{\partial t} - A \frac{\partial}{\partial x} \left( \rho^n \frac{\partial \omega^n}{\partial x} \right) + A \frac{\omega^n}{\rho^n} \right] \sin(\pi jx) \, dx = 0, \quad j = 1, 2, \ldots, n,
\]

(4.8)

\[
\int_0^1 \left[ \frac{\partial \theta^n}{\partial t} + K \rho^n \frac{\partial \nu^n}{\partial x} - \rho^n \left( \frac{\partial \nu^n}{\partial x} \right)^2 - \rho^n \left( \frac{\partial \omega^n}{\partial x} \right)^2 - \frac{\omega^n}{\rho^n} \right]
\]

\[
- D \frac{\partial}{\partial x} \left( \rho^n \frac{\partial \theta^n}{\partial x} \right) \right] \cos(\pi kx) \, dx = 0, \quad k = 0, 1, 2, \ldots, n.
\]

(4.9)

From (4.6) and (4.2) it follows

\[
\rho^n(x, t) = \rho_0(x) \left( 1 + \rho_0(x) \int_0^t \frac{\partial \nu^n}{\partial x}(x, \tau) \, d\tau \right)^{-1}
\]

\[
= \rho_0(x) \left( 1 + \rho_0(x) \sum_{i=1}^n (i\pi) \cos(\pi ix) \int_0^t \nu_i^\prime(\tau) \, d\tau \right)^{-1},
\]

(4.10)

and because of (2.25), for sufficiently small \(T_n\) we have

\[
\rho^n(x, t) > 0, \quad (x, t) \in [0, 1] \times [0, T_n].
\]

(4.11)

Therefore the conditions (4.8) and (4.9) have a sense. Let \(v_0, \omega_0 (i, j = 1, 2, \ldots)\) and \(\theta_0 (k = 0, 1, 2, \ldots)\) be the Fourier coefficients of the functions \(v_0, \omega_0\) and \(\theta_0\), respectively:

\[
v_0i = 2 \int_0^1 v_0(x) \sin(\pi ix) \, dx, \quad i = 1, 2, \ldots,
\]

\[
\omega_0j = 2 \int_0^1 \omega_0(x) \sin(\pi jx) \, dx, \quad j = 1, 2, \ldots,
\]

\[
\theta_00 = \int_0^1 \theta_0(x) \, dx, \quad \theta_0k = 2 \int_0^1 \theta_0(x) \cos(\pi kx) \, dx, \quad k = 1, 2, \ldots;
\]
let

\[ v_0^n(x) = \sum_{i=1}^{n} v_{0i} \sin(\pi i x), \]  
\[ \omega_0^n(x) = \sum_{j=1}^{n} \omega_{0j} \sin(\pi j x), \]  
\[ \theta_0^n(x) = \sum_{k=0}^{n} \theta_{0k} \cos(\pi k x). \]  

The initial conditions for \( v^n, \omega^n \) and \( \theta^n \) we take in the form:

\[ v^n(x, 0) = v_0^n(x), \]  
\[ \omega^n(x, 0) = \omega_0^n(x), \]  
\[ \theta^n(x, 0) = \theta_0^n(x). \]  

Let

\[ z_r^n(t) = \int_0^t v_r^n(\tau) d\tau, \quad r = 1, 2, \ldots, n. \]  

Taking into account (4.2)–(4.4), (4.10) and (4.18), from (4.7)–(4.9) we obtain for \( \{ (v_j^n, \omega_j^n, \theta_k^n, z_r^n) : i, j, r = 1, 2, \ldots, n, \ k = 0, 1, 2, \ldots, n \} \) a Cauchy problem:

\[ v_j^n = \phi_j^n(v_r^n, \ldots, v_r^n, \omega_0^n, \ldots, \omega_0^n, \theta_0^n, \theta_0^n, \ldots, \theta_0^n, z_1^n, \ldots, z_n^n), \]  
\[ \omega_j^n = \psi_j^n(v_r^n, \ldots, v_r^n, \omega_0^n, \ldots, \omega_0^n, \theta_0^n, \theta_0^n, \ldots, \theta_0^n, z_1^n, \ldots, z_n^n), \]  
\[ \theta_k^n = \lambda_k \Pi_k^n(v_r^n, \ldots, v_r^n, \omega_1^n, \ldots, \omega_1^n, \theta_0^n, \theta_1^n, \ldots, \theta_1^n, z_1^n, \ldots, z_n^n), \]  
\[ z_r^n = v_r^n, \]  
\[ v_r^n(0) = v_{0i}, \]  
\[ \omega_r^n(0) = \omega_{0j}, \]  
\[ \theta_k^n(0) = \theta_{0k}, \]  
\[ z_r^n(0) = 0, \]

where \( \lambda_0 = 1, \lambda_k = 2 \) for \( k = 1, 2, \ldots, n \), and
\[
\phi^n = 2 \int_0^1 \left[ \frac{\partial}{\partial x} \left( \rho^n \frac{\partial \psi^n}{\partial x} \right) - K \frac{\partial}{\partial x} \left( \rho^n \theta^n \right) \right] \sin(\pi x) \, dx.
\] (4.27)

\[
\psi^n = 2 \int_0^1 A \left[ \frac{\partial}{\partial x} \left( \rho^n \frac{\partial \psi^n}{\partial x} \right) - \omega^n \right] \sin(\pi x) \, dx,
\] (4.28)

\[
11^n_k = \int_0^1 \left[ -K \rho^n \frac{\partial \omega^n}{\partial x} + \rho^n \left( \frac{\partial \omega^n}{\partial x} \right)^2 + \frac{\partial}{\partial x} \left( \rho^n \frac{\partial \theta^n}{\partial x} \right) + \left( \omega^n \right)^2 \rho^n \right] \cos(kx) \, dx.
\] (4.29)

With the help of the Cauchy–Picard theorem (c.f. [20]), one can easily conclude that the following statements are valid.

**Lemma 4.1.** For each \( n \in \mathbb{N} \) there exists \( T_n \in \mathbb{R}^+ \) such that the Cauchy problem (4.19) and (4.26) has a unique solution, defined on \([0, T_n]\); the functions \( \psi^n, \omega^n \) and \( \theta^n \), defined by the formulas (4.2)–(4.4), belong to the class \( C^\infty(\mathcal{Q}_n) \), \( \mathcal{Q}_n = [0, 1] \times [0, T_n] \) and satisfy the conditions (4.13)–(4.17).

**Lemma 4.2.** There exists \( T_n \in \mathbb{R}^+ \) such that function \( \rho^n \), defined by (4.10), satisfies the condition

\[
\frac{m}{2} < \rho^n(x, t) < 2M \quad \text{in} \quad \overline{\mathcal{Q}_n}.
\] (4.30)

**5. A priori estimates**

Our purpose is to find out \( T_0 \in \mathbb{R}^+ \) such that for each \( n \in \mathbb{N} \) there exists a solution of the problem (4.10) and (4.26), defined on \([0, T_n]\). It will be sufficient to find out uniform (in \( n \in \mathbb{N} \)) a priori estimates for a function \( \psi^n, \omega^n, \theta^n, \omega^{(j)} \), defined through Lemmas 4.1. and 4.2. In that what follows, \( C > 0 \) denotes a \( \psi^n \)–\( \omega^n \) constant, not depending on \( n \in \mathbb{N} \).

**Lemma 5.1.** For \( t \in [0, T_n] \) it holds the inequality

\[
\|\omega^n(t)\|^2 + \int_0^t \left( \|\frac{\partial \omega^n}{\partial x}(\tau)\|^2 + \|\omega^n(\tau)\|^2 \right) \, d\tau \leq C.
\] (5.1)

**Proof.** Multiplying (4.8) by \( \omega_j^n \) and summing over \( j = 1, 2, \ldots, n \), after integration by parts we obtain

\[
\frac{1}{2A} \frac{d}{dt} \|\omega^n(t)\|^2 + \int_0^1 \left[ \rho^n(x, t) \left( \frac{\partial \omega^n}{\partial x}(x, t) \right)^2 + \frac{1}{\rho^n(x, t)} \left( \omega^n(x, t) \right)^2 \right] \, dx = 0.
\]
Integrating over \([0, t]\), \(0 < t \leq T_n\), and taking into account (4.16), we have
\[
\frac{1}{2A} \|\omega^n(t)\|^2 + \int_0^t \int_0^1 \left[ \rho^n(x,t) \left( \frac{\partial \omega^n}{\partial x}(x,t) \right)^2 + \frac{1}{\rho^n(x,t)} (\omega^n(x,t))^2 \right] \, dx \, dt \\
= \frac{1}{2A} \|\omega^n_t\|^2 \leq \frac{1}{2A} \|\omega_0\|^2,
\]
and using (4.30) we get (5.1).

**Lemma 5.2.** For \(t \in [0, T_n]\) it holds the inequality
\[
\left| \int_0^1 \theta^n(x,t) \, dx \right| \leq C \left( 1 + \left\| \frac{\partial \nu^n}{\partial x}(t) \right\| \right)^2. \tag{5.2}
\]

**Proof.** Multiplying (4.7) by \(v_i^n\) and summing over \(i = 1, 2, \ldots, n\), after integration by parts and using (4.9) for \(k = 0\), we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|\nu^n(t)\|^2 + \int_0^1 \theta^n(x,t) \, dx \right) = \int_0^1 \frac{1}{\rho^n(x,t)} (\omega^n(x,t))^2 \, dx \\
+ \int_0^1 \rho^n(x,t) \left( \frac{\partial \omega^n}{\partial x}(x,t) \right)^2 \, dx.
\]
Taking into account (4.15), (4.17), (4.30), (5.1) and the inequality
\[
\|\nu^n\| \leq 2^{-\frac{1}{2}} \left\| \frac{\partial \nu^n}{\partial x} \right\|
\]
we obtain (5.2).

**Lemma 5.3.** For \((x, t) \in \overline{Q}_n\) it holds the inequality
\[
|\theta^n(x,t)| \leq C \left( 1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\| + \left\| \frac{\partial \nu^n}{\partial x}(t) \right\|^2 \right). \tag{5.3}
\]

**Proof.** Let \(t \in [0, T_n]\) and \(x_1(t), x_2(t) \in [0, 1]\), such that
\[
m_n(t) = \min_{x \in [0,1]} \theta^n(x,t) = \theta^n(x_1(t),t), \\
M_n(t) = \max_{x \in [0,1]} \theta^n(x,t) = \theta^n(x_2(t),t).
\]
For \(x \in [0, 1]\) it holds
\[
\theta^n(x,t) - m_n(t) = \int_{x_1(t)}^x \frac{\partial \theta^n}{\partial x}(x,t) \, dx \leq \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|
\]
and hence
\[ \theta^n(x,t) \leq \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| + m_n(t) \leq \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| + \int_0^1 \theta^n(x,t) \, dx. \]

Analogously we have
\[ \theta^n(x,t) - M_n(t) = \int_{\mathcal{S}(t)} \frac{\partial \theta^n}{\partial x} (x,t) \, dx \geq - \left\| \frac{\partial \theta^n}{\partial x} (t) \right\|, \]
and
\[ \theta^n(x,t) \geq - \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| + M_n(t) \geq - \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| - \int_0^1 \theta^n(x,t) \, dx. \]

So, it holds
\[ |\theta^n(x,t)| \leq \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| + \int_0^1 \theta^n(x,t) \, dx; \]
using (5.2) we get (5.3).

**Lemma 5.4.** For \( t \in [0, T_n] \) it holds the inequality
\[ \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| \leq C \left( 1 + \int_0^t \left\| \frac{\partial^2 \nu^n}{\partial x^2} (\tau) \right\|^2 \, d\tau \right). \quad (5.4) \]

**Proof.** The conclusion follows immediately from (4.10).

**Lemma 5.5.** For \( t \in [0, T_n] \) it holds
\[ \frac{d}{dt} \left( \left\| \frac{\partial \nu^n}{\partial x} (t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x} (t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x} (t) \right\|^2 \right) \]
\[ + \left\| \frac{\partial^2 \nu^n}{\partial x^2} (t) \right\|^2 + \left\| \frac{\partial^2 \omega^n}{\partial x^2} (t) \right\|^2 + \left\| \frac{\partial^2 \theta^n}{\partial x^2} (t) \right\|^2 \]
\[ \leq C \left( 1 + \left\| \frac{\partial \nu^n}{\partial x} (t) \right\|^8 + \left\| \frac{\partial \omega^n}{\partial x} (t) \right\|^8 + \left\| \frac{\partial \theta^n}{\partial x} (t) \right\|^8 + \left( \int_0^t \left\| \frac{\partial^2 \nu^n}{\partial x^2} (\tau) \right\|^2 \, d\tau \right)^4 \right). \quad (5.5) \]

**Proof.** Multiplying (4.7), (4.8) and (4.9) respectively by \((\pi i)^2 \nu^n_i, (\pi j)^2 \omega^n_j\) and \((\pi k)^2 \theta^n_k\) and taking into account (4.2)-(4.4), after summation over \(i,j,k = 1,2, \ldots, n\) and addition of the obtained equalities, we get
\[ \frac{1}{2} \frac{d}{dt} \left( \left\| \frac{\partial \nu^n}{\partial x} (t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x} (t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x} (t) \right\|^2 \right) + \int_0^1 \rho^n(x,t) \left( \frac{\partial^2 \nu^n}{\partial x^2} (x,t) \right)^2 \]
\[ + A \left( \frac{\partial^2 \omega^n}{\partial x^2} (x,t) \right)^2 + D \left( \frac{\partial^2 \theta^n}{\partial x^2} (x,t) \right)^2 \right) \, dx = \sum_{i=1}^{10} I_i(t), \quad (5.6) \]
where

\[ I_1(t) = - \int_0^1 \frac{\partial \rho^n}{\partial x} \frac{\partial \nu^n}{\partial x} \frac{\partial^2 \nu^n}{\partial x^2} \, dx, \quad I_2(t) = K \int_0^1 \frac{\partial \rho^n}{\partial x} \theta^n \frac{\partial^2 \nu^n}{\partial x^2} \, dx, \]

\[ I_3(t) = K \int_0^1 \rho^n \frac{\partial \theta^n}{\partial x} \frac{\partial^2 \nu^n}{\partial x^2} \, dx, \quad I_4(t) = A \int_0^1 \frac{1}{\rho^n} \theta^n \frac{\partial^2 \omega^n}{\partial x^2} \, dx, \]

\[ I_5(t) = -A \int_0^1 \frac{\partial \rho^n}{\partial x} \frac{\partial \omega^n}{\partial x} \frac{\partial^2 \omega^n}{\partial x^2} \, dx, \quad I_6(t) = K \int_0^1 \rho^n \theta^n \frac{\partial^2 \theta^n}{\partial x^2} \, dx, \]

\[ I_7(t) = -\int_0^1 \rho^n \left( \frac{\partial \nu^n}{\partial x} \right)^2 \frac{\partial^2 \theta^n}{\partial x^2} \, dx, \quad I_8(t) = -D \int_0^1 \frac{\partial \nu^n}{\partial x} \frac{\partial \theta^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} \, dx, \]

\[ I_9(t) = -\int_0^1 \frac{1}{\rho^n} (\omega^n)^2 \frac{\partial^2 \theta^n}{\partial x^2} \, dx, \quad I_{10}(t) = -\int_0^1 \rho^n \left( \frac{\partial \omega^n}{\partial x} \right)^2 \frac{\partial^2 \theta^n}{\partial x^2} \, dx. \]

Taking into account (5.1)–(5.4) and the inequalities

\[ |f|^2 \leq 2 \|f\| \left\| \frac{\partial f}{\partial x} \right\|, \quad \left\| \frac{\partial f}{\partial x} \right\|^2 \leq 2 \left\| \frac{\partial f}{\partial x} \right\| \left\| \frac{\partial^2 f}{\partial x^2} \right\|, \tag{5.7} \]

\[ \|f\| \leq 2^{-\frac{1}{2}} \left\| \frac{\partial f}{\partial x} \right\|, \quad \left\| \frac{\partial f}{\partial x} \right\| \leq 2^{-\frac{1}{2}} \left\| \frac{\partial^2 f}{\partial x^2} \right\| \tag{5.8} \]

(for a function \( f \) vanishing at \( x = 0 \) and \( x = 1 \) or with the first derivative vanishing at the same points), one can estimate the functions \( I_1(t) - I_{10}(t) \). For instance,

\[ I_1(t) \leq \max_{x \in [0,1]} \left| \frac{\partial \nu^n}{\partial x}(x,t) \right| \left\| \frac{\partial \rho^n}{\partial x}(t) \right\| \left\| \frac{\partial^2 \nu^n}{\partial x^2}(t) \right\| \]

\[ \leq 2^\frac{1}{2} \left\| \frac{\partial \nu^n}{\partial x}(t) \right\|^\frac{1}{2} \left\| \frac{\partial^2 \nu^n}{\partial x^2}(t) \right\| \left\| \frac{\partial \rho^n}{\partial x}(t) \right\| \]

\[ \leq C \left\| \frac{\partial \nu^n}{\partial x}(t) \right\|^\frac{1}{2} \left\| \frac{\partial^2 \nu^n}{\partial x^2}(t) \right\| \left( 1 + \left( \int_0^t \left\| \frac{\partial^2 \nu^n}{\partial x^2}(\tau) \right\|^2 \, d\tau \right)^\frac{1}{2} \right); \]

applying the Young inequality, we get

\[ I_1(t) \leq \varepsilon \left\| \frac{\partial^2 \nu^n}{\partial x^2}(t) \right\|^2 + C \left[ 1 + \left\| \frac{\partial \nu^n}{\partial x}(t) \right\|^4 + \left( \int_0^t \left\| \frac{\partial^2 \nu^n}{\partial x^2}(\tau) \right\|^2 \, d\tau \right)^4 \right], \]
where \( \varepsilon > 0 \) is arbitrary. In an analogous way one obtains the inequalities:

\[
I_2(t) \leq \varepsilon \left\| \frac{\partial^2 \nu^n}{\partial x^2}(t) \right\|^2 + C \left[ 1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^4 + \left( \int_0^t \left\| \frac{\partial^2 \nu^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^4 \right],
\]

\[
I_3(t) \leq \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2,
\]

\[
I_4(t) \leq \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C,
\]

\[
I_5(t) \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left[ 1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^4 + \left( \int_0^t \left\| \frac{\partial^2 \nu^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^4 \right],
\]

\[
I_6(t) \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left( 1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^4 + \left\| \frac{\partial \nu^n}{\partial x}(t) \right\|^8 \right),
\]

\[
I_7(t) \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + \varepsilon \left\| \frac{\partial^2 \nu^n}{\partial x^2}(t) \right\|^2 + C \left( 1 + \left\| \frac{\partial \nu^n}{\partial x}(t) \right\|^8 \right),
\]

\[
I_8(t) \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left[ 1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^4 + \left( \int_0^t \left\| \frac{\partial^2 \nu^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^4 \right],
\]

\[
I_9(t) \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left( 1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right),
\]

\[
I_{10}(t) \leq \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + \varepsilon \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + C \left( 1 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^8 \right).
\]

Inequality (5.5) follows from (5.6) and (4.30). \( \square \)

**Lemma 5.6.** There exists \( T_0 \in \mathbb{R}_+ \), such that for each \( n \in \mathbb{N} \) the Cauchy problem (4.19)–(4.26) has a unique solution, defined on \([0, T_0]\). Moreover, the functions \( \nu^n, \omega^n, \theta^n \) and \( \rho^n \) satisfy the inequalities

\[
\max_{t \in [0, T_0]} \left( \left\| \frac{\partial \nu^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) \leq C,
\]

\[
\int_0^{T_0} \left( \left\| \frac{\partial^2 \nu^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 \right) dt \leq C,
\]

\[
\max_{t \in [0, T_0]} \left\| \frac{\partial \rho^n}{\partial x}(t) \right\| \leq C,
\]

\[
\frac{m}{2} \leq \rho^n(x, t) \leq 2M, \quad (x, t) \in \overline{Q}_0, \quad Q_0 = Q_{T_0}.
\]
Proof. Let
\[ y_n(t) = \left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 + \int_0^t \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 \, d\tau. \]  
(5.12)

According to (5.5) it holds
\[ \dot{y}_n \leq C(1 + y_n^4). \]  
(5.13)

Because of (4.15)–(4.17) we have
\[ y_n(0) = \left\| \frac{d v^n_0}{d x} \right\|^2 + \left\| \frac{d \omega^n_0}{d x} \right\|^2 + \left\| \frac{d \theta^n_0}{d x} \right\|^2 \leq \left\| \frac{d v_0}{d x} \right\|^2 + \left\| \frac{d \omega_0}{d x} \right\|^2 + \left\| \frac{d \theta_0}{d x} \right\|^2, \]
\[ \text{i.e.} \]
\[ y_n(0) \leq C. \]  
(5.14)

Let \([0, T'], T' \in \mathbb{R}_+\), be an existence interval of the Cauchy problem
\[ \dot{y} = C(1 + y^4) \]  
(5.15)
\[ y(0) = C. \]  
(5.16)

From (5.13)–(5.16) it follows
\[ y_n(t) \leq y(t), \quad t \in [0, T']. \]  
(5.17)

Let \(0 < T_0 < T'\). From (5.12) and (5.17) we obtain
\[ \max_{t \in [0, T_0]} \left( \left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) + \int_0^{T_0} \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 \, d\tau \leq C. \]  
(5.18)

and, using (5.5),
\[ \frac{d}{dt} \left( \left\| \frac{\partial v^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \omega^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) \]
\[ + \left\| \frac{\partial^2 \omega^n}{\partial x^2}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x^2}(t) \right\|^2 \]  
\[ \leq C; \]
taking into account (4.15)–(4.17) we obtain (5.9). From (5.9) and (5.4) it follows (5.10). According to (4.10) we have
\[ \rho^n(x, t) \leq \frac{M}{1 - M \int_0^t \left| \frac{\partial v^n}{\partial x}(x, \tau) \right| \, d\tau}. \]

With the help of (5.7), (5.8) and (5.9) we find that
\[ \int_0^t \left| \frac{\partial v^n}{\partial x}(x, \tau) \right| \, d\tau \leq \sqrt{2 \left( \max_{t \in [0, T_0]} \left\| \frac{\partial^2 v^n}{\partial x^2}(t) \right\|^2 \right)^{\frac{1}{4}} \left( \int_0^{T_0} \left\| \frac{\partial^2 v^n}{\partial x^2}(\tau) \right\|^2 \, d\tau \right)^{\frac{1}{4}}} T_0^{\frac{1}{4}} \leq C T_0^{\frac{1}{4}}. \]

Let \(T_0 < \min\{T', (2M)^{-\frac{1}{4}}C^{-\frac{1}{2}}\}\); then for \((x, t) \in \overline{Q}_0\) we have
\[ \rho^n(x, t) \leq 2M. \]
For such $T_0$ and $(x, t) \in \overline{Q}_0$, from (4.10) we obtain analogously
$$\rho^n(x, t) \geq \frac{m}{2}.$$  

From (4.2)–(4.4) and (5.9) one can easy conclude that for $t \in [0, T_0]$ it holds
$$\sum_{i=1}^{n} \left[ |\nabla i^n(t)| + |\omega_i^n(t)| + |\theta_i^n(t)| \right] \leq C. \quad (5.19)$$

From (4.21) and (4.29) we have
$$\theta_0^n(t) = \int_{0}^{1} \int_{0}^{1} \left[ -K \rho^n \theta^n \frac{\partial \rho^n}{\partial x} + \rho^n \left( \frac{\partial \rho^n}{\partial x} \right)^2 + \frac{(\omega^n)^2}{\rho^n} + \rho^n \left( \frac{\partial \omega^n}{\partial x} \right)^2 \right] dx \, dt + \theta_{00}.$$  

With the help of (5.3), (5.9), (5.11) and (5.7), (5.8), for $t \in [0, T_0]$ we obtain
$$|\theta_0^n(t)| \leq C. \quad (5.20)$$

From (5.19) and (5.20) we conclude that the solution of the problem (4.19)–(4.26) is defined on $[0, T_0]$.

**Lemma 5.7.** Let $T_0$ be defined by Lemma 5.6. Then for each $n \in \mathbb{N}$ it holds
$$\int_{0}^{T_0} \left( \left\| \frac{\partial \rho^n}{\partial t} (\tau) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial t} (\tau) \right\|^2 + \left\| \frac{\partial \rho^n}{\partial t} (\tau) \right\|^2 + \left\| \frac{\partial \rho^n}{\partial t} (\tau) \right\|^2 \right) \, d\tau \leq C. \quad (5.21)$$

**Proof.** Multiplying (4.7) by $\frac{\partial \rho^n}{\partial t} (t)$ and summing over $i = 1, 2, \ldots, n$, we obtain
$$\left\| \frac{\partial \rho^n}{\partial t} (t) \right\|^2 = \int_{0}^{1} \left( \frac{\partial \rho^n}{\partial x} \frac{\partial \rho^n}{\partial x} + \rho^n \frac{\partial^2 \rho^n}{\partial x^2} - K \frac{\partial \rho^n}{\partial x} \frac{\partial \theta^n}{\partial x} \frac{\partial \rho^n}{\partial t} - K \rho^n \frac{\partial \theta^n}{\partial x} \frac{\partial \rho^n}{\partial t} \right) \, dx$$
$$\leq C \left( \max_{x \in [0,1]} \left\| \frac{\partial \rho^n}{\partial x} (x, t) \right\| \left\| \frac{\partial \rho^n}{\partial x} (t) \right\| \left\| \frac{\partial \rho^n}{\partial t} (t) \right\| + \left\| \frac{\partial^2 \rho^n}{\partial x^2} (t) \right\| \left\| \frac{\partial \rho^n}{\partial t} (t) \right\| \right.$$  
$$+ \max_{x \in [0,1]} \left\| \theta^n (x, t) \right\| \left\| \frac{\partial \rho^n}{\partial x} (t) \right\| \left\| \frac{\partial \rho^n}{\partial t} (t) \right\| + \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| \left\| \frac{\partial \rho^n}{\partial t} (t) \right\|. \right.$$  

Applying (5.7), (5.8), (5.3) and (5.4) we find that
$$\left\| \frac{\partial \rho^n}{\partial t} (t) \right\|^2 \leq C \left[ \left\| \frac{\partial \rho^n}{\partial t} (t) \right\| \left\| \frac{\partial^2 \rho^n}{\partial x^2} (t) \right\| \left( 1 + \left( \int_{0}^{1} \left\| \frac{\partial^2 \rho^n}{\partial x^2} (\tau) \right\|^2 \, d\tau \right)^{\frac{1}{2}} \right) \right.$$  
$$+ \left\| \frac{\partial^2 \rho^n}{\partial x^2} (t) \right\| \left\| \frac{\partial \rho^n}{\partial x} (t) \right\| + \left\| \frac{\partial \rho^n}{\partial x} (t) \right\| \left( 1 + \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| \right.$$  
$$+ \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| \left( 1 + \left( \int_{0}^{1} \left\| \frac{\partial^2 \rho^n}{\partial x^2} (\tau) \right\|^2 \, d\tau \right)^{\frac{1}{2}} \right) \right.$$  
$$+ \left\| \frac{\partial \theta^n}{\partial x} (t) \right\| \left. \right\| \frac{\partial \rho^n}{\partial x} (t) \right\| \left\| \frac{\partial \rho^n}{\partial x} (t) \right\|. \right.$$
With the help of Young inequality and (5.9) one can easily conclude that
\[
\int_0^{T_0} \left\| \frac{\partial \nu^n}{\partial t}(\tau) \right\|^2 d\tau \leq C.
\]

In the same way from (4.8) and (4.9) we obtain the estimates for \( \left\| \frac{\partial \omega^n}{\partial t} \right\| \) and \( \left\| \frac{\partial \theta^n}{\partial t} \right\| \), respectively. The estimate for \( \left\| \frac{\partial \rho^n}{\partial t} \right\| \) follows from (4.6) and (5.9).

From Lemmas 5.6. and 5.7. we obtain immediately the next result.

**Proposition 5.1.** Let \( T_0 \in \mathbb{R}_+ \) be defined by Lemma 5.6. Then for the sequence \( \{ (\rho^n, \nu^n, \omega^n, \theta^n) : n \in \mathbb{N} \} \) the following statements hold true:

(i) \( \{ \rho^n \} \) is bounded in \( L^\infty(Q_0), L^\infty(0, T_0; H^1([0, 1])) \) and \( H^1(Q_0) \);

(ii) \( \{ \nu^n \}, \{ \omega^n \}, \{ \theta^n \} \) are bounded in \( L^\infty(0, T_0; H^1([0, 1])), H^1(Q_0), \) and \( L^2(0, T_0; H^2([0, 1])) \).

### 6. The proof of Theorem 2.2.

In proofs that follow we use some well-known facts of Functions Analysis (e.g. [3]).

Let \( T_0 \in \mathbb{R}_+ \) be defined by Lemma 5.6. Theorem 2.2. is a consequence of the following lemmas.

**Lemma 6.1.** There exists a function
\[
\rho \in L^\infty(0, T_0; H^1([0, 1])) \cap H^1(Q_0) \cap C(\overline{Q}_0)
\]
and a subsequence of \( \{ \rho^n \} \) (for simplicity denoted again as \( \{ \rho^n \} \)), such that

\[
\rho^n \rightharpoonup \rho \text{ weakly-* in } L^\infty(0, T_0; H^1([0, 1])), \tag{6.1}
\]

weakly in \( H^1(Q_0) \), \hspace{1cm} \tag{6.2}

strongly in \( C(\overline{Q}_0) \). \hspace{1cm} \tag{6.3}

The function \( \rho \) satisfies the conditions

\[
\frac{m}{2} \leq \rho \leq 2M \text{ in } \overline{Q}_0, \tag{6.4}
\]

\[
\rho(x, 0) = \rho_0(x), \hspace{1cm} x \in [0, 1]. \tag{6.5}
\]

**Proof:** The conclusions (6.1) and (6.2) follow immediately from Proposition 5.1. Let \( (x, t), (x', t') \in \overline{Q}_0 \). Then

\[
|\rho^n(x, t) - \rho^n(x', t')| \leq |\rho^n(x, t) - \rho^n(x', t)| + |\rho^n(x', t) - \rho^n(x', t')|.
\]
Using (4.6) and Proposition 5.1, we obtain

\[
|\rho^n(x, t) - \rho^n(x', t)| \leq \int x' \left| \frac{\partial \rho^n}{\partial x}(\xi, t) \right| d\xi \leq C|x - x'|^{1/2},
\]

\[
|\rho^n(x', t) - \rho^n(x'', t)| \leq \int x' \left| \frac{\partial \rho^n}{\partial x}(\xi', t) \right| d\xi \leq C \int x' \left| \frac{\partial \nu^n}{\partial t} (\xi', \tau) \right| d\tau
\]

\[
\leq C \int t \left\| \nu^n(\tau) \right\|_{H^2([0,1])} d\tau \leq C|t - t'|^{1/2}.
\]

The statement (6.3) follows now from the Arzela–Ascoli theorem. The conditions (6.4) and (6.5) follow from (5.11) and (4.6), respectively.

**Lemma 6.2.** There exist functions

\[
v, \omega, \theta \in L^\infty(0, T_0; H^1([0,1])) \cap H^1(Q_0) \cap L^2(0, T_0; H^2([0,1]))
\]

and a subsequence of \{v^n, \omega^n, \theta^n\} (denoted again as \{v^n, \omega^n, \theta^n\}), such that

\[
(v^n, \omega^n, \theta^n) \rightharpoonup^* (v, \omega, \theta) \text{ weakly-\ast in } L^\infty(0, T_0; H^1([0,1])),
\]

\[
(v^n, \omega^n, \theta^n) \rightharpoonup (v, \omega, \theta) \text{ weakly in } H^1(Q_0),
\]

\[
(v^n, \omega^n, \theta^n) \rightharpoonup (v, \omega, \theta) \text{ strongly in } L^2(Q_0),
\]

\[
(v^n, \omega^n, \theta^n) \rightharpoonup (v, \omega, \theta) \text{ weakly in } L^2(0, T_0; H^2([0,1])).
\]

The functions \(v, \omega\) and \(\theta\) satisfy the conditions

\[
v(0, t) = v(1, t) = \omega(0, t) = \omega(1, t) = 0, \quad t \in [0, T_0],
\]

\[
\frac{\partial \theta}{\partial x}(0, t) = \frac{\partial \theta}{\partial x}(1, t) = 0 \quad \text{a.e. in } ]0, T_0[,
\]

\[
v(x, 0) = v_0(x), \quad \omega(x, 0) = \omega_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in [0, 1].
\]

**Proof.** The conclusions follow from Proposition 5.1 and embedding properties (see Remark 2.1).

**Lemma 6.3.** The functions \(\rho, v, \omega\) and \(\theta\), defined by Lemmas 6.1 and 6.2., satisfy the equations (2.14)–(2.17) a.e. in \(Q_0\).

**Proof.** Let \(\{\rho^n, v^n, \omega^n, \theta^n\} : n \in \mathbb{N}\) be subsequence defined by Lemmas 6.1 and 6.2. The equation (2.14) follows then immediately from (4.6). Let us transform
the equations (4.7)–(4.9) (integrating by parts) to slightly different forms:

\[
\int_{0}^{1} \left[ \frac{\partial \nu^n}{\partial t} \sin(\pi ix) + \pi i \rho^n \left( \frac{\partial \nu^n}{\partial x} - K \theta^n \right) \cos(\pi ix) \right] dx = 0,
\]

\[
\int_{0}^{1} \left[ \left( \frac{\partial \omega}{\partial t} + A \frac{\omega^n}{\rho^n} \right) \sin(\pi jx) + A \pi j \rho^n \frac{\partial \omega^n}{\partial x} \cos(\pi jx) \right] dx = 0,
\]

\[
\int_{0}^{1} \left[ \left( \frac{\partial \theta^n}{\partial t} + K \rho^n \theta^n \frac{\partial \nu^n}{\partial x} - \frac{(\omega^n)^2}{\rho^n} - (\rho^n - \rho) \left( \frac{\partial \nu^n}{\partial x} \right)^2 - (\rho^n - \rho) \left( \frac{\partial \omega^n}{\partial x} \right)^2 \right.ight.
\]
\[
+ \rho \omega^n \frac{\partial^2 \omega^n}{\partial x^2} + \frac{\partial \rho}{\partial x} \omega^n \frac{\partial \omega^n}{\partial x} + \frac{\partial \rho}{\partial x} \nu^n \frac{\partial \nu^n}{\partial x} + \rho \nu^n \frac{\partial^2 \nu^n}{\partial x^2} \right) \cos(\pi kx)
\]
\[- \pi k \left( \rho \omega^n \frac{\partial \omega^n}{\partial x} + D \rho \frac{\partial \theta^n}{\partial x} + \rho \nu^n \frac{\partial \nu^n}{\partial x} \right) \sin(\pi kx) \right] dx = 0.
\]

Taking limits (when \( n \to \infty \)), we obtain

\[
\int_{0}^{1} \left[ \frac{\partial \nu}{\partial t} \sin(\pi ix) + \pi \rho \left( \frac{\partial \nu}{\partial x} - K \theta \right) \cos(\pi ix) \right] dx = 0,
\]

\[
\int_{0}^{1} \left[ \left( \frac{\partial \omega}{\partial t} + A \frac{\omega}{\rho} \right) \sin(\pi jx) + A \pi j \rho \frac{\partial \omega}{\partial x} \cos(\pi jx) \right] dx = 0,
\]

\[
\int_{0}^{1} \left[ \left( \frac{\partial \theta}{\partial t} + K \rho \theta \frac{\partial \nu}{\partial x} - \frac{\omega^2}{\rho} + \rho \omega \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial \rho}{\partial x} \omega \frac{\partial \omega}{\partial x} + \frac{\partial \rho}{\partial x} \nu \frac{\partial \nu}{\partial x} + \rho \nu \frac{\partial^2 \nu}{\partial x^2} \right) \cos(\pi kx)
\]
\[- \pi k \left( \rho \nu \frac{\partial \nu}{\partial x} + \rho \omega \frac{\partial \omega}{\partial x} + D \rho \frac{\partial \theta}{\partial x} \right) \sin(\pi kx) \right] dx = 0.
\]

Now, integrating by parts and taking into account (6.10) and (6.11), we get the equations (2.15)–(2.17).

**Lemma 6.4.** There exists \( T_0 \in \mathbb{R}_+ \) such that the function \( \theta \), defined by Lemma 6.2., satisfies the condition

\[
\theta > 0 \quad \text{in} \quad \overline{Q}_0. \quad (6.13)
\]

**Proof.** Because of the inclusion \( \theta \in C(\overline{Q}_0) \) (see Remark 2.1.), for each \( \varepsilon > 0 \) there exists \( T_0 \in \mathbb{R}_+ \), such that for \( (x, t) \in \overline{Q}_0 \) it holds

\[
|\theta(x, t) - \theta(x, 0)| = |\theta(x, t) - \theta_0(x)| < \varepsilon,
\]

or

\[
\theta(x, t) > \theta_0(x) - \varepsilon \geq m - \varepsilon.
\]

\[\square\]
Remark 6.1. In the second part of this work we intend to prove (with use of Theorem 2.2.) that a generalised solution of the problem (2.14)–(2.24) exists in $Q_T$ for each $T \in \mathbb{R}_+$. 

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REFERENCES


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