A NOTE ON STEFFENSEN'S AND IYENGAR'S INEQUALITY

Vera Čuljak and Neven Elezović, Zagreb, Croatia

Abstract. Taking Steffensen's inequality as a starting point, we obtain some generalizations of classical Iyengar's inequality.

1. Introduction

If $f$ is a differentiable on $[a, b]$ and $|f'(x)| \leq M$, then classical Iyengar's inequality [2] states that it holds

$$\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2} (b - a) \right| \leq \frac{M(b - a)^2}{4} - \frac{1}{4M}(f(b) - f(a))^2. \quad (1)$$

Recently, Agarwal and Dragomir [1] applied Steffensen's inequality to obtain inequality which is stronger than Iyengar's one:

$$\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{[f(b) - f(a) - m(b - a)][M(b - a) - f(b) + f(a)]}{2(M - m)(b - a)} \quad (2)$$

where $m \leq f'(x) \leq M$ for all $x \in [a, b]$. In fact, (2) reduces to (1) if one takes $-m = M = \sup_{a \leq x \leq b} |f'(x)|$.

Inequality (1) has been generalized in various ways. From the point of this note, the following one will interest us ([3], Theorem 1).

THEOREM 1. Let the function $f : [a, b] \to \mathbb{R}$ satisfies the following conditions

(i) $f^{(n-1)} \in C_a$ (with constant $M$ and $0 < \alpha \leq 1$) for some $n \in \mathbb{N}$;
(ii) $f^{(k)}(a) = f^{(k)}(b) = 0$, $(k = 1, \ldots, n - 1)$.

Then it holds

$$\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{M(b - a)^{\alpha+n-1}}{\alpha + n} \left( \zeta^\alpha + n - 1 \frac{q}{2} [1 + (\alpha + n - 1)(2\zeta - 1)] \right), \quad (3)$$


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where \( \zeta \) is the real root of the equation
\[
\zeta^{a+n-1} - (1 - \zeta)^{a+n-1} = q
\]
and
\[
q = \frac{(\alpha + n - 1)^{(n-1)}}{M(b - a)^{a+n-1}} \left( f(b) - f(a) \right), \quad p^{(m)} = p(p - 1) \cdots (p - m + 1), \quad m \in \mathbb{N}.
\]

We shall use Steffensen's inequality to obtain result similar, but simpler than (3).

2. Main results

The starting point is Hayashi's modification of the well-known Steffensen's inequality:

**THEOREM 2.** Let \( F : [a, b] \rightarrow \mathbb{R} \) be a nonincreasing mapping on \([a, b]\) and \( G : [a, b] \rightarrow \mathbb{R} \) an integrable mapping on \([a, b]\) with \( 0 \leq G(x) \leq A \), for all \( x \in [a, b] \). Then, the following inequality holds
\[
A \int_a^b F(x)G(x)dx \leq A \int_a^{a+\lambda} F(x)dx,
\]
where \( \lambda = \frac{1}{A} \int_a^b G(x)dx. \)

We shall suppose that \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) satisfies following conditions:

1° \( f \in C^{(n-1)}(I^*), [a, b] \subseteq I^* \), and
\[
m \leq f^{(n)}(x) \leq M, \quad \forall x \in [a, b]; \tag{5}
\]

2° \( f^{(k)}(a) = f^{(k)}(b) \) for odd \( k \geq 1, k < n \) and \( f^{(k)}(a) = -f^{(k)}(b) \) for even \( k \geq 2, k < n \).

**THEOREM 3.** Let \( n \) be an even number and let \( f \) satisfies conditions 1° and 2°.
Then it holds
\[
\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2} (b - a) - \frac{M + m}{2^{n+1}(n + 1)!} (b - a)^{n+1} \right|
\leq \frac{M - m}{(n + 1)!} \left( \frac{b - a}{2} \right)^{n+1} \left[ 1 - \left( \frac{|\Delta|}{M - m} \right)^{n+1} \right], \tag{6}
\]
where
\[
\Delta = \frac{2}{b - a} \left( f^{(n-1)}(a) - 2f^{(n-1)} \left( \frac{a + b}{2} \right) + f^{(n-1)}(b) \right). \tag{7}
\]

**Proof.** Denote \( c = (a + b)/2 \). We shall apply Theorem 2 for
\[
F(x) = \begin{cases} 
(x - c)^n, & a \leq x \leq c, \\
-(x - c)^n, & c \leq x \leq b. 
\end{cases} \tag{8}
\]
and
\[
G(x) = \begin{cases} 
M - f^{(n)}(x), & a \leq x \leq c, \\
 f^{(n)}(x) - m, & c \leq x \leq b. 
\end{cases}
\]  
(9)

Then it holds
\[
0 \leq G(x) \leq M - m =: A
\]
and
\[
\lambda = \frac{1}{A} \left( \int_a^c (M - f^{(n)}(x)) \, dx + \int_c^b (f^{(n)}(x) - m) \, dx \right) \\
= \frac{b - a}{2} \left( 1 + \frac{\Delta}{M - m} \right),
\]  
(10)

where \( \Delta \) is defined in (7).

Two cases are possible: (a) \( \Delta \leq 0 \); and (b) \( \Delta > 0 \).

Case (a). In the case \( \Delta \leq 0 \) it holds \( \lambda \leq (b - a)/2 \). Therefore, the left term of Hayashi inequality (4) is
\[
\alpha_1 = (M - m) \int_{b-\lambda}^b F(x) \, dx = (M - m) \int_{b-\lambda}^b [-(x - c)^n] \, dx \\
= -\frac{M - m}{n + 1} \left( \frac{b - a}{2} \right)^{n+1} \left[ 1 + \left( \frac{\Delta}{M - m} \right)^{n+1} \right].
\]  
(11)

It is easy to see that
\[
\alpha_2 = (M - m) \int_a^{a+\lambda} F(x) \, dx = -\alpha_1.
\]

In the case (b), it holds \( \lambda > (b - a)/2 \) and similar calculation gives
\[
\alpha_1 = -\frac{M - m}{n + 1} \left( \frac{b - a}{2} \right)^{n+1} \left[ 1 - \left( \frac{\Delta}{M - m} \right)^{n+1} \right].
\]  
(12)

Now, we need only to calculate the middle term in Steffensen’s inequality, using integration by parts and properties 2\( \circ \) of function \( f \):
\[
I = \int_a^c (x - c)^n (M - f^{(n)}(x)) \, dx - \int_c^b (x - c)^n (f^{(n)}(x) - m) \, dx \\
= -n! \left[ \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} (b - a) - \frac{(b - a)^{n+1}}{2^{n+1}(n+1)!} (M + m) \right].
\]

This proves the theorem.

**Corollary 1.** If \( f \) satisfies 1\( \circ \) and 2\( \circ \), then for even \( n \in \mathbb{N} \) it holds
\[
\left| \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} (b - a) \right| \\
\leq \frac{\max \{ |M|, |m| \} (b - a)^{n+1}}{2^{n+1}(n+1)!} - \frac{|\Delta|^{n+1} (b - a)^{n+1}}{2^{n+1}(M - m)^n(n+1)!}.
\]  
(13)
Theorem 4. Let \( n \geq 1 \) be an odd number and let \( f \) satisfies conditions \( 1^\circ \) and \( 2^\circ \). Then it holds

\[
\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b - a) \right| \leq \frac{M - m}{(n + 1)!} \left( \frac{b - a}{2} \right)^{n+1} \left[ 1 - \left( \frac{M + m - 2\Delta}{M - m} \right)^{n+1} \right],
\]

where

\[
\Delta = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a}.
\]

Proof. We use the same idea as in Theorem 3. But, in this case is sufficient to take

\[ F(x) = -(x - c)^n \]

and

\[ G(x) = f^{(n)}(x) - m \]

Therefore, \( A = M - m \) again and since \( f \) satisfies \( 2^\circ \), we have

\[
\lambda = \frac{1}{M - m} \int_a^b (f^{(n)}(x) - m)dx = \frac{\Delta - m}{M - m} (b - a),
\]

\[ \alpha_1 = -(M - m) \int_{b - \lambda}^b (x - c)^n dx \]

\[ = -\frac{M - m}{n + 1} \left( \frac{b - a}{2} \right)^{n+1} \left[ 1 - \left( \frac{M + m - 2\Delta}{M - m} \right)^{n+1} \right]. \]

Since \( \alpha_2 = -\alpha_1 \) and the middle term in Steffensen's inequality is equal to

\[
- \int_a^b (x - c)^n [f^{(n)}(x) - m]dx = -n!(x - c)f(x) \bigg|_a^b + n! \int_a^b f(x)dx
\]

\[ = n! \left( \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b - a) \right) \]

the theorem follows.

Remark. Let \( f \) satisfies conditions of Theorem 1 with \( \alpha = 1 \). As a corollary of Theorem 1, it is proved in [3] that

\[
\left| \int_a^b f(x)dx \right| \leq \frac{M(b - a)^{n+1}}{2^n(n + 1)!}.
\]

holds, if an additional condition \( f(a) = f(b) = 0 \) is assumed. But, such an inequality holds under weaker conditions \( 1^\circ \) and \( 2^\circ \). If \( n \) is even, this follows from Corollary 1, and for odd \( n \) it is sufficient to take \( M = \sup |f^{(n)}(x)| \) and \( m = -M \) in Theorem 4. Therefore, for all \( n \in \mathbb{N} \) we have:

Corollary 2. If \( f \) satisfies \( 1^\circ \) and \( 2^\circ \) then it holds

\[
\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b - a) \right| \leq \frac{M(b - a)^{n+1}}{2^n(n + 1)!}.
\]

where \( M = \sup |f^{(n)}(x)|, x \in [a, b] \).
REFERENCES


Vera Ćuljak
Department of Mathematics
Faculty of Civil Engineering
Kačičeva 26
10 000 Zagreb, Croatia
e-mail: vera@master.grad.hr

Neven Elezović
Department of Applied Mathematics
Faculty of Electrical Engineering and Computing
Unska 3
10 000 Zagreb, Croatia
e-mail: neven.elez@fer.hr