GLASNIK MATEMATIČKI
Vol. 33(53) (1998), 173–184

ON THE ZEROS OF POLYNOMIALS AND RELATED ANALYTIC FUNCTIONS

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Abstract. Let $P(z)$ be a polynomial of degree n with real or complex coefficients. In this paper we obtain a ring shaped region containing all the zeros of $P(z)$. Our results include, as special cases, several known extensions of Eneström–Kakeya theorem on the zeros of a polynomial. We shall also obtain zero free regions for certain class of analytic functions.

1. Introduction and statements of results

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0,$$

then according to a famous result due to Eneström and Kakeya [9, p. 136], the polynomial $P(z)$ does not vanish in $|z| > 1$.

Applaying this result to $P(tz)$, the following more general result is immediate.

THEOREM A. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$t^n a_n \geqslant t^{n-1} a_{n-1} \geqslant \ldots \geqslant t a_1 \geqslant a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leqslant t$.

In the literature [2, 5, 7, 8] there exist some extensions and generalizations of Eneström–Kakeya theorem. Govil and Rahman [5] generalized this theorem to the polynomial with complex coefficients.

While refining the result of Govil and Rahman [5], Govil and Jain [4] proved the following result.

THEOREM B. *Let $P(z) = \sum_{k=0}^{n} a_k z^k \not\equiv 0$ be a polynomial with complex coefficients such that*

$$|\arg a_k - \beta| \leqslant \alpha \leqslant \frac{\pi}{2}, \quad k = 0, 1, \ldots, n$$

for some β, and

$$|a_n| \geqslant |a_{n-1}| \geqslant \ldots \geqslant |a_1| \geqslant |a_0|.$$

Mathematics subject classification (1991): 30C10, 30C15.
Key words and phrases: polynomials, zeros, analytic functions.

Then $P(z)|$ has all its zeros in the ring shaped region given by

$$R_3 \leqslant |\alpha| \leqslant R_2.$$

Here

$$R_2 = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{\frac{1}{2}},$$

and

$$R_3 = \frac{1}{2M_2^2}\left[-R_2^2|b|(M_2 - |a_0|) + \left\{4|a_0|R_2^2M_2^3 + R_2^4|b|^2(M_2 - |a_0|)^2\right\}^{\frac{1}{2}}\right],$$

where

$$M_1 = |a_n|R,$$
$$M_2 = |a_n|R_2^2[R + R_2 - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha)],$$
$$c = |a_n - a_{n-1}|,$$
$$b = a_1 - a_0,$$

and

$$R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|.$$

As an extension of Theorem A, Dewan and Bikham [3] have recently proved the following result.

THEOREM C. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $t > 0$ and $0 < k \leqslant n$,*

$$t^n a_n \leqslant t^{n-1}a_{n-1} \leqslant \ldots \leqslant t^k a_k \geqslant t^{k-1}a_{k-1} \geqslant \ldots \geqslant ta_1 \geqslant a_0.$$

Then $P(z)$ has all its zeros in the circle

$$|z| \leqslant \frac{t}{|a_n|}\left\{\left(\frac{2a_k}{t^{n-k}} - a_n\right) + \frac{1}{t^n}\left(|a_0| - a_0\right)\right\}.$$

The main aim of this paper is to prove the following more general result (Theorem 1) which includes Theorem A, B and C as special cases. These theorems and many other such results can be established from Theorem 1 by a fairly uniform procedure. We shall also study the zeros of certain related analytic functions.

We start by proving the following:

THEOREM 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$ be a polynomial of degree n. If for some real number $t > 0$*

$$\operatorname*{Max}_{|z|=R} |ta_0 z^{n+1} + (ta_1 - a_0)z^n + \ldots + (ta_n - a_{n-1})z| \leqslant M_1 \tag{1}$$

and

$$\operatorname*{Max}_{|z|=R} |-a_n z^{n+1} + (ta_n - a_{n-1})z^n + \ldots + (ta_1 - a_0)z| \leqslant M_2, \tag{2}$$

where R is any positive real number. Then all zeros of $P(z)$ lie in the ring shaped region

$$r_2 \leqslant |z| \leqslant r_1 \tag{3}$$

where

$$r_1 = \frac{2M_1^2}{\{R^4|ta_n - a_{n-1}|^2(M_1 - |a_n|)^2 + 4|a_n|R^2M_1^3\}^{\frac{1}{2}} - |ta_n - a_{n-1}|(M_1 - |a_n|)R^2} \tag{4}$$

and

$$r_2 = \frac{1}{2M_2^2}\left[\{R^4|ta_1 - a_0|^2(M_2 - t|a_0|)^2 + 4M_2^3R^2t|a_0|\}^{\frac{1}{2}} - |ta_1 - a_0|(M_2 - t|a_0|)R^2\right]. \tag{5}$$

Remark 1. It can be easily verified that

$$\begin{aligned} r_1 &= \frac{2M_1^2}{-|ta_n - a_{n-1}|(M_1 - |a_n|)R^2 + \{R^4|ta_n - a_{n-1}|^2(M_1 - |a_n|)^2 + 4|a_n|R^2M_1^3\}^{\frac{1}{2}}} \\ &= \frac{|ta_n - a_{n-1}|(M_1 - |a_n|)R^2 + \{R^4|ta_n - a_{n-1}|^2(M_1 - |a_n|)^2 + 4|a_n|R^2M_1^3\}^{\frac{1}{2}}}{2|a_n|M_1R^2} \\ &= \frac{|ta_n - a_{n-1}|}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{|ta_n - a_{n-1}|^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|R^2}\right\}^{\frac{1}{2}}. \end{aligned} \tag{6}$$

If we take $R = (1/t)$ in (6), then we get

$$r_1 = \frac{|ta_n - a_{n-1}|}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{|ta_n - a_{n-1}|^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1t^2}{|a_n|}\right\}^{\frac{1}{2}}. \tag{7}$$

Suppose now $P(z) = \sum_{j=1}^{n} a_j z^j$ satisfies the conditions of Theorem B, than it can be easily seen (for reference see [5]) that

$$|a_j - a_{j-1}| \leqslant \{(|a_j| - |a_{j-1}|)\cos\alpha + (|a_j| + |a_{j-1}|)\sin\alpha\},$$

so that for $R = t = 1$, we get from (1),

$$\begin{aligned} &\operatorname*{Max}_{|z|=R} |ta_0 z^{n+1} + (ta_1 - a_0)z^n + \ldots + (ta_n - a_{n-1})z| \\ &\qquad \leqslant \sum_{j=1}^{n} |a_j - a_{j-1}| + |a_0| \end{aligned}$$

$$\leqslant \sum_{j=1}^{n}(|a_j| - |a_{j-1}|)\cos\alpha + \sum_{j=1}^{n}(|a_j| + |a_{j-1}|)\sin\alpha + |a_0|$$

$$= |a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{j=0}^{n-1}|a_j| - |a_0|(\cos\alpha + \sin\alpha - 1)$$

$$\leqslant |a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|$$

$$= |a_n|r = M_1, \quad \text{say}$$

where $r = \cos\alpha + \sin\alpha + \dfrac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|$.

Also from (7) with $t = 1$, we have

$$r_1 = \frac{|a_n - a_{n-1}|}{2}\Big(\frac{1}{|a_n|} - \frac{1}{M_1}\Big) + \left\{\frac{|a_n - a_{n-1}|^2}{4}\Big(\frac{1}{|a_n|} - \frac{1}{M_1}\Big)^2 + \frac{M_1}{|a_n|}\right\}^{\frac{1}{2}}.$$

Clearly $r_1 \geqslant 1$ and it follows by a similar argument as above that

$$\operatorname*{Max}_{|z|=R}| - a_n z^{n+1} + (ta_n - a_{n-1})z^n + \ldots + (ta_1 - a_0)z|$$

$$\leqslant |a_n|r_1^{n+1} + r_1^n\sum_{j=1}^{n}|a_j - a_{j-1}|$$

$$\leqslant |a_n|r_1^n\Big\{r_1 + r - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha)\Big\} = M_2, \quad \text{say.}$$

Now, from (7) for $t = 1$, we get $r_1 = R_2$ and from (5) for $R = R_2$, $t = 1$ we get $r_2 = R_3$. Consequently, it follows by Theorem 1 that all the zeros of $P(z)$ lie in $R_3 \leqslant |z| \leqslant R_2$, which is precisely the conclusion of Theorem B. Similarly, many other such results, in particular Theorem 2 of [3] and Theorem 2 of [4] easily follows from Theorem 1 by a fairly similar procedure.

Next, we use Theorem 1 to prove the following result, which includes Theorem C as a special case and is also an extension of a result due to Mohammad [10].

THEOREM 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some $t > 0$*

$$\operatorname*{Max}_{|z|=R}|ta_0 z^n + (ta_1 - a_0)z^{n-1} + \ldots + (ta_n - a_{n-1})| \leqslant M_3, \tag{8}$$

where R is any positive real number, then all the zeros of $P(z)$ lie in

$$|z| \leqslant \operatorname{Max}\left\{\frac{M_3}{|a_n|}, \frac{1}{R}\right\}.$$

Remark 2. If $P(z) = \sum_{j=0}^{n} a_j z^j$ satisfies the conditions of Theorem C, then for

$R = (1/t)$, with $a_{-1} = 0$, we have

$$\operatorname*{Max}_{|z|=\frac{1}{t}} \left| \sum_{k=0}^{n} (ta_k - a_{k-1}) z^{n-k} \right| \leqslant \sum_{k=0}^{n} \frac{|ta_k - a_{k-1}|}{t^{n-k}} = M_3, \quad \text{say.}$$

Since

$$\frac{1}{R} = t = \left| \sum_{k=0}^{n} \frac{ta_k - a_{k-1}}{a_n t^{n-k}} \right| \leqslant \sum_{k=0}^{n} \frac{|ta_k - a_{k-1}|}{|a_n| t^{n-k}} = \frac{M_3}{|a_n|}.$$

It follows by Theorem 2 that all the zeros of $P(z)$ lie in

$$|z| \leqslant \frac{M_3}{|a_n|} \leqslant \sum_{k=0}^{n} \frac{|ta_k - a_{k-1}|}{|a_n| t^{n-k}}. \tag{9}$$

Now a simple calculation shows that

$$\begin{aligned} \sum_{k=0}^{n} \frac{|ta_k - a_{k-1}|}{|a_n| t^{n-k}} &= \sum_{k=0}^{\lambda} \frac{|ta_k - a_{k-1}|}{|a_n| t^{n-k}} + \sum_{k=\lambda+1}^{n} \frac{|ta_k - a_{k-1}|}{|a_n| t^{n-k}} \\ &= \frac{t}{|a_n|} \left\{ \left(\frac{2a_\lambda}{t^{n-\lambda}} - a_n \right) + \frac{1}{t^n} (|a_0| - a_0) \right\}, \end{aligned}$$

and therefore from (9), we precisely get the conclusion of Theorem C.

Again, if $P(z)$ is a polynomial of degree n such that for some $t > 0$

$$0 \leqslant a_0 \leqslant ta_1 \leqslant \ldots \leqslant t^\lambda a_\lambda \geqslant t^{\lambda+1} a_{\lambda+1} \geqslant \ldots \geqslant t^n a_n,$$

then from (8), we have

$$\begin{aligned} \operatorname*{Max}_{|z|=R} \left| \sum_{k=0}^{n} (ta_k - a_{k-1}) z^{n-k} \right| &\leqslant \sum_{k=0}^{n} |ta_k - a_{k-1}| R^{n-k} \\ &= \sum_{k=0}^{\lambda} (ta_k - a_{k-1}) R^{n-k} + \sum_{k=\lambda+1}^{n} (a_{k-1} - ta_k) R^{n-k} \\ &= \frac{1}{R} (2a_\lambda R^{n-\lambda} - a_n) + \left(t - \frac{1}{R} \right) \left(\sum_{k=0}^{\lambda} a_k R^{n-k} - \sum_{k=\lambda+1}^{n} a_k R^{n-k} \right) = M_4. \end{aligned} \tag{10}$$

Using (10) in Theorem 2, we immediately get the following result, which is a generalisation of Eneström–Kakeya Theorem.

COROLLARY 1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n. If for some* $t > 0$

$$0 \leqslant a_0 \leqslant ta_1 \leqslant \ldots \leqslant t^\lambda a_\lambda \geqslant t^{\lambda+1} a_{\lambda+1} \geqslant \ldots \geqslant t^n a_n,$$

than all the zeros of $P(z)$ *lie in* $|z| \leqslant \operatorname{Max}\left(R_1, \frac{1}{R}\right)$, *where*

$$R_1 = \frac{1}{R|a_n|} (2a_\lambda R^{n-\lambda} - a_n) + \left(t - \frac{1}{R} \right) \frac{1}{|a_n|} \left(\sum_{k=0}^{\lambda} a_k R^{n-k} - \sum_{k=\lambda+1}^{n} a_k R^{n-k} \right).$$

If we take $\lambda = n$ in Corollary 1, we get

COROLLARY 2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n. If for some* $t > 0$

$$0 \leqslant a_0 \leqslant ta_1 \leqslant \ldots \leqslant t^n a_n,$$

than all the zeros of $P(z)$ *lie in* $|z| \leqslant \mathrm{Max}\left(R_1, \frac{1}{R}\right)$, *where*

$$R_1 = \frac{1}{R} + \left(t - \frac{1}{R}\right) \sum_{k=0}^{n} \left(\frac{a_k}{a_n}\right) R^{n-k}.$$

For $R = \frac{1}{t}$, Corollary 2 reduces to Theorem A.

We now turn to the study of zeros of certain related analytic functions.

THEOREM 3. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ *be analytic in* $|z| \leqslant R$. *If for some positive real number* $t \leqslant R$

$$\underset{|z|=R}{\mathrm{Max}} \left| \sum_{j=1}^{\infty} (a_{j-1} - ta_j) z^{j-1} \right| \leqslant M, \tag{11}$$

than $f(z)$ *does not vanish in* $|z| < r$, *where*

$$r = \frac{1}{2M^2} \left[\left\{ (Mr - t|a_0|)^2 |a_0 - ta_1|^2 + 4t|a_0| RM^3 \right\}^{\frac{1}{2}} - (MR - t|a_0|)|a_0 - ta_1| \right]. \tag{12}$$

By a similar argument as in the proof of Theorem 2, it can be easily verified that if $t|a_0| \leqslant MR$, then from (12),

$$r \geqslant \frac{t|a_0|}{M}, \tag{13}$$

and if $t|a_0| > MR$, then $f(z)$ does not vanish in

$$|z| \leqslant R. \tag{14}$$

By combining (13) and (14), the following corollary follows immediately.

COROLLARY 3. *If* $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ *be analytic in* $|z| \leqslant R$ *and*

$$\underset{|z|=R}{\mathrm{Max}} \left| \sum_{j=1}^{\infty} (a_{j-1} - ta_j) z^{j-1} \right| \leqslant M$$

then $f(z)$ *does not vanish in*

$$|z| < \mathrm{Min}\left\{ \frac{t|a_0|}{M}, R \right\}. \tag{15}$$

Finally, we present the following extension of Theorem 5 of [1].

THEOREM 4. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ *be analytic in* $|z| \leqslant t$. *If for some finite non-negative integer k*

$$a_0 \leqslant ta_1 \leqslant \ldots \leqslant t^k a_k \geqslant t^{k+1} a_{\lambda+1} \geqslant \ldots,$$

then $f(z)$ does not vanish in

$$|z| < \frac{t}{\left(2t^k\left|\frac{a_k}{a_0}\right| - 1\right) + \frac{2}{|a_0|}\sum_{j=1}^{\infty}\left|a_j - |a_j|\right|t^j}.$$

If $a_j > 0$ and $k = 0$, then Theorem 4 reduces to Theorem 5 of [1].

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Govil, Rahman and Schmeisser [6].

LEMMA 1. *If $f(z)$ is analytic in $|z| \leqslant 1$, $f(0) = a$ where $|a| < 1$, $f'(0) = b$, $|f(z)| \leqslant 1$ on $|z| = 1$, then for $|z| \leqslant 1$,*

$$|f(z)| \leqslant \frac{(1-|a|)|z|^2 + |b|\,|z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b|\,|z| + (1-|a|)}.$$

The example

$$f(z) = \frac{a + \frac{b}{1+a}z - z^2}{1 - \frac{b}{1+a}z - az^2}$$

shows that the estimate is sharp.

From Lemma 1, one can easily deduce the following:

LEMMA 2. *If $f(z)$ is analytic in $|z| \leqslant R$, $f(0) = 0$, $f'(0) = b$ and $|f(z)| \leqslant M$ for $|z| = R$, then*

$$|f(z)| \leqslant \frac{M|z|}{R^2} \cdot \frac{M|z| + R^2|b|}{M + |b|\,|z|} \qquad \text{for } |z| \leqslant R.$$

3. Proofs of the theorems

Proof of Theorem 1. Consider the polynomial

$$F(z) = (t - z)P(z) = -a_n z^{n+1} + (ta_n - a_{n-1})z^n + \ldots + ta_0. \tag{16}$$

We have

$$G(z) = z^{n+1}F(1/z) = -a_n + (ta_n - a_{n-1})z + \ldots + ta_0 z^{n+1},$$

so that

$$|G(z)| \geqslant |a_n| - |H(z)|, \tag{17}$$

where

$$H(z) = (ta_n - a_{n-1})z + (ta_{n-1} - a_{n-2})z^2 + \ldots + ta_0 z^{n+1}.$$

Clearly, $H(0)=0$ and $H'(0)=ta_n-a_{n-1}$. Since by (1) $|H(z)|\leqslant M_1$, for $|z|=R$, therefore, it follows by Lemma 2, that

$$|H(z)|\leqslant \frac{M_1|z|}{R^2}\cdot\frac{M_1|z|+R^2|ta_n-a_{n-1}|}{M_1+|ta_n-a_{n-1}|\,|z|},\quad \text{for } |z|\leqslant R.$$

Using this in (17) we get

$$\begin{aligned}|G(z)|&\geqslant |a_n|-\frac{M_1|z|(M_1|z|+R^2|ta_n-a_{n-1}|)}{R^2(M_1+|ta_n-a_{n-1}|\,|z|)}\\&=\frac{|a_n|R^2M_1+R^2|ta_n-a_{n-1}|(|a_n|-M_1)|z|-M_1^2|z|^2}{R^2(M_1+|ta_n-a_{n-1}|\,|z|)}>0,\end{aligned}$$

if

$$M_1^2|z|^2+R^2|ta_n-a_{n-1}|(M_1-|a_n|)|z|-|a_n|R^2M_1<0.$$

This gives $|G(z)|>0$, if

$$\begin{aligned}|z|&<\frac{\{R^4|ta_n-a_{n-1}|^2(M_1-|a_n|)^2+4|a_n|R^2M_1^3\}^{\frac{1}{2}}-R^2|ta_n-a_{n-1}|(M_1-|a_n|)}{2M_1^2}\\&=\frac{1}{r_1}.\end{aligned}\qquad \text{(by (4))}$$

Consequently, all zeros of $G(z)$ lie in $|z|\geqslant \frac{1}{r_1}$. As $F(z)=z^{n+1}G(1/z)$ we conclude that all the zeros of $F(z)$ lie in $|z|\leqslant r_1$. Since every zero of $P(z)$ is also a zero of $F(z)$, it follows that all the zeros of $P(z)$ lie in

$$|z|\leqslant r_1. \tag{18}$$

Again, from (16), we have

$$|F(z)|\geqslant |ta_0|-|T(z)|, \tag{19}$$

where

$$T(z)=-a_nz^{n+1}+(ta_n-a_{n-1})z^n+\ldots+(ta_1-a_0)z.$$

Clearly $T(0)=0$ and $T'(0)=ta_1-a_0$. Since by (2), $|T(z)|\leqslant M_2$ for $|z|=R$, therefore, it follows by Lemma 2, that

$$|T(z)|\leqslant \frac{M_2|z|}{R^2}\,\frac{M_2|z|+R^2|ta_1-a_0|}{M_2+|ta_1-a_0||z|},\quad \text{for } |z|\leqslant R.$$

So that from (19) we have

$$\begin{aligned}|F(z)|&\geqslant |ta_0|-\frac{M_2|z|(M_2|z|+R^2|ta_1-a_0|)}{R^2(M_2+|ta_1-a_0|\,|z|)}\\&=\frac{t|a_0|R^2M_2+R^2|ta_1-a_0|(t|a_0|-M_2)|z|-M_2^2|z|^2}{R^2(M_2+|ta_1-a_0|\,|z|)}>0,\end{aligned}$$

if

$$M_2^2|z|^2+R^2|ta_1-a_0|(M_2-t|a_0|)|z|-t|a_0|R^2M_2<0.$$

Thus $F(z) > 0$, if

$$
\begin{aligned}
|z| &< \frac{1}{2M_2^2}\Big[\Big\{R^4|ta_1 - a_0|^2(M_2 - t|a_0|)^2 + 4t|a_0|R^2M_2^3\Big\}^{\frac{1}{2}} \\
&\qquad\qquad - R^2|ta_1 - a_0|(M_2 - t|a_0|)\Big] \\
&= r_2. \qquad\qquad \text{(by (5))}
\end{aligned}
$$

Since every zero of $P(z)$ is also a zero of $F(z)$, we conclude that all zeros of $P(z)$ lie in

$$|z| \geqslant r_2. \tag{20}$$

The desired result follows by combining (18) and (20).

Proof of Theorem 2. From (1) and (8) we get

$$\max_{|z|=R} |ta_0z^{n+1} + (ta_1 - a_0)z^n + \ldots + (ta_n - a_{n-1})z| \leqslant M_3R = M_1, \quad \text{say.}$$

Replacing M_1 by M_3R in (4) it follows from Theorem 1 that

$$r_1 = \frac{2M_3^2}{\{|ta_n - a_{n-1}|^2(M_3R - |a_n|)^2 + 4|a_n|M_3^3R\}^{\frac{1}{2}} - |ta_n - a_{n-1}|(M_3R - |a_n|)}. \tag{21}$$

Now, first we suppose that $|a_n| \leqslant M_3R$, then $M_3R - |a_n| \geqslant 0$. Since $|ta_n - a_{n-1}| \leqslant M_3$, therefore, we have

$$|ta_n - a_{n-1}|(M_3R - |a_n|) \leqslant M_3(M_3R - |a_n|).$$

Or, equivalently

$$|a_n|M_3 + |ta_n - a_{n-1}|(M_3R - |a_n|) \leqslant M_3^2R,$$

which on multiplication by $4M_3|a_n|$, gives

$$4M_3^2|a_n|^2 + 4M_3|a_n|\,|ta_n - a_{n-1}|(M_3R - |a_n|) \leqslant 4|a_n|M_3^3R.$$

Adding $|ta_n - a_{n-1}|^2(M_3R - |a_n|)^2$ both sides, we get

$$\{2M_3|a_n| + |ta_n - a_{n-1}|(M_3R - |a_n|)\}^2 \leqslant |ta_n - a_{n-1}|^2(M_3R - |a_n|)^2 + 4|a_n|M_3^3R.$$

Or,

$$2M_3|a_n| \leqslant \{|ta_n - a_{n-1}|^2(M_3R - |a_n|)^2 + 4|a_n|M_3^3R\}^{\frac{1}{2}} - |ta_n - a_{n-1}|(M_3R - |a_n|),$$

from which we conclude that

$$r_1 \leqslant \frac{M_3}{|a_n|}. \tag{22}$$

Hence it follows by Theorem 1 that all the zeros of $P(z)$ lie in the circle $|z| \leqslant (M_3/|a_n|)$.

Next, we suppose that $|a_n| > M_3R$, then this clearly implies from (8),

$$|ta_0z^{n+1} + (ta_1 - a_0)z^n + \ldots + (ta_n - a_{n-1})z| < |a_n| \qquad \text{for } |z| = R.$$

Using Rouche's theorem, it follows that the polynomial

$$G(z) = ta_0 z^{n+1} + (ta_1 - a_0)z^n + \ldots + (ta_n - a_{n-1})z + a_n$$

does not vanish in $|z| < R$. This implies that the polynomial $F(z) = z^{n+1}G(1/z)$ does not vanish in $|z| > \frac{1}{R}$. Since every zero of $P(z)$ is also a zero of $F(z)$, we conclude that all zeros of $P(z)$ lie in the circle

$$|z| \leqslant \frac{1}{R}. \tag{23}$$

From (22) and (23) it follows that all the zeros of $P(z)$ lie in

$$|z| \leqslant \operatorname{Max}\left\{\frac{M_3}{|a_n|}, \frac{1}{R}\right\}.$$

This proves Theorem 2 completely.

Proof of Theorem 3. It is easy to observe that $\lim_{k\to\infty} a_k t^k = 0$. Now, consider the function

$$F(z) = (z-t)f(z) = -ta_0 + z\sum_{j=1}^{\infty}(a_{j-1} - ta_j)z^{j-1} = -ta_0 + G(z), \tag{24}$$

where

$$G(z) = z\sum_{j=1}^{\infty}(a_{j-1} - ta_j)z^{j-1}.$$

Here $G(0) = 0$, $G'(0) = a_0 - ta_1$ and since

$$|G(z)| \leqslant R\left|\sum_{j=1}^{\infty}(a_{j-1} - ta_j)z^{j-1}\right| \leqslant MR \qquad \text{for } |z| = R.$$

Therefore, it follows by Lemma 2, that

$$|G(z)| \leqslant \frac{M|z|(M|z| + |a_0 - ta_1|R)}{MR + |a_0 - ta_1|\,|z|} \qquad \text{for } |z| \leqslant R.$$

Using this in (24), we get

$$\begin{aligned} |F(z)| &\geqslant |ta_0| - \frac{M|z|(M|z| + |a_0 - ta_1|R)}{MR + |a_0 - ta_1|\,|z|} \\ &= \frac{|ta_0|MR + (t|a_0| - MR)|a_0 - ta_1|\,|z| - M^2|z|^2}{MR + |a_0 - ta_1|\,|z|} > 0, \end{aligned}$$

if

$$M^2|z|^2 + (MR - t|a_0|)\,|a_0 - ta_1|\,|z| - t|a_0|MR < 0.$$

This gives $|F(z)| > 0$, if

$$|z| < \frac{1}{2M^2}\left[\left\{(MR - t|a_0|)^2|a_0 - ta_1|^2 + 4t|a_0|M^3R\right\}^{\frac{1}{2}} - (MR - t|a_0|)\,|a_0 - ta_1|\right] = r.$$

(by (12))

Therefore $F(z)$ does not vanish in $|z| < r$, from which it follows that $f(z)$ does not vanish in $|z| < r$. This completes the proof of Theorem 3.

Proof of Theorem 4. It is clear that $\lim_{j\to\infty} t^j a_j = 0$. Since

$$|a_0| = \left|\sum_{j=1}^{\infty}(a_{j-1} - ta_j)t^{j-1}\right| \leqslant \operatorname*{Max}_{|z|=t}\left|\sum_{j=1}^{\infty}(a_{j-1} - ta_j)z^{j-1}\right| = M, \quad \text{say.}$$

Therefore $\dfrac{|a_0|}{M} \leqslant 1$, and hence

$$\operatorname{Min}\left\{\frac{t|a_0|}{M}, t\right\} = \frac{t|a_0|}{M}.$$

Using this in (15), with $R = t$, it follows that $f(z)$ does not vanish in

$$|z| < \frac{t|a_0|}{M}. \tag{25}$$

Now, for $|z| = t$ we have

$$\begin{aligned}
M = \operatorname*{Max}_{|z|=t}\left|\sum_{j=1}^{\infty}(a_{j-1} - ta_j)z^{j-1}\right| &\leqslant \sum_{j=1}^{\infty}|a_{j-1} - ta_j|t^{j-1}\\
&\leqslant \sum_{j=1}^{\infty}\big|t|a_j| - |a_{j-1}|\big|t^{j-1} + \sum_{j=1}^{\infty}\big|t(a_j - |a_j|) - (a_{j-1} - |a_{j-1}|)\big|t^{j-1}\\
&= \sum_{j=1}^{k}\big(t|a_j| - |a_{j-1}|\big)t^{j-1} + \sum_{j=k+1}^{\infty}\big(|a_{j-1} - t|a_j|\big)t^{j-1}\\
&\qquad + \sum_{j=1}^{\infty}\Big|t(a_j - |a_j|) - (a_{j-1} - |a_{j-1}|)\Big|t^{j-1}\\
&= 2t^k|a_k| - |a_0| + \sum_{j=1}^{\infty}\Big|t(a_j - |a_j|) - (a_{j-1} - |a_{j-1}|)\Big|t^{j-1}\\
&\leqslant 2t^k|a_k| - |a_0| + 2\sum_{j=1}^{\infty}\Big|a_j - |a_j|\Big|t^j,
\end{aligned}$$

therefore, it follows from (25), that $f(z)$ does not vanish in

$$|z| < \frac{t|a_0|}{2t^k|a_k| - |a_0| + 2\sum_{j=1}^{\infty}\big|a_j - |a_j|\big|t^j}.$$

This proves the Theorem 4 completely.

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(Received June 6, 1997)
(Revised January 19, 1998)

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