ON THE ZEROS OF POLYNOMIALS
AND RELATED ANALYTIC FUNCTIONS

Abdul Aziz and W. M. Shah, Srinagar, India

Abstract. Let $P(z)$ be a polynomial of degree $n$ with real or complex coefficients. In this paper we obtain a ring shaped region containing all the zeros of $P(z)$. Our results include, as special cases, several known extensions of Eneström–Kakeya theorem on the zeros of a polynomial. We shall also obtain zero free regions for certain class of analytic functions.

1. Introduction and statements of results

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ such that
\[ a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0, \]
then according to a famous result due to Eneström and Kakeya [9, p. 136], the polynomial $P(z)$ does not vanish in $|z| > 1$.

Applying this result to $P(tz)$, the following more general result is immediate.

**Theorem A.** If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n$ such that
\[ r^n a_n \geq r^{n-1} a_{n-1} \geq \ldots \geq r a_1 \geq a_0 > 0, \]
then all the zeros of $P(z)$ lie in $|z| \leq t$.


While refining the result of Govil and Rahman [5], Govil and Jain [4] proved the following result.

**Theorem B.** Let $P(z) = \sum_{k=0}^{n} a_k z^k \neq 0$ be a polynomial with complex coefficients such that
\[ |\arg a_k - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad k = 0, 1, \ldots, n \]
for some $\beta$, and
\[ |a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|. \]


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Then $P(z)$ has all its zeros in the ring shaped region given by

$$R_3 \leq |\alpha| \leq R_2.$$

Here

$$R_2 = \frac{c}{2} \left( \frac{1}{|a_n|} - \frac{1}{M_1} \right) + \left\{ \frac{c^2}{4} \left( \frac{1}{|a_n|} - \frac{1}{M_1} \right)^2 + \frac{M_1}{|a_n|} \right\}^{\frac{1}{2}},$$

and

$$R_3 = \frac{1}{2M_2^2} \left[ -2R^2 b |M_2 | - |a_0 | \right] + \left\{ 4|a_0 |^2 R_2 + R_2^4 |b|^2 (M_2 | - |a_0 |)^2 \right\}^{\frac{1}{2}},$$

where

$$M_1 = |a_n| R,$$

$$M_2 = |a_n| R_2^2 R + R_2 - \frac{|a_0 |}{|a_n|} (\cos \alpha + \sin \alpha),$$

$$c = |a_n - a_{n-1}|,$$

$$b = a_1 - a_0,$$

and

$$R = \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{k=0}^{n-1} |a_k|.$$

As an extension of Theorem A, Dewan and Bikham [3] have recently proved the following result.

**Theorem C.** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ such that for some $j = 0$ and $0 < k \leq n$,

$$t^n a_n \leq t^{n-1} a_{n-1} \leq \ldots \leq t^k a_k \geq t^{k-1} a_{k-1} \geq \ldots \geq t a_1 \geq a_0.$$

Then $P(z)$ has all its zeros in the circle

$$|z| \leq \left( \frac{2a_k}{t^n-k} - a_n \right) + \frac{1}{t^n} (|a_0| - a_0).$$

The main aim of this paper is to prove the following more general result (Theorem 1) which includes Theorem A, B and C as special cases. These theorems and many other such results can be established from Theorem 1 by a fairly uniform procedure. We shall also study the zeros of certain related analytic functions.

We start by proving the following:

**Theorem 1.** Let $P(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ be a polynomial of degree $n$. If for some real number $t > 0$

$$\max_{|z|=R} \left| t a_0 z^{n+1} + (t a_1 - a_0) z^n + \ldots + (t a_n - a_{n-1}) z \right| \leq M_1$$

(1)

and
where $R$ is any positive real number. Then all zeros of $P(z)$ lie in the ring shaped region

$$r_2 \leq |z| \leq r_1$$

where

$$r_1 = \frac{2M_1^2}{\{R^4|t_{a_n} - a_{n-1}|(M_1 - |a_n|)^2 + 4|a_n|R^2M_1^3\}^{\frac{1}{2}} - |t_{a_n} - a_{n-1}|(M_1 - |a_n|)R^2}$$

and

$$r_2 = \frac{1}{2M_2^2} \left[ \{R^4|t_{a_1} - a_0|^2(M_2 - t|a_0|)^2 + 4M_2^2R^2t|a_0|\}^{\frac{1}{2}} - |t_{a_1} - a_0|(M_2 - t|a_0|)R^2 \right].$$

**Remark 1.** It can be easily verified that

$$r_1 = \frac{2M_1^2}{\{R^4|t_{a_n} - a_{n-1}|(M_1 - |a_n|)^2 + 4|a_n|R^2M_1^3\}^{\frac{1}{2}} - |t_{a_n} - a_{n-1}|(M_1 - |a_n|)R^2}$$

If we take $R = (1/t)$ in (6), then we get

$$r_1 = \frac{|t_{a_n} - a_{n-1}|(1/|a_n| - 1/M_1)}{2} + \left\{ \frac{|t_{a_n} - a_{n-1}|^2}{4} \left( \frac{1}{|a_n|} - \frac{1}{M_1} \right)^2 + \frac{M_1}{|a_n|} \right\}^{\frac{1}{2}}.$$

Suppose now $P(z) = \sum_{j=1}^{n} a_jz^j$ satisfies the conditions of Theorem B, than it can be easily seen (for reference see [5]) that

$$|a_j - a_{j-1}| \leq \{(|a_j| - |a_{j-1}|) \cos \alpha + (|a_j| + |a_{j-1}|) \sin \alpha \},$$

so that for $R = t = 1$, we get from (1),

$$\max_{|z| = R} |t_{a_0}z^{n+1} + (t_{a_1} - a_0)z^n + \ldots + (t_{a_n} - a_{n-1})z|$$

$$\leq \sum_{j=1}^{n} |a_j - a_{j-1}| + |a_0|.$$
\[
\left| \sum_{j=1}^{n} (|a_j| - |a_{j-1}|) \cos \alpha + \sum_{j=1}^{n} (|a_j| + |a_{j-1}|) \sin \alpha + |a_0| \right|
\]

\[
= |a_n|(\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| - |a_0|(\cos \alpha + \sin \alpha - 1)
\]

\[
\leq |a_n|(\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j|
\]

\[
= |a_n|r = M_1, \quad \text{say}
\]

where \( r = \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j| \).

Also from (7) with \( t = 1 \), we have

\[
r_1 = \frac{|a_n - a_{n-1}|}{2} \left( \frac{1}{|a_n|} - \frac{1}{M_1} \right) + \left\{ \frac{|a_n - a_{n-1}|^2}{4} \left( \frac{1}{|a_n|} - \frac{1}{M_1} \right)^2 + \frac{M_1}{|a_n|} \right\}^{\frac{1}{2}}.
\]

Clearly \( r_1 \geq 1 \) and it follows by a similar argument as above that

\[
\max_{|z|=R} |a_n \bar{z}^{n+1} + (ta_n - a_{n-1}) \bar{z}^{n} + \ldots + (ta_1 - a_0)z| \leq |a_n|r_1^{n+1} + r_1^n \sum_{j=1}^{n} |a_j - a_{j-1}|
\]

\[
\leq |a_n|\left( r_1 + r - \frac{|a_0|}{|a_n|}(\cos \alpha + \sin \alpha) \right) = M_2, \quad \text{say}.
\]

Now, from (7) for \( t = 1 \), we get \( r_1 = R_2 \) and from (5) for \( R = R_2, t = 1 \) we get \( r_2 = R_3 \). Consequently, it follows by Theorem 1 that all the zeros of \( P(z) \) lie in \( R_3 \leq |z| \leq R_2 \), which is precisely the conclusion of Theorem B. Similarly, many other such results, in particular Theorem 2 of [3] and Theorem 2 of [4] easily follows from Theorem 1 by a fairly similar procedure.

Next, we use Theorem 1 to prove the following result, which includes Theorem C as a special case and is also an extension of a result due to Mohammad [10].

**Theorem 2.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \). If for some \( t > 0 \)

\[
\max_{|z|=R} |ta_0z^n + (ta_1 - a_0)z^{n-1} + \ldots + (ta_n - a_{n-1})| \leq M_3, \quad (8)
\]

where \( R \) is any positive real number, then all the zeros of \( P(z) \) lie in

\[
|z| \leq \max \left\{ \frac{M_3}{|a_n|}, \frac{1}{R} \right\}.
\]

**Remark 2.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) satisfies the conditions of Theorem C, then for
R = (1/r), with \( a_{-1} = 0 \), we have

\[
\text{Max}_{|z| = \frac{1}{R}} \left| \sum_{k=0}^{n} (ta_k - a_{k-1})z^{n-k} \right| \leqslant \sum_{k=0}^{n} \left| \frac{ta_k - a_{k-1}}{t^{n-k}} \right| = M_3, \text{ say.}
\]

Since

\[
\frac{1}{R} = \tau = \left| \sum_{k=0}^{n} \frac{ta_k - a_{k-1}}{a_n t^{n-k}} \right| \leqslant \sum_{k=0}^{n} \left| \frac{ta_k - a_{k-1}}{a_n t^{n-k}} \right| = M_3 \frac{t}{|a_n|}.
\]

It follows by Theorem 2 that all the zeros of \( P(z) \) lie in

\[
|z| \leqslant \frac{M_3}{|a_n|} \sum_{k=0}^{n} \left| \frac{ta_k - a_{k-1}}{a_n t^{n-k}} \right|.
\]

(9)

Now a simple calculation shows that

\[
\sum_{k=0}^{n} \left| \frac{ta_k - a_{k-1}}{a_n t^{n-k}} \right| = \sum_{k=0}^{\lambda} \left| \frac{ta_k - a_{k-1}}{a_n t^{n-k}} \right| + \sum_{k=\lambda+1}^{n} \left| \frac{ta_k - a_{k-1}}{a_n t^{n-k}} \right|
\]

\[
= \frac{t}{|a_n|} \left\{ \frac{2a_\lambda}{t^{n-\lambda} - a_n} + \frac{1}{t^n} (|a_0| - a_0) \right\},
\]

and therefore from (9), we precisely get the conclusion of Theorem C.

Again, if \( P(z) \) is a polynomial of degree \( n \) such that for some \( t > 0 \)

\[
0 \leqslant a_0 \leqslant ta_1 \leqslant \ldots \leqslant t^\lambda a_\lambda \geqslant t^{\lambda+1} a_{\lambda+1} \geqslant \ldots \geqslant t^n a_n,
\]

then from (8), we have

\[
\text{Max}_{|z| = R} \left| \sum_{k=0}^{n} (ta_k - a_{k-1})z^{n-k} \right| \leqslant \sum_{k=0}^{n} |ta_k - a_{k-1}|R^{n-k}
\]

\[
= \sum_{k=0}^{\lambda} (ta_k - a_{k-1})R^{n-k} + \sum_{k=\lambda+1}^{n} (a_{k-1} - ta_k)R^{n-k}
\]

\[
= \frac{1}{R} (2a_\lambda R^{n-\lambda} - a_n) + \left( t - \frac{1}{R} \right) \left( \sum_{k=0}^{\lambda} a_k R^{n-k} - \sum_{k=\lambda+1}^{n} a_k R^{n-k} \right) = M_4.
\]

(10)

Using (10) in Theorem 2, we immediately get the following result, which is a generalisation of Eneström–Kakeya Theorem.

**COROLLARY 1.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \). If for some \( t > 0 \)

\[
0 \leqslant a_0 \leqslant ta_1 \leqslant \ldots \leqslant t^\lambda a_\lambda \geqslant t^{\lambda+1} a_{\lambda+1} \geqslant \ldots \geqslant t^n a_n,
\]

than all the zeros of \( P(z) \) lie in \( |z| \leqslant \text{Max} \left( R_1, \frac{1}{R} \right) \), where

\[
R_1 = \frac{1}{R|a_n|} (2a_\lambda R^{n-\lambda} - a_n) + \left( t - \frac{1}{R} \right) \frac{1}{|a_n|} \left( \sum_{k=0}^{\lambda} a_k R^{n-k} - \sum_{k=\lambda+1}^{n} a_k R^{n-k} \right).
\]

If we take \( \lambda = n \) in Corollary 1, we get
Corollary 2. Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \). If for some \( t > 0 \)
\[ 0 \leq a_0 \leq ta_1 \leq \ldots \leq t^n a_n, \]
then all the zeros of \( P(z) \) lie in \( |z| \leq \max (R_1, \frac{1}{R}) \), where
\[ R_1 = \frac{1}{R} + \left( t - \frac{1}{R} \right) \sum_{k=0}^{n} \left( \frac{a_k}{a_n} \right) R^{n-k}. \]

For \( R = \frac{1}{t} \), Corollary 2 reduces to Theorem A.

We now turn to the study of zeros of certain related analytic functions.

Theorem 3. Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0 \) be analytic in \( |z| \leq R \). If for some positive real number \( t \leq R \)
\[ \max_{|z|=R} \left| \sum_{j=1}^{\infty} (a_{j-1} - ta_j) z^{j-1} \right| \leq M, \]
then \( f(z) \) does not vanish in \( |z| < r \), where
\[ r = \frac{1}{2M^2} \left[ \left( (MR - t|a_0|)^2|a_0 - ta_1|^2 + 4t|a_0|RM^3 \right)^{\frac{1}{2}} - (MR - t|a_0|)|a_0 - ta_1| \right]. \]

By a similar argument as in the proof of Theorem 2, it can be easily verified that if \( t|a_0| \leq MR \), then from (12),
\[ r \geq \frac{t|a_0|}{M}, \]
and if \( t|a_0| > MR \), then \( f(z) \) does not vanish in
\[ |z| \leq R. \]

By combining (13) and (14), the following corollary follows immediately.

Corollary 3. If \( f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0 \) be analytic in \( |z| \leq R \) and
\[ \max_{|z|=R} \left| \sum_{j=1}^{\infty} (a_{j-1} - ta_j) z^{j-1} \right| \leq M \]
then \( f(z) \) does not vanish in
\[ |z| < \min \left\{ \frac{t|a_0|}{M}, R \right\}. \]

Finally, we present the following extension of Theorem 5 of [1].

Theorem 4. Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0 \) be analytic in \( |z| \leq t \). If for some finite non-negative integer \( k \)
\[ a_0 \leq ta_1 \leq \ldots \leq t^k a_k \geq t^{k+1} a_{k+1} \geq \ldots, \]
then $f(z)$ does not vanish in

$$|z| < \frac{t}{\left(2^{k} \frac{|a_{k}|}{a_{0}} - 1\right) + \frac{2}{|a_{0}|} \sum_{j=1}^{\infty} |a_{j} - |a_{j}|t^{j}} .$$

If $a_{j} > 0$ and $k = 0$, then Theorem 4 reduces to Theorem 5 of [1].

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Govil, Rahman and Schmeisser [6].

**Lemma 1.** If $f(z)$ is analytic in $|z| \leq 1$, $f(0) = a$ where $|a| < 1$, $f'(0) = b$, $|f(z)| \leq 1$ on $|z| = 1$, then for $|z| \leq 1$,

$$|f(z)| \leq \frac{(1 - |a|)|z|^{2} + |b||z| + |a|(1 - |a|)}{|a|(1 - |a|)|z|^{2} + |b||z| + (1 - |a|)} .$$

The example

$$f(z) = \frac{a + \frac{b}{1 + a}z - z^{2}}{1 - \frac{b}{1 + a}z - az^{2}}$$

shows that the estimate is sharp.

From Lemma 1, one can easily deduce the following:

**Lemma 2.** If $f(z)$ is analytic in $|z| \leq R$, $f(0) = 0$, $f'(0) = b$ and $|f(z)| \leq M$ for $|z| = R$, then for $|z| \leq R$,

$$|f(z)| \leq \frac{M|z|}{R^{2}} \cdot \frac{M|z| + R^{2}|b|}{M + |b||z|} .$$

3. Proofs of the theorems

**Proof of Theorem 1.** Consider the polynomial

$$F(z) = (t - z)P(z) = -a_{n}z^{n+1} + (ta_{n} - a_{n-1})z^{n} + \ldots + ta_{0} . \quad (16)$$

We have

$$G(z) = z^{n+1}F(1/z) = -a_{n} + (ta_{n} - a_{n-1})z + \ldots + ta_{0}z^{n+1} ,$$

so that

$$|G(z)| \geq |a_{n}| - |H(z)| , \quad (17)$$

where

$$H(z) = (ta_{n} - a_{n-1})z + (ta_{n-1} - a_{n-2})z^{2} + \ldots + ta_{0}z^{n+1} .$$
Clearly, \( H(0) = 0 \) and \( H'(0) = ta_n - a_{n-1} \). Since by (1) \( |H(z)| \leq M_1 \), for \( |z| = R \), therefore, it follows by Lemma 2, that
\[
|H(z)| \leq \frac{M_1|z|}{R^2} \cdot \frac{M_1|z| + R^2|ta_n - a_{n-1}|}{M_1 + |ta_n - a_{n-1}| |z|}, \quad \text{for } |z| \leq R.
\]
Using this in (17) we get
\[
|G(z)| \geq |a_n| - \frac{M_1|z|(M_1|z| + R^2|ta_n - a_{n-1}|)}{R^2(M_1 + |ta_n - a_{n-1}| |z|)}
= \frac{|a_n|R^2M_1 + R^2|ta_n - a_{n-1}|(|a_n| - M_1)|z| - M_1^2|z|^2}{R^2(M_1 + |ta_n - a_{n-1}| |z|)} > 0,
\]
if
\[
M_1^2|z|^2 + R^2|ta_n - a_{n-1}|(|a_n| - M_1)|z| - |a_n|R^2M_1 < 0.
\]
This gives \( |G(z)| > 0 \), if

\[
|z| < \frac{\left[R^2|ta_n - a_{n-1}|^2(M_1 - |a_n|)^2 + 4|a_n|R^2M_1^2\right]^\frac{1}{2} - R^2|ta_n - a_{n-1}|(M_1 - |a_n|)}{2M_1^2} = \frac{1}{r_1}.
\]
(by (4))

Consequently, all zeros of \( G(z) \) lie in \( |z| \geq \frac{1}{r_1} \). As \( F(z) = z^n + G(1/z) \) we conclude that all the zeros of \( F(z) \) lie in \( |z| \leq r_1 \). Since every zero of \( P(z) \) is also a zero of \( F(z) \), it follows that all the zeros of \( P(z) \) lie in
\[
|z| \leq r_1. \tag{18}
\]

Again, from (16), we have
\[
|F(z)| \geq |ta_0| - |T(z)|, \tag{19}
\]
where
\[
T(z) = -a_nz^{n+1} + (ta_n - a_{n-1})z^n + \ldots + (ta_1 - a_0)z.
\]
Clearly \( T(0) = 0 \) and \( T'(0) = ta_1 - a_0 \). Since by (2), \( |T(z)| \leq M_2 \) for \( |z| = R \), therefore, it follows by Lemma 2, that
\[
|T(z)| \leq \frac{M_2|z|}{R^2} \cdot \frac{M_2|z| + R^2|ta_1 - a_0|}{M_2 + |ta_1 - a_0||z|}, \quad \text{for } |z| \leq R.
\]
So that from (19) we have
\[
|F(z)| \geq |ta_0| - \frac{M_2|z|(M_2|z| + R^2|ta_1 - a_0|)}{R^2(M_2 + |ta_1 - a_0||z|)}
= \frac{|a_0|R^2M_2 + R^2|ta_1 - a_0|(t|a_0| - M_2)|z| - M_2^2|z|^2}{R^2(M_2 + |ta_1 - a_0||z|)} > 0,
\]
if
\[
M_2^2|z|^2 + R^2|ta_1 - a_0|(M_2 - t|a_0|)|z| - t|a_0|R^2M_2 < 0.
\]
Thus \( F(z) > 0 \), if

\[
|z| < \frac{1}{2M_2^2} \left[ \left\{ R^4 |ta_1 - a_0|^2 (M_2 - t|a_0|)^2 + 4t|a_0| R^2 M_2^3 \right\}^{\frac{1}{2}}
\right.
\]

\[
\left. - R^2 |ta_1 - a_0| (M_2 - t|a_0|) \right]\]

\[
= r_2. \quad \text{(by (5))}
\]

Since every zero of \( P(z) \) is also a zero of \( F(z) \), we conclude that all zeros of \( P(z) \) lie in

\[
|z| \geq r_2. \quad \text{(20)}
\]

The desired result follows by combining (18) and (20).

\[\text{Proof of Theorem 2.}\]

From (1) and (8) we get

\[
\left| \frac{ta_0 z^{n+1} + (ta_1 - a_0) z^n + \ldots + (ta_n - a_{n-1}) z}{|z|=R} \right| \leq M_3 R = M_1, \quad \text{say.}
\]

Replacing \( M_1 \) by \( M_3 R \) in (4) it follows from Theorem 1 that

\[
r_1 = \frac{2M_3}{\{ |ta_n - a_{n-1}|^2 (M_3 R - |a_n|)^2 + 4|a_n|M_3^2 R \}^{\frac{1}{2}} - |ta_n - a_{n-1}| (M_3 R - |a_n|) \}.
\]

(21)

Now, first we suppose that \( |a_n| \leq M_3 R \), then \( M_3 R - |a_n| \geq 0 \). Since \( |ta_n - a_{n-1}| \leq M_3 \), therefore, we have

\[
|ta_n - a_{n-1}| (M_3 R - |a_n|) \leq M_3 (M_3 R - |a_n|).
\]

Or, equivalently

\[
|a_n|M_3 + |ta_n - a_{n-1}| (M_3 R - |a_n|) \leq M_3^2 R,
\]

which on multiplication by \( 4M_3 |a_n| \), gives

\[
4M_3^2 |a_n|^2 + 4M_3 |a_n| |ta_n - a_{n-1}| (M_3 R - |a_n|) \leq 4|a_n|M_3^3 R.
\]

Adding \( |ta_n - a_{n-1}|^2 (M_3 R - |a_n|)^2 \) both sides, we get

\[
\{2M_3 |a_n| + |ta_n - a_{n-1}| (M_3 R - |a_n|)\}^2 \leq |ta_n - a_{n-1}|^2 (M_3 R - |a_n|)^2 + 4|a_n|M_3^2 R.
\]

Or,

\[
2M_3 |a_n| \leq \{ |ta_n - a_{n-1}|^2 (M_3 R - |a_n|)^2 + 4|a_n|M_3^2 R \}^{\frac{1}{2}} - |ta_n - a_{n-1}| (M_3 R - |a_n|),
\]

from which we conclude that

\[
r_1 \leq \frac{M_3}{|a_n|}. \quad \text{(22)}
\]

Hence it follows by Theorem 1 that all the zeros of \( P(z) \) lie in the circle \( |z| \leq (M_3/|a_n|) \).

Next, we suppose that \( |a_n| > M_3 R \), then this clearly implies from (8),

\[
|ta_0 z^{n+1} + (ta_1 - a_0) z^n + \ldots + (ta_n - a_{n-1}) z| < |a_n| \quad \text{for} \quad |z| = R.
\]
Using Rouche's theorem, it follows that the polynomial
\[ G(z) = ta_0 z^{n+1} + (ta_1 - a_0)z^n + \ldots + (ta_n - a_{n-1})z + a_n \]
does not vanish in \(|z| < R\). This implies that the polynomial \( F(z) = z^{n+1}G(1/z) \)
does not vanish in \(|z| > \frac{1}{R}\). Since every zero of \( P(z) \) is also a zero of \( F(z) \), we
conclude that all zeros of \( P(z) \) lie in the circle
\[ |z| \leq \frac{1}{R}. \] (23)

From (22) and (23) it follows that all the zeros of \( P(z) \) lie in
\[ |z| \leq \text{Max}\left\{ \frac{M_3}{|a_n|}, \frac{1}{R} \right\}. \]

This proves Theorem 2 completely.

**Proof of Theorem 3.** It is easy to observe that \( \lim_{k \to \infty} a_k t^k = 0 \). Now, consider the
function
\[ F(z) = (z-t)f(z) = -ta_0 + z\sum_{j=1}^{\infty} (a_j - ta_j)z^{j-1} = -ta_0 + G(z), \] (24)
where
\[ G(z) = z\sum_{j=1}^{\infty} (a_j - ta_j)z^{j-1}. \]

Here \( G(0) = 0, G'(0) = a_0 - ta_1 \) and since
\[ |G(z)| \leq R \sum_{j=1}^{\infty} |a_j - ta_j|z^{j-1} \leq MR \quad \text{for } |z| = R. \]

Therefore, it follows by Lemma 2, that
\[ |G(z)| \leq \frac{M|z|(M|z| + |a_0 - ta_1|R)}{MR + |a_0 - ta_1||z|} \quad \text{for } |z| \leq R. \]

Using this in (24), we get
\[ |F(z)| \geq |ta_0| - \frac{M|z|(M|z| + |a_0 - ta_1|R)}{MR + |a_0 - ta_1||z|} = \frac{|ta_0|MR + (t|a_0| - MR)|a_0 - ta_1||z| - M^2|z|^2}{MR + |a_0 - ta_1||z|} > 0, \]
if
\[ M^2|z|^2 + (MR - t|a_0|)|a_0 - ta_1||z| - t|a_0|MR < 0. \]
This gives \(|F(z)| > 0\), if
\[ |z| < \frac{1}{2M^2} \left[ ((MR - t|a_0|)^2|a_0 - ta_1|^2 + 4t|a_0|M^3R)^{\frac{1}{2}} - (MR - t|a_0|)|a_0 - ta_1| \right] = r. \]
(by (12))
Therefore $F(z)$ does not vanish in $|z| < r$, from which it follows that $f(z)$ does not vanish in $|z| < r$. This completes the proof of Theorem 3.

Proof of Theorem 4. It is clear that $\lim_{j \to \infty} j^t a_j = 0$. Since

$$|a_0| = \left| \sum_{j=1}^{\infty} (a_{j-1} - t a_j) z^{j-1} \right| \leq \max_{|z|=t} \left| \sum_{j=1}^{\infty} (a_{j-1} - t a_j) z^{j-1} \right| = M,$$

say. Therefore $\frac{|a_0|}{M} < 1$, and hence

$$\min \left\{ \frac{t|a_0|}{M}, t \right\} = \frac{t|a_0|}{M}.$$

Using this in (15), with $R = t$, it follows that $f(z)$ does not vanish in

$$|z| < \frac{t|a_0|}{M}.$$ (25)

Now, for $|z| = t$ we have

$$M = \max_{|z|=t} \left| \sum_{j=1}^{\infty} (a_{j-1} - t a_j) z^{j-1} \right| \leq \sum_{j=1}^{\infty} |a_{j-1} - t a_j| t^{j-1}$$

$$= \sum_{j=1}^{k} |t a_j - a_{j-1}| t^{j-1} + \sum_{j=k+1}^{\infty} \left( |a_{j-1} - t a_j| - |a_{j-1} - |a_{j-1}| t^{j-1} \right)$$

$$= 2 t^k |a_k| - |a_0| + \sum_{j=1}^{\infty} \left( |a_{j-1} - t a_j| - |a_{j-1} - |a_{j-1}| t^{j-1} \right)$$

$$\leq 2 t^k |a_k| - |a_0| + 2 \sum_{j=1}^{\infty} |a_j - |a_j| t^j,$$

therefore, it follows from (25), that $f(z)$ does not vanish in

$$|z| < \frac{t|a_0|}{2 t^k |a_k| - |a_0| + 2 \sum_{j=1}^{\infty} |a_j - |a_j| t^j}.$$ (26)

This proves the Theorem 4 completely.

REFERENCES


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