ON GENERALIZED CAUCHY AND PEXIDER FUNCTIONAL EQUATIONS OVER A FIELD

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Abstract. Let \( \mathbb{K} \) be a commutative field and \((\mathbb{P}, +)\) be a uniquely 2-divisible group (not necessarily abelian). We characterize all functions \( T: \mathbb{K} \to \mathbb{P} \) such that the Cauchy difference \( T(s + t) - T(t) - T(s) \) depends only on the product \( st \) for all \( s, t \in \mathbb{K} \). Further, we apply this result to describe solutions of the functional equation \( F(s + t) = K(st) \circ H(s) \circ G(t) \), where the unknown functions \( F, K, H, G \) map the field \( \mathbb{K} \) into some function spaces arranged so that the compositions make sense. Conditions are established under which the equation can be reduced to a corresponding generalized Cauchy equation, and the general solution is given. Finally, we solve the equation \( F(s + t) = K(st) + H(s) + G(t) \) for functions \( F, K, H, G \) mapping \( \mathbb{K} \) into \( \mathbb{P} \). The paper generalizes some results from [11], [13] and, up to some extent, from [2].

1. Introduction

Consider the following functional equation

\[ F(s + t) = K(st) + H(s) + G(t), \]

where the functions \( F, K, H \) and \( G \) to be determined map an algebraic structure (with two binary operations) into another one (with one binary operation). The problem of describing the general solution of (1) (under the additional assumption \( G = H \) in the class of real functions defined on positive reals had been originally raised by Z. Daróczy (see [8]), and was solved by Gy. Maksa in [14]. In [7], J. A. Baker solved (1) for functions \( F, K, H, G \) mapping positive reals into a uniquely 2-divisible abelian group.

B. R. Ebanks, PL. Kannappan and P. K. Sahoo in [11] described all functions \( T, A \) mapping a commutative field \( \mathbb{K} \) into a uniquely 2-divisible abelian group \( \mathbb{P} \) such that

\[ T(s + t) - T(t) - T(s) = A(st), \quad s, t \in \mathbb{K}. \]

With this description, they obtained the general solution of (1) for functions \( F, K, H, G \) mapping the field \( \mathbb{K} \) into the group \( \mathbb{P} \). The general solution of (1), for the case of functions \( F, K, H, G \) mapping the set of reals into itself, can be also found in [13].

In this paper, among other things, we generalize the results from [11] and [13] presenting general solutions of (1) and (2) for functions \( F, K, H, G, T, A \) which
map a commutative field into a uniquely 2-divisible group (not necessarily abelian).
We will consider (1) in an iterative form i.e. for functions \( F, K, H, G \) defined on a
field \( \mathbb{K} \) and taking values in some function spaces, with composition of functions as
a binary operation. Then (1) can be rewritten as follows

\[
F(s + t) = K(st) \circ H(s) \circ G(t), \quad s, t \in \mathbb{K}.
\]

(GPE)

Similarly, (2) takes the form

\[
T(s + t) \circ T(t)^{-1} \circ T(s)^{-1} = A(st), \quad s, t \in \mathbb{K}.
\]

(GCE)

We shall establish conditions allowing to reduce (GPE) to (GCE), and solve (GPE)
for \( F, K, H, G \) mapping \( \mathbb{K} \) into a group of functions. Since any group can be
considered as a group of transformations of a set, the latter result implies a solution
of (1) for functions \( F, K, H, G \) taking values in an arbitrary abstract group.

The iterative Pexider equation \( F(s + t) = H(s) \circ G(t) \), and some its general­
izations, over groupoids have been examined in [2], [3], [4], [5], [6].

Some related results concerning equation (2) can be found in [1], [9], [10], [13].

2. Solutions of equation (2)

In this section, taking inspiration from paper [11], we will give a description of
all functions \( A, T \) satisfying (2), mapping a field into a group (possibly non-abelian).
Throughout the paper, \( \mathbb{K} \) means a commutative field.

Let us start with the following

**Lemma 1.** Let \( P \) be a group and \( D \) be a subgroup of \( P \). Suppose that functions
\( A : \mathbb{K} \to D, T : \mathbb{K} \to P \) satisfy (2). Then the function \( \varphi : \mathbb{K} \to D \) defined by
\( \varphi(t) := A(t) - A(0) \) satisfies the equation

\[
\varphi(-s^2) = \varphi(-s^2 - r) + \varphi(r), \quad s \in \mathbb{K} \setminus \{0\}, \quad r \in \mathbb{K}.
\]

(3)

**Proof.** From equation (2), since \( \mathbb{K} \) is a commutative field, we get

\[
T(t) + T(s) = T(s) + T(t), \quad s, t \in \mathbb{K}.
\]

(4)

Hence, by (2) again,

\[
A(t) + A(s) = A(s) + A(t), \quad s, t \in \mathbb{K}.
\]

(5)

Using (4) it is easy to verify that function \( M : \mathbb{K}^2 \to P \) defined by

\[
M(s, t) := T(s + t) - T(t) - T(s)
\]

satisfies

\[
M(s, t) + M(s + t, w) = M(s, t + w) + M(t, w), \quad s, t, w \in \mathbb{K}.
\]
Consequently, by (2), function $A$ satisfies

$$A(st) + A((s + t)w) = A(s(t + w)) + A(tw), \quad s, t, w \in \mathbb{K}. \quad (6)$$

Now, using substitutions given in [11] (Section 2, p. 252) and our formula (5) we shall reduce equation (6) to (3). Put $t = -s$ in (6). The result is that

$$A(-s^2) + A(0) = A(-s^2 + sw) + A(-sw), \quad s, w \in \mathbb{K}. \quad (7)$$

If $s \neq 0$, then setting $w = -r/s$ in (7), we obtain

$$A(-s^2) + A(0) = A(-s^2 - r) + A(r), \quad s \in \mathbb{K} \setminus \{0\}, \quad r \in \mathbb{K}.$$  

Hence, by (5), we can write

$$A(-s^2) - A(0) = \{A(-s^2 - r) - A(0)\} + \{A(r) - A(0)\},$$

for $s \in \mathbb{K} \setminus \{0\}, \ r \in \mathbb{K}$, which means that $\varphi(t) := A(t) - A(0)$ satisfies equation (3). $\Box$

Recall that a group $P$ is said to have 2-torsion if $2x = 0$ for some element $x \neq 0$ in $P$. Further, a group $P$ is called uniquely 2-divisible if the equation $2x = y$ admits a unique solution $x \in P$ for each $y \in P$.

For any prime $p$, by $\mathbb{F}_p$ we denote the field of $p$ elements.

**LEMMA 2.** Suppose that $\mathbb{K}$ is neither $\mathbb{F}_2$, $\mathbb{F}_3$ nor an infinite field of characteristic 2 and $P$ is a group. A map $\varphi : \mathbb{K} \to P$ satisfies (3) and

$$\varphi(t) + \varphi(s) = \varphi(s) + \varphi(t), \quad s, t \in \mathbb{K}, \quad (8)$$

if and only if $\varphi$ is an additive function.

**LEMMA 3.** Let $\mathbb{K}$ be an infinite field of characteristic 2 and $P$ be a group with no 2-torsion. Then $\varphi : \mathbb{K} \to P$ satisfies (3) and (8) if and only if $\varphi(t) = 0$ for $t \in \mathbb{K}$.

The above two lemmas can be proved in much the same way as Theorem 1 and Lemma 3 in [11], respectively. There is only one difference. Here we assume condition (8) on function $\varphi$, while in [11] the commutativity of group $P$ is supposed. Thus, (8) must be used whenever the commutativity of group $P$ is needed. For the reader convenience, we will sketch the proofs.

**Proof of Lemma 2.** Let $\mathbb{K}$ be a field with at least four elements, and assume that the characteristic of $\mathbb{K}$ is not equal to 2. Fix $s, t \in \mathbb{K} \setminus \{0\}$. Arguing as in [11] (the proof of Lemma 1), we obtain

$$\varphi(-s^2 + t^2) = \varphi(-s^2) - \varphi(-t^2). \quad (9)$$

Now, replacing in (3), $r$ by $-t^2 - r$ for $t \in \mathbb{K} \setminus \{0\}, \ r \in \mathbb{K}$ and using (3), (8) and (9), we get

$$\varphi(-s^2 + t^2 + r) = \varphi(-s^2) - \varphi(-t^2 - r)$$

$$= \varphi(-s^2) - [\varphi(-t^2) - \varphi(r)] = \varphi(-s^2) - \varphi(-t^2) + \varphi(r)$$

$$= \varphi(-s^2 + t^2) + \varphi(r).$$
Hence, in the same way as in [11], we obtain
\[ \varphi(s + t) = \varphi(s) + \varphi(t), \quad s, t \in \mathbb{K}, \quad s \in \mathbb{K} \setminus \{-1, 1\}, \quad t \in \mathbb{K}, \] (10)
which yields \( \varphi(-s) = -\varphi(s) \) for \( s \in \mathbb{K} \setminus \{-1, 1\} \). Further, following [11], we get
\[-\{\varphi(s) + \varphi(-1)\} = -\varphi(s) + \varphi(1) \quad \text{for} \quad s \in \mathbb{K} \setminus \{-1, 0, 1, 2\}.\]
This implies, by (8), \( -\varphi(-1) = \varphi(1) \) i.e., (10) holds for \( s \in \{-1, 1\} \). Thus, \( \varphi \) is an odd function. Since the rest of the proof does not differ essentially from the proof of Theorem 1 in [11], we omit it. □

**Proof of Lemma 3.** Arguing as in the proof of Lemma 3 in [11], we get
\[ \varphi(s^2) - \varphi(t^2) - \varphi(r) = \varphi(s^2) - \{\varphi(t^2) - \varphi(r)\} \]
for \( s, t \in \mathbb{K} \setminus \{0\} \) such that \( s + t \neq 0 \) and \( r \in \mathbb{K} \). Hence, by (8), we have \( 2\varphi(r) = 0 \) for \( r \in \mathbb{K} \), which implies \( \varphi(r) = 0 \) for \( t \in \mathbb{K} \). The converse is evident, and the proof is finished. □

**Remark 1.** (cf. [11], Remark 1) If \( \mathbb{K} \) is a field of characteristic 2 and \( P \) is a group with no 2-torsion, then the map \( \varphi : \mathbb{K} \to P \) is additive if and only if \( \varphi \) is the zero map. In fact, putting \( s = t \) in \( \varphi(s + t) = \varphi(s) + \varphi(t) \), we obtain \( 2\varphi(t) = \varphi(0) = 0 \).

Now, we can give a description of functions \( A, T \) satisfying equation (2).

**Theorem 1.** Assume that \( \mathbb{K} \) is neither \( \mathbb{F}_2 \) nor \( \mathbb{F}_3 \), \( P \) is a group and \( D \) is a uniquely 2-divisible subgroup of \( P \). Then the maps \( A : \mathbb{K} \to D \) and \( T : \mathbb{K} \to P \) satisfy equation (2) if and only if they have the form
\[ A(t) = 2A_1(t) + c, \quad t \in \mathbb{K}, \] (11)
\[ T(t) = A_1(t^2) + A_2(t) - c, \quad t \in \mathbb{K}, \] (12)
where \( A_1, A_2 \) are additive functions mapping \( \mathbb{K} \) into \( D, P \), respectively, satisfying the equation
\[ A_1(s) + A_2(t) = A_2(t) + A_1(s), \quad s, t \in \mathbb{K} \] (13)
and \( c \in D \) is such that
\[ c + A_i(t) = A_i(t) + c \quad \text{for} \quad t \in \mathbb{K}, \quad i \in \{1, 2\}. \] (14)

**Proof.** For the proof of the "if" part, note that by the additivity of functions \( A_1 \) and \( A_2 \) (since \( \mathbb{K} \) is a commutative field), we get
\[ A_i(s) + A_i(t) = A_i(t) + A_i(s), \quad i \in \{1, 2\}, \quad s, t \in \mathbb{K}. \] (15)
Using the additivity of \( A_1, A_2, (13), (14) \) and (15) the rest of that part of the proof is a simple verification.
We proceed to show the "only if" part. Let us begin, as in the proof of Lemma 1, by observing that $T$ satisfies (4). Consequently, by (2), we get (5) and

$$T(s) + A(t) = A(t) + T(s), \quad s, t \in \mathbb{K}. \tag{16}$$

Hence, in particular, we have

$$T(s) + A(0) = A(0) + T(s), \quad s \in \mathbb{K}. \tag{17}$$

Note that, by Lemma 1, $\varphi(t) = A(t) - A(0)$ satisfies (3). Furthermore, it is easy to see by (5), that $\varphi$ also satisfies (8).

Consider first the case when $\mathbb{K}$ is neither $\mathbb{F}_3$ nor a field of characteristic 2. Then, from Lemma 2 we infer that $\varphi : \mathbb{K} \to D$ is an additive function. Following [11] (see the proof of Theorem 2), let $A_1 : \mathbb{K} \to D$ be an additive map such that $2A_1(t) = \varphi(t)$ for $t \in \mathbb{K}$. Then by the definition of $\varphi$, we get (11), where $c = A(0)$.

Combining (5) and (11) we see that

$$-c + A(t) = A(t) - c = 2A_1(t), \quad t \in \mathbb{K}. \tag{18}$$

Computing $A(t)$ from the above equalities and comparing the obtained formulae, we have

$$2A_1(t) + c = c + 2A_1(t), \quad t \in \mathbb{K}. \tag{19}$$

Further, substituting (11) into (16) and using (17) (recall that $c = A(0)$), we get

$$T(s) + 2A_1(t) = 2A_1(t) + T(s), \quad s, t \in \mathbb{K}. \tag{19}$$

Since the characteristic of $\mathbb{K}$ is not equal to 2, by the fact that $2A_1(t) = A_1(2t)$ for $t \in \mathbb{K}$, from (18) and (19) we infer, respectively

$$A_1(s) + c = c + A_1(s), \quad s \in \mathbb{K}, \tag{20}$$

$$T(s) + A_1(t) = A_1(t) + T(s), \quad s, t \in \mathbb{K}. \tag{21}$$

Using (11) we can rewrite equation (2) as follows (cf. [11], the proof of Theorem 2):

$$T(s + t) - T(t) - T(s) = 2A_1(st) + c = A_1(2st) + c$$

$$= A_1((s + t)^2 - s^2 - t^2) + c = \{A_1((s + t)^2) - c\} - \{A_1(s^2) - c\} - \{-c + A_1(t^2)\}$$

for $s, t \in \mathbb{K}$. Now, it is easily seen, by (17), (20) and (21), that the map $A_2 : \mathbb{K} \to P$ defined by

$$A_2(t) := T(t) - A_1(t^2) + c$$

is additive. This provides (12). Finally, it is easy to check that (12), (17), (20) and (21) jointly with the additivity of function $A_1$, imply (13) and (14).

To end the proof, consider the remaining case when $\mathbb{K}$ is a field of characteristic 2 with at least four elements. Then by Lemma 2 and Remark 1 (in the case when $\mathbb{K}$ is a finite field) or by Lemma 3 (if $\mathbb{K}$ is an infinite field) we infer that $\varphi(t) = 0$ for
t ∈ ℜ. Consequently, \( A(t) = A(0) \) for \( t ∈ ℜ \). Setting \( c = A(0) \), we get (11) with \( A_1(t) = 0 \) for \( t ∈ ℜ \).

On account of (2), we obtain
\[
T(s + t) - T(t) - T(s) = c \quad \text{for} \quad s, t ∈ ℜ.
\]
Hence, by (17), we can write
\[
T(s + t) + c = \{T(s) + c\} + \{T(t) + c\}, \quad s, t ∈ ℜ,
\]
which shows that function \( A_2 : ℜ \to P \) defined by
\[
A_2(t) := T(t) + c, \quad t ∈ ℜ
\]
is additive. Note that (12) holds with \( A_1(t) = 0 \) for \( t ∈ ℜ \). It is evident that (13) and (14) are satisfied. □

Remark 2. It is obvious that in case of an abelian group \( P \) conditions (13) and (14) can be dropped. If moreover \( P = D \) then our Theorem 1 reduces to the corresponding result from [11] (Theorem 2) (see also [13], Corollary 1).

The next corollary follows easily from Theorem 1 and will be used later.

Corollary 1. Let \( ℜ, P \) and \( D \) be the same as in Theorem 1. Then an additive function \( A : ℜ \to D \) and \( T : ℜ \to P \) satisfy (2) if and only if they are given by
\[
A(t) = 2A_1(t), \quad t ∈ ℜ,
\]
\[
T(t) = A_1(t^2) + A_2(t), \quad t ∈ ℜ,
\]
where \( A_1, A_2 \) are additive functions mapping \( ℜ \) into \( D, P \), respectively, satisfying (13).

Remark 3. In [11] one can find examples (see Examples 1, 2, 3 and Remark 2) showing the sharpness of the results stated in Lemma 2 and Theorem 1.

3. Solutions of (GPE) and (1)

In the sequel, \( W, X, Y, Z \) stand for arbitrary nonempty sets. By \( \text{In} (X, Y) \) (\( \text{Sur} (X, Y) \), \( \text{Bij} (X, Y) \)) we denote the set of all injections (surjections, bijections) of a set \( X \) into (onto) itself. For simplicity of notation, we write \( \text{In} X, \text{Sur} X, \text{Bij} X \) in the case when \( X = Y \). As usual \( Y^X \) means the set of all functions mapping \( X \) into \( Y \). \( \text{Ran} f \) denotes the range of the function \( f \) and \( \text{id} X \) stands for the identity function on the set \( X \).

By abuse of language, a function \( \varphi \) mapping a field \( ℜ \) into a group of functions \( (P, \circ) \) (with composition of functions as the group operation) satisfying
\[
\varphi(s + t) = \varphi(s) \circ \varphi(t), \quad s, t ∈ ℜ
\]
is said to be additive.
We now can formulate and prove our results concerning the generalized Pexider equation (GPE), which state conditions allowing to reduce (GPE) to (GCE).

**Theorem 2.** Suppose that the functions $F$, $K$, $H$, $G$ mapping $\mathbb{K}$ into $Z^W$, $Z^X$, $Bij (X, Y)$, $X^W$, resp., satisfy (GPE). If $K(0) \in In (Y, Z)$ and $G(0) \in Sur (W, X)$ then there exist functions $a \in Sur (W, Z_0)$, $b \in Bij (X, Z_0)$, $c \in Bij (Y, Z_0)$, $A, T : \mathbb{K} \rightarrow Bij Z_0$, where $Z_0 := \text{Ran } K(0)$, such that (GCE) holds, $A$ satisfies

$$A(-s^2) = A(-s^2 - r) \circ A(r) \quad \text{for } s \in \mathbb{K} \setminus \{0\}, \quad r \in \mathbb{K}$$

and

$$\begin{align*}
F(t) &= T(t) \circ a, \\
K(t) &= A(t) \circ c, \\
H(t) &= c^{-1} \circ T(t) \circ b, \\
G(t) &= b^{-1} \circ T(t) \circ a, \\
\end{align*}$$

(23)

Moreover, if $\mathbb{K}$ is neither $\mathbb{F}_2$, $\mathbb{F}_3$ nor an infinite field of characteristic 2 then $A : \mathbb{K} \rightarrow (Bij Z_0, \circ)$ is an additive function.

Conversely, if $a \in Z^W_0$, $b \in Bij (X, Z_0)$, $c \in Bij (Y, Z_0)$ for a nonempty set $Z_0 \subset Z$ and $A, T : \mathbb{K} \rightarrow Bij Z_0$ satisfy (GCE), then the functions $F$, $K$, $H$, $G$ given by (23) satisfy (GPE).

**Proof.** To simplify the notation, we put $F_t := F(t)$, $K_t := K(t)$, $H_t := H(t)$, $G_t := G(t)$ for $t \in \mathbb{K}$. Assume that the functions $F, K, H, G$ such that $H_t \in Bij (X, Y)$, $t \in \mathbb{K}$, $K_0 \in In (Y, Z)$ and $G_0 \in Sur (W, X)$ satisfy (GPE). Setting alternately $t = 0$ and $s = 0$ into (GPE), we get respectively

$$\begin{align*}
F_s &= K_0 \circ H_s \circ G_0, \\
F_t &= K_0 \circ H_0 \circ G_t, \\
\end{align*}$$

(24)

(25)

Comparing the right hand sides of (24) and (25) for $s = t$, by the injectivity of $K_0$, we obtain

$$H_t \circ G_0 = H_0 \circ G_t, \quad t \in \mathbb{K}.$$  

Hence

$$G_t = H_0^{-1} \circ H_t \circ G_0, \quad t \in \mathbb{K}.$$  

(26)

On the other side, by (25), we have

$$G_t = H_0^{-1} \circ K_0^{-1} \circ F_t, \quad t \in \mathbb{K}.$$  

(27)

Introduce on $W$ an equivalence relation "$\equiv$" putting

$$w \equiv v \quad \text{if and only if } \quad G_0(w) = G_0(v)$$

Let $g$ be an invertible mapping such that $g([w]) \in [w]$ for $w \in W$, where $[w]$ stands for the equivalence class containing $w$. Set

$$\widetilde{G}_0 := G_0 \circ g \quad \text{and} \quad \widetilde{F}_t := F_t \circ g, \quad t \in \mathbb{K}.$$
It is evident that $\widetilde{G}_0$ is a bijection of the factor set $W/\varrho$ of $W$ modulo $\varrho$ onto $X$. Further, since $H_t \in \text{In}(X, Y)$ for $t \in \mathbb{K}$ and $K_0 \in \text{In}(Y, Z)$, by (24), we get
\[ F_t(w) = F_t(v) \iff G_0(w) = G_0(v) \]
for every $t \in \mathbb{K}$, $v, w \in W$. Thus $\widetilde{F}_t \in \text{In}(W/\varrho, Z)$ for $t \in \mathbb{K}$. Now, in view of (24), we can write
\[ \widetilde{F}_t = K_0 \circ H_t \circ \widetilde{G}_0, \quad t \in \mathbb{K}, \]
which implies
\[ H_t = K_0^{-1} \circ \widetilde{F}_t \circ \widetilde{G}_0^{-1}, \quad t \in \mathbb{K}. \tag{28} \]
Substituting (28) and (27) into (GPE), we obtain
\[ F_{s+t} = K_{st} \circ K_0^{-1} \circ \widetilde{F}_s \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} \circ F_t, \quad s, t \in \mathbb{K}. \]
Hence
\[ \widetilde{F}_{s+t} = K_{st} \circ K_0^{-1} \circ \widetilde{F}_s \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} \circ \widetilde{F}_t, \quad s, t \in \mathbb{K}. \tag{29} \]
Observe that, by (24), since $H_t \in \text{Sur}(X, Y)$ for $t \in \mathbb{K}$ and $G_0 \in \text{Sur}(W, X)$, we get
\[ \text{Ran } F_t = \text{Ran } K_0 = Z_0, \quad t \in \mathbb{K} \]
and consequently, $\widetilde{F}_t \in \text{Bij}(W/\varrho, Z_0)$ for $t \in \mathbb{K}$. Define $T : \mathbb{K} \to \text{Bij}Z_0$, by
\[ T_t := \widetilde{F}_t \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1}, \quad t \in \mathbb{K}. \tag{30} \]
Using (30) and (29) we can write
\[
T_{s+t} = \widetilde{F}_{s+t} \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} = K_{st} \circ K_0^{-1} \circ \widetilde{F}_s \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} \\
\quad \circ \widetilde{F}_t \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} = K_{st} \circ K_0^{-1} \circ T_s \circ T_t.
\]
Thus $T$ and $K$ satisfy the following functional equation
\[ T_{s+t} = K_{st} \circ K_0^{-1} \circ T_s \circ T_t, \quad s, t \in \mathbb{K}. \tag{31} \]
Define $A : \mathbb{K} \to Z^{Z_0}$ setting
\[ A_t := K_t \circ K_0^{-1}, \quad t \in \mathbb{K}. \tag{32} \]
Then from (31), we get
\[ T_{s+t} = A_{st} \circ T_s \circ T_t, \quad s, t \in \mathbb{K}, \tag{33} \]
i.e. $A$ and $T$ satisfy (GCE). It is clear that $A_t \in \text{Bij}Z_0$ for every $t \in \mathbb{K}$. Now, since from (32) we get $A_0 = id_{Z_0}$, by Lemma 1 (for $P = D = (\text{Bij}Z_0, o)$) we conclude that $A$ satisfies (22).

We are going to show that formulae (23) hold. On account of (24) and (30), we have
\[ H_t = K_0^{-1} \circ T_t \circ K_0 \circ H_0, \quad t \in \mathbb{K}. \tag{34} \]
Putting (34) into (26), we get
\[ G_t = H_0^{-1} \circ K_0^{-1} \circ T_t \circ K_0 \circ H_0 \circ G_0, \quad t \in \mathbb{K}. \]  
(35)

Next, substitution of (35) into (25) gives
\[ F_t = T_t \circ K_0 \circ H_0 \circ G_0, \quad t \in \mathbb{K}. \]  
(36)

In view of (32), we obtain
\[ K_t = A_t \circ K_0, \quad t \in \mathbb{K}. \]  
(37)

Finally, setting
\[ a := K_0 \circ H_0 \circ G_0, \quad b := K_0 \circ H_0, \quad c := K_0 \]
it is easily seen, that \( a \in \text{Sur}(W, Z_0), b \in \text{Bij}(X, Z_0), c \in \text{Bij}(Y, Z_0), \) and from (34), (35), (36), (37) (switching to the previous notation) we get formulae (23).

To finish the first part of the proof it remains to show that if \( \mathbb{K} \) is neither \( \mathbb{F}_2, \mathbb{F}_3 \) nor an infinite field of characteristic 2, then \( A \) is an additive function. To see this, observe that from (33) we get,
\[ T_s \circ T_t = T_t \circ T_s, \quad s, t \in \mathbb{K}. \]
(since, the commutativity of \( \mathbb{K} \) yields \( T_{s+t} = T_{t+s} \) and \( A_{st} = A_{ts} \)). Consequently, by (33) again, we obtain
\[ A_s \circ A_t = A_t \circ A_s, \quad s, t \in \mathbb{K}. \]

Applying Lemma 2 we get the additivity of function \( A \).

The converse statement is easy to check. \( \Box \)

From the above proof one can easily infer the following

**COROLLARY 2.** Let \( R \) be a ring with unity. Suppose that functions \( F, K, H, G \) mapping \( R \) into \( W^W, Z^Z, \text{Bij}(X, Y), X^W, \) respectively, satisfy (GPE). If \( K(0) \in \text{In}(Y, Z) \) and \( G(0) \in \text{Sur}(W, X) \) then there exist functions \( a \in \text{Sur}(W, Z_0), b \in \text{Bij}(X, Z_0), c \in \text{Bij}(Y, Z_0), A, T : \mathbb{K} \rightarrow \text{Bij}Z_0 (Z_0 = \text{Ran}K(0)) \), satisfying (GCE) and such that (23) holds.

Conversely, if \( a \in Z_0^W \), \( b \in \text{Bij}(X, Z_0), c \in \text{Bij}(Y, Z_0) \) for a nonempty set \( Z_0 \subset Z \) and \( A, T : R \rightarrow \text{Bij}Z_0 \) satisfy (GCE), then the functions \( F, K, H, G \) given by (23) satisfy (GPE).

**Remark 4.** Observe that in the case when \( Y = Z \) and \( K(t) = id_Y \) for \( t \in R \) (\( R \) means a ring with unity) then (GPE) takes the form of the well-known Pexider equation which, by the above corollary, can be reduced to the Cauchy equation. In fact, we have \( A(t) = id_Y \) for \( t \in R \) (cf. (32)) and (GCE) reduces to the Cauchy equation. This particular result has been obtained in a more general situation in [2] (see Theorem 1), where the Pexider equation on a groupoid has been examined.

Theorem 2 jointly with Lemma 3 and Corollary 1 allow us to prove the following theorem.
THEOREM 3. Suppose that $K$ is neither $F_2$ nor $F_3$ and $D$ is a uniquely 2-divisible subgroup of the group $(\text{Bij} Y, \circ)$. Then functions $F, K, H, G$ mapping $K$ into $Y^W, D, \text{Bij} (X, Y), X^W$, respectively, such that $G(0) \in \text{Sur} (W, X)$ satisfy (GPE) if and only if they are given by

\[
\begin{align*}
F(t) &= A_1(t^2) \circ A_2(t) \circ a, \\
K(t) &= A_1^2(t) \circ c, \\
H(t) &= c^{-1} \circ A_1(t^2) \circ A_2(t) \circ b, \\
G(t) &= b^{-1} \circ A_1(t^2) \circ A_2(t) \circ a, \quad t \in K
\end{align*}
\] (38)

where $A_1 : K \to D$, $A_2 : K \to (\text{Bij} Y, \circ)$ are additive functions satisfying

\[A_1(s) \circ A_2(t) = A_2(t) \circ A_1(s), \quad s, t \in K,\]

and $a \in \text{Sur} (W, Y)$, $b \in \text{Bij} (X, Y)$, $c \in \text{Bij} Y$ are constants.

Proof. Suppose that functions $F, K, H, G$ mapping $K$ into $Y^W, D, \text{Bij} (X, Y), X^W$, respectively, such that $G(0) \in \text{Sur} (W, X)$ satisfy (GPE). Then, in view of Theorem 2, there exist $a \in \text{Sur} (W, Y), b \in \text{Bij} (X, Y), c \in \text{Bij} Y$ (note that $Z_0 = \text{Ran} K(0) = Y$ in this case), $A : K \to D$ (cf. (32)), $T : K \to \text{Bij} Y$ such that (GCE) and (23) hold. Moreover, by Theorem 2 or Lemma 3 (in case of an infinite field of characteristic 2), we infer that $A$ is an additive function. Applying Corollary 1, for $P = (\text{Bij} Y, \circ)$, we get from (23) the required formulae (38).

The converse is straightforward. \qed

From Theorem 3, we obtain the general solution of (1). Namely, the subsequent result holds.

COROLLARY 3. Assume that $K$ is neither $F_2$ nor $F_3$ and $(D, +)$ is a uniquely 2-divisible group. Then the maps $F, K, H, G : K \to D$ satisfy (1) if and only if they are given by

\[
\begin{align*}
F(t) &= A_1(t^2) + A_2(t) + a, \\
K(t) &= 2A_1(t) + c, \\
H(t) &= -c + A_1(t^2) + A_2(t) + b, \\
G(t) &= -b + A_1(t^2) + A_2(t) + a, \quad t \in K
\end{align*}
\] (39)

where $A_1, A_2 : K \to D$ are additive functions satisfying (13) and $a, b, c \in D$ are constants.

Proof. By the well-known Cayley theorem (see e.g. [12]) any group can be considered as a group of bijections of a set. Moreover, since $D$ is a uniquely 2-divisible group, the corresponding group of transformations (the group of left (right) translations of $D$) share the property as well. Thus, there is no loss of generality assuming that $D$ is a uniquely 2-divisible group of bijections of a set onto itself. Now, Theorem 3 (switching to additive notation) yields the statement. \qed

Remark 5. Corollary 3 generalizes corresponding results from [11] and [13] (Corollary 2 and Theorem 1, respectively), which actually inspired the paper.
Namely, it is easily seen that in the case when $D$ is an abelian group, the general solution of (1), given by (39), reduces to this obtained in [11] and [13]. However, the methods of solution of (1) presented in those papers fail in the case when functions $F, K, H, G$ take values in a non-abelian group.

REFERENCES


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