CURVES IN \(n\)-DIMENSIONAL \(k\)-ISOTROPIC SPACE

Željka Milin Šipuš, Zagreb, Croatia, and Blaženka Divjak, Varaždin, Croatia

Abstract. In this paper we develop the theory of curves in \(n\)-dimensional \(k\)-isotropic space \(I^n_k\). We derive explicit expressions and geometrical interpretations for the curvatures of a curve.

1. Introduction

The \(n\)-dimensional \(k\)-isotropic space \(I^n_k\) was introduced by H. Vogler and H. Wresnig in [17]. We follow the notations and the terminology used in that paper. The special cases of \(I^2, I^3, I^4\) were thoroughly studied in [2], [3], [4], [9], [10] [12], [13], [14], [15], [16]. The case of \(I^n_1\) was introduced in [11], and studied in [1] and [5]. The theory of curves in \(n\)-dimensional flag space \(I^{n-1}\) was studied in [7] and in [8]. A general approach to the theory of curves in Cayley/Klein spaces is given in [6].

In this paper we develop the theory of curves in \(I^n_k\). We construct the Frenet frame of an admissible curve and calculate the explicit expressions of the curvatures of such a curve. We derive also the geometrical interpretation of these curvatures and investigate the curves having some of their curvatures equal to zero. Finally we describe the conditions, in terms of curvatures, if a curve lies in an \(l\)-isotropic \(m\)-plane.

Let \(A\) denote an \(n\)-dimensional affine space and \(V\) its corresponding vector space. The space \(V\) is decomposed in a direct sum

\[
V = U_1 \oplus U_2
\]

such that \(\dim U_2 = k\), \(\dim U_1 = n - k\). Let \(B_2 = \{b_{n-k+1}, \ldots, b_n\}\) be a basis for the subspace \(U_2\). In \(U_2\) a flag of vector spaces \(U_2 := C_1 \supset \ldots \supset C_l \supset C_{l+1} \supset \ldots \supset C_k := [b_n], C_l = [b_{n-k+l}, \ldots, b_n]\) is defined. According to it we distinguish the following classes of vectors: the Euclidean vectors as the vectors in \(V \setminus U_2\) and the isotropic vectors of degree \(l\) or \(l\)-isotropic vectors, \(l = 1, \ldots, k\), as the vectors in \(U_2\), \(x = \sum_{m=1}^{k} x_{n-k+m} b_{n-k+m}\), for which holds

\[
x_{n-k+1} = \ldots = x_{n-k+l-1} = 0, x_{n-k+l} \neq 0.
\]


Key words and phrases: \(n\)-dimensional \(k\)-isotropic space, curve, curvature of a curve.
By $\pi_i : V \to U_i$, $i = 1, 2$, we denote the canonical projections. The scalar product $\cdot : U_1 \times U_1 \to \mathbb{R}$ is extended in the following way on the whole $V$ by

$$x \cdot y = \pi_1(x) \cdot \pi_1(y).$$

(2)

Therefore the isotropic vectors are orthogonal (scalar product vanishes) to all other vectors, especially also to themselves.

For $x \in V$ we define its isotropic length by $||x|| := ||\pi_1(x)||$. But if $x$ is an $l$-isotropic vector, then its isotropic length is 0, and therefore we introduce as isotropic length the $l$th-range of $x$, i.e. $[x]_l := x_{n-k+l}$, $l = 1, \ldots, k$.

The group of motions of $I^k_n$ is given by the matrix

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix},$$

(3)

where $A$ is an orthogonal $(n - k, n - k)$-matrix, $\det A = 1$, $B$ a real $(k, n - k)$-matrix and $C$ a real lower triangular $(k, k)$-matrix such that $c_{n-k+l}^{n-k+l} = 1$.

2. Hyperplanes in $I^k_n$

We distinguish the following classes of hyperplanes in $I^k_n$. We say that a hyperplane in $I^k_n$ given by an equation

$$u_0 + u_1x_1 + \ldots + u_nx_n = 0$$

is of type $l$ or $l$-isotropic, $l = 0, \ldots, k$, if $u_{n-l} \neq 0$ and $u_{n-l+1} = \ldots = u_n = 0$. Especially, for $l = 0$ we say that a hyperplane is non-isotropic and for $l = k$ that it is completely isotropic.

Proposition 1. Let $H$ be an $l$-isotropic hyperplane, $l = 0, \ldots, k - 1$. Then there are no $(k-l)$-isotropic vectors in $H$. Furthermore, there exists a basis consisting of $n - k$ Euclidean vectors and of one of $m$-isotropic vectors, $m = 1, \ldots, k$, $m \neq k - l$, but also a basis consisting of $n - l - 1$ Euclidean vectors and of one of $m$-isotropic vectors, $m = k - l + 1, \ldots, k$.

In every basis of $H$ the number of Euclidean vectors varies from $n - k$ to $n - l - 1$; there are at most $k - m$ $m$-isotropic vectors, if $m \leq k - l - 1$, and at most $k - m + 1$ $m$-isotropic vectors, if $m \geq k - l + 1$.

Proof. Let $H$ be an $l$-isotropic hyperplane given by

$$u_0 + u_1x_1 + \ldots + u_{n-l}x_{n-l} = 0, \quad u_{n-l} \neq 0.$$
Then its equation can be written in the following form

\[\begin{vmatrix}
X_1 & \ldots & X_{n-k} & \ldots & X_{n-l-1} & X_{n-l} + \frac{\mu_0}{u_{n-l}} & X_{n-l+1} & \ldots & X_n \\
u_{n-l} & \ldots & 0 & \ldots & 0 & -u_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & u_{n-l} & \ldots & 0 & -u_{n-k} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots & u_{n-l} & -u_{n-l-1} & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{vmatrix} = 0. \tag{4}\]

From (4) it can be seen that there are no \((k - l)\)-isotropic vectors in an \(l\)-isotropic hyperplane, \(l = 0, \ldots, k - 1\). Furthermore, it can also be seen that there exist the mentioned bases for \(H\); the first follows directly from (4), the others by making linear combinations of the vectors of the first mentioned basis.

**Corollary 1.** In a non-isotropic hyperplane there are no \(k\)-isotropic vectors. Furthermore, there exists a basis consisting of \(n - 1\) Euclidean vectors, but also a basis consisting of \(n - k\) Euclidean vectors and of one of \(m\)-isotropic vectors, \(m = 1, \ldots, k - 1\). In every basis the number of Euclidean vectors varies from \(n - k\) to \(n - 1\), there are at most \(k - m\) \(m\)-isotropic vectors, \(m = 1, \ldots, k - 1\).

**Corollary 2.** In a completely isotropic hyperplane exist all \(m\)-isotropic directions, \(m = 1, \ldots, k\). There exists a basis consisting of \(n - k - 1\) Euclidean vectors and of one of \(m\)-isotropic vectors, \(m = 1, \ldots, k\). Generally, every basis consists of \(n - k - 1\) Euclidean vectors, and of at most \(k - m + 1\) \(m\)-isotropic vectors, \(m = 1, \ldots, k\).

### 3. Curves in \(I_n^k\)

**Definition 1.** Let \(I \subseteq \mathbb{R}\) be an open interval and \(\varphi : I \to I_n^k\) a vector function given in affine coordinates by

\[\vec{O}X(t) = (x_1(t), \ldots, x_n(t)) := x(t),\]

where \(\varphi(t) = X\) is a point in \(A\).

The set of points \(c \in I_n^k\) is called a \(C^r\)-curve, \(r \geqslant 1\), if there is an open interval \(I \subseteq \mathbb{R}\) and a \(C^r\)-mapping \(\varphi : I \to I_n^k\) such that \(\varphi(t) = c\).

A \(C^r\)-curve is regular if \(\dot{x}(t) \neq 0\), \(t \in I\).
A $C^r$-curve is simple if it is regular and $\varphi$ is injective.

One can easily see that the notions of $C^r$-curve, regular $C^r$-curve and simple $C^r$-curve are invariant under the group of motions of $I^k_n$.

**Definition 2.** A point $P_0(t_0)$ of a regular $C^n$-curve is called an inflection point of order $l$, $l = 2, \ldots, n-1$, if the set of vectors

$$\{\dot{x}(t_0), \ldots, x^{(l-1)}(t_0)\}$$

is linearly independent and the set of vectors

$$\{\ddot{x}(t_0), \ldots, x^{(l)}(t_0)\}$$

is linearly dependent.

If a curve has no inflection points of any order $l$, $l = 2, \ldots, n-1$, it is said to be non-degenerated.

The notion of an inflection point of order $l$ is a geometrical notion i.e. it does not depend on parametrization and is invariant under the group of motions. Moreover, it is a differential invariant of order $l$.

**4. Osculating planes**

**Definition 3.** Let $c$ be a simple $C^r$-curve given by $x = x(t)$ and $P(t) \in c$ an inflection point of order $r$. The osculating $m$-plane, $m = 1, \ldots, r-1$, at the point $P$ is $m$-dimensional plane in $I^k_n$ spanned by the vectors $\dot{x}(t), \ldots, x^{(m)}(t)$.

If $c$ is a non-degenerated simple $C^n$-curve, then the osculating hyperplane of $c$ at $P(t)$ is the hyperplane spanned by $\dot{x}(t), \ldots, x^{(n-1)}(t)$. Its equation is given by

$$\det(x - x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)) = 0, \quad (5)$$

where $x$ denotes a position vector of an arbitrary point of the osculating hyperplane.

**Proposition 2.** Let $c : I \rightarrow I^k_n$ be a simple $C^{l+1}$-curve on which all of the points are inflection points of order $l + 1$, $l = 1, \ldots, n-1$. Then there exists an $l$-plane which contains the curve $c$.

**Definition 4.** A curve $c$ is said to be an admissible $C^r$-curve, $r \geq n-1$, if $\pi_1(c)$ is non-degenerated and $c$ is a simple, non-degenerated $C^r$-curve without $l$-isotropic osculating hyperplanes, $l = 1, \ldots, k$.

**Theorem 1.** A $C^r$-curve $c$, $r \geq n-1$, is admissible if and only if

$$\begin{vmatrix}
\dot{x}_1(t) & \ldots & \dot{x}_{n-1}(t) \\
\vdots & \vdots & \vdots \\
x^{(n-1)}_1(t) & \ldots & x^{(n-1)}_{n-1}(t)
\end{vmatrix} \neq 0, \quad t \in I, \quad (6)$$
An admissible curve has neither $l$-isotropic tangents nor $l$-isotropic osculating $m$-planes, $l = 1, \ldots, k$, $m = 2, \ldots, n - 1$.

Proof. If $c$ is admissible, then the statement obviously holds. Conversely, if (6) holds, then $c$ is non-degenerated. Furthermore $c$ is regular because otherwise it would be $\mathbf{x}(t) = 0$, $t \in I$, and so the first row of the determinant (6) would consist of zeros. If $c$ has $l$-isotropic tangents, then the first row of the determinant (7) would be zero. In every $l$-isotropic $m$-plane, $l = 1, \ldots, k$, there is $k$-isotropic direction. Therefore if $c$ has osculating $l$-isotropic $m$-plane, (6) would be zero.

5. Frenet’s equations of a curve in $I^k_n$

Definition 5. Let $c : [a, b] \rightarrow I^k_n$ be an admissible curve. Then

$$s := \int_a^b ||\dot{\mathbf{x}}||dt = \int_a^b |\pi_1(\mathbf{x})|dt$$

is called the isotropic arc length of the curve $c$ from $\mathbf{x}(a)$ to $\mathbf{x}(b)$.

One can notice that the isotropic arc length of an admissible curve $c$ coincides with the Euclidean arc length of the projection $\pi_1(c)$ to the basic space.

Proposition 3. Every admissible $C^l$-curve $c$ can be reparametrized by the arc length $s$ and $s$ is the arc length on $c$ exactly when $||\dot{\mathbf{x}}(s)|| = 1$.

Let $c : I \rightarrow I^k_n$ be a curve parametrized by the arc length. Notice that $c$ is also admissible. Now we can define the $n$-frame $\{\mathbf{t}_1(s), \ldots, \mathbf{t}_n(s)\}$ of a curve $c$ in a point $\mathbf{x}(s)$. It should be an orthonormal basis of $V$ like it is defined in [17].

By applying the Gram-Schmidt orthogonalization process to the set

$$\{\mathbf{x}', \ldots, \mathbf{x}^{(n-k)}\}$$

we get the orthonormal set of vectors $\{\mathbf{t}_1, \ldots, \mathbf{t}_{n-k}\}$

$$\mathbf{t}_1 := \mathbf{x}'$$

$$\mathbf{b}_m := \mathbf{x}^{(m)} - \sum_{i=1}^{m-1} (\mathbf{x}^{(m)} \cdot \mathbf{t}_i) \mathbf{t}_i$$

$$\mathbf{t}_m := \frac{\mathbf{b}_m}{||\mathbf{b}_m||}, \quad m = 2, \ldots, n - k.$$

One can see that the frame $\{\pi_1(\mathbf{t}_1), \ldots, \pi_1(\mathbf{t}_{n-k})\}$ is the Frenet $(n - k)$-frame of the curve $\pi_1(c)$.

If we put $\tilde{U}_1 = [\mathbf{t}_1, \ldots, \mathbf{t}_{n-k}]$, then $\tilde{U}_1 \cap U_2 = \{0\}$, and therefore we have the following decomposition $V = \tilde{U}_1 \oplus U_2$. Now we should define the basis of $U_2$...
consisting of one unit 1-isotropic vector, \ldots, one unit \( k \)-isotropic vector. Let us suppose that \( x_{n-k+1}^{(n-k+1)}(s) \neq 0 \). If \( x_{n-k+1}^{(n-k+1)}(s) = 0 \) then there must exist some other coordinate \( x_{n-k+i} \) such that \( x_{n-k+i}^{(n-k+1)}(s) \neq 0 \) and we can form the vector \( t_{n-k+1} \) by it. Now we define

\[
t_{n-k+1} := (0, \ldots, 0, 1, \frac{x_{n-k+2}}{x_{n-k+1}}, \ldots, \frac{x_{n-k+1}}{x_{n-k+1}}).
\]

Obviously \( t_{n-k+1} \) is an unit 1-isotropic vector.

Let us also define

\[
\kappa_{n-k+1}(s) = \left( \frac{x_{n-k+2}}{x_{n-k+1}}, \ldots, \frac{x_{n-k+1}}{x_{n-k+1}} \right).
\]

If \( \kappa_{n-k+1}(s) \neq 0 \), we can put

\[
t_{n-k+2} := (0, \ldots, 0, 0, 1, \frac{x_{n-k+3}}{x_{n-k+1}}, \ldots, \frac{x_{n-k+1}}{x_{n-k+1}}),
\]

which is an unit 2-isotropic vector. Now we introduce

\[
\kappa_{n-k+2}(s) = \left( \frac{x_{n-k+1}}{x_{n-k+1}}, \ldots, \frac{x_{n-k+1}}{x_{n-k+1}} \right).
\]

Continuing the process, under the assumptions \( \kappa_{n-k+j}(s) \neq 0, \ldots, \kappa_{n-k+j}(s) \neq 0 \), we define the \((j + 1)\)-isotropic vector

\[
t_{n-k+j+1} = (0, \ldots, 0, 1, \left( x_{n-k+j+2}^{(n-k+1)} : x_{n-k+1}^{(n-k+1)} \right)' : \kappa_{n-k+1})', \ldots, \kappa_{n-k+j}, \ldots,
\]

and

\[
\kappa_{n-k+j+1} = \left( \left( x_{n-k+j+2}^{(n-k+1)} : x_{n-k+1}^{(n-k+1)} \right)' : \kappa_{n-k+1} \right)' : \kappa_{n-k+2}, \ldots, \kappa_{n-k+j} \right)'
\]

\[j = 1, \ldots, k - 2.\]

The last vector is equal to

\[t_n = (0, \ldots, 0, 1).\]

Obviously the following theorem is true.
THEOREM 2. (Frenet's Equations)
Let \( c \) be an admissible curve in \( I_n^k \) parametrized by the arc length and let \( \{ t_1, \ldots, t_n \} \) be its Frenet n-frame. Then there exist functions \( \kappa_1, \ldots, \kappa_{n-1} : I \to \mathbb{R} \) such that the following equations hold

\[
\begin{align*}
t_1' &= \kappa_1 t_2, \\
t_i' &= -\kappa_{i-1} t_{i-1} + \kappa_i t_{i+1}, \quad i = 2, \ldots, n - k, \\
t_{n-k+j}' &= \kappa_{n-k+j} t_{n-k+j+1}, \quad j = 1, \ldots, k - 1, \\
t_n' &= 0.
\end{align*}
\]

6. Explicit expressions of the curvatures of a curve in \( I_n^k \)

Let us derive now the explicit expressions of the curvatures of an admissible curve \( c \) parametrized by its arc length. Since \( \kappa_i, i = 1, \ldots, n - k - 1 \), are the curvatures of the projection \( \pi_1(c) \) of the curve \( c \), we have

\[
\kappa_i^2(s) = \frac{\Gamma(x', \ldots, x^{(i-1)}) \Gamma(x', \ldots, x^{(i+1)})}{\Gamma^2(x', \ldots, x^{(i)})}, \quad i = 1, \ldots, n - k - 1,
\]

where \( \Gamma \) denotes Gram's determinant with a scalar product defined in (2).

The expressions for the curvatures \( \kappa_{n-k+1}, \ldots, \kappa_{n-1} \) are given by the construction of the Frenet frame in the previous section. We can obtain the explicit expression for the curvature \( \kappa_{n-k} \) in the following way. Using Frenet's equations we get

\[
\begin{align*}
x' &= t_1, \\
x^{(i)} &= a_{i1} t_1 + \ldots + a_{i-1} t_{i-1} + \kappa_1 \cdots \kappa_{i-1} t_i, \quad i = 2, \ldots, n.
\end{align*}
\]

Therefore it holds

\[
\begin{align*}
det(x', \ldots, x^{(n)}) &= \kappa_1^{n-1} \cdots \kappa_{n-1} \\
det(\pi_1(x'), \ldots, \pi_1(x^{(n-k)})) &= \kappa_1^{n-k-1} \cdots \kappa_{n-k-1}.
\end{align*}
\]

Now we have

\[
\kappa_{n-k}^k = \frac{det(x', \ldots, x^{(n)})}{det(\pi_1(x'), \ldots, \pi_1(x^{(n-k)}))^{k+1} \kappa_{n-k+1} \cdots \kappa_{n-1}}.
\]

By substituting the expressions for \( \kappa_i, i = 1, \ldots, n - k - 1 \), and by noticing that

\[
det(\pi_2(x^{(n-k+1)}), \ldots, \pi_2(x^{(n)})) = (x_{n-k+1}^{(n-k+1)})^k \kappa_{n-k+1} \cdots \kappa_{n-1}
\]

we get the following expression for \( \kappa_{n-k} \)

\[
\kappa_{n-k}^k = \frac{det(x', \ldots, x^{(n)}) \Gamma(x', \ldots, x^{(n-k-1)}) / 2 (x_{n-k+1}^{(n-k+1)})^k}{det(\pi_1(x'), \ldots, \pi_1(x^{(n-k)}))^{k+1} det(\pi_2(x^{(n-k+1)}), \ldots, \pi_2(x^{(n)}))}.
\]
Let us notice that for the curvatures \( \kappa_{n-k}, \ldots, \kappa_{n-1} \) we can also derive the following explicit expressions. It is easy to show that

\[
\begin{vmatrix}
x_1' & \cdots & x_{n-k+j}' \\
\vdots & \ddots & \vdots \\
x_1^{(n-k+j)} & \cdots & x_{n-k+j}^{(n-k+j)}
\end{vmatrix} = \kappa_1^{n-k+j-1} \cdots \kappa_{n-k+j-1}
\]

(9)

\[
j = 2, \ldots, k - 1,
\]

holds. By using (9) and by considering that

\[
(\kappa_1 \cdots \kappa_{n-k-1})^2 = \frac{\Gamma(x', \ldots, x^{(n-k)})}{\Gamma(x', \ldots, x^{(n-k-1)})}
\]

we get

\[
\kappa_{n-k}^2 = \frac{\frac{\Gamma(x', \ldots, x^{(n-k-1)})}{\Gamma(x', \ldots, x^{(n-k)})}}{\Gamma^2(x', \ldots, x^{(n-k)})}
\]

(10)

and by induction

\[
\kappa_{n-k+j} = \frac{\frac{\Gamma(x', \ldots, x^{(n-k+j-1)})}{\Gamma(x', \ldots, x^{(n-k+j)})}}{\Gamma^2(x', \ldots, x^{(n-k+j)})}
\]

(11)

\[
j = 1, \ldots, k - 1.
\]

Let us now suppose that \( V \) is endowed with a scalar product \( \cdot : V \times V \to \mathbb{R} \) such that its restriction to \( U_1 \) coincides with the already defined scalar product \( \cdot : U_1 \times U_1 \to \mathbb{R} \). We shall use the same notation for the scalar product on \( V \) as for the degenerated scalar product defined in (2). Let us also introduce the following notation. Let \( \Gamma_{n-k+i}(y_1, \ldots, y_m) \), \( i = 1, \ldots, k \), denote the Gram's determinant of the projections of the vectors \( y_1, \ldots, y_m \) onto the \( (n-k+i) \)-dimensional subspace of \( V \) spanned by the first \( n-k+i \) coordinate vectors and \( \Gamma_{n-k}(y_1, \ldots, y_m) = \Gamma(y_1, \ldots, y_m) \). Then the expression (10) can be written as

\[
\kappa_{n-k}^2 = \frac{\Gamma_{n-k+1}(x', \ldots, x^{(n-k+1)})\Gamma(x', \ldots, x^{(n-k-1)})}{\Gamma^2(x', \ldots, x^{(n-k)})}
\]

(12)
and the expressions (11) as

$$
\kappa_{n-k+j}^2 = \frac{\Gamma_{n-k+j+1}(x', \ldots, x^{(n-k+j+1)}) \Gamma_{n-k+j-1}(x', \ldots, x^{(n-k+j-1)})}{\Gamma_{n-k+j}(x', \ldots, x^{(n-k+j)})},
$$

(13)

$$
j = 1, \ldots, k - 1.
$$

We can prove the following theorem.

**Theorem 3.** Let $\kappa_1, \ldots, \kappa_{n-1} : I \to \mathbb{R}$ be differentiable functions different from 0 such that $\kappa_1, \ldots, \kappa_{n-k-2} > 0$. Then there exists, up to isotropic motions, a unique admissible curve $c$ parametrized by the arc length such that $\kappa_1, \ldots, \kappa_{n-1}$ are its curvatures.

**Proof.** Under these assumptions, there exists, up to an Euclidean motion, a unique projection $\pi_1(c)$ of the curve $c$ in the Euclidean space $U_1$ parametrized by the arc length such that $\kappa_1, \ldots, \kappa_{n-k-1}$ are its curvatures. Furthermore, (9) implies

$$
\begin{vmatrix}
x_1' & \cdots & x_{n-k+1}' \\
\vdots & \ddots & \vdots \\
x_1^{(n-k+1)} & \cdots & x_{n-k+1}^{(n-k+1)}
\end{vmatrix} = \kappa_1^{n-k} \cdots \kappa_{n-k}.
$$

Expansion by the last column of this determinant gives a linear differential equation with differentiable coefficients for the function $x_{n-k+1}(s)$ which enables us to find that function. By similar reasoning, for already found functions $x_1, \ldots, x_{n-k+j-1}$, the expression (9) enables us to find the functions $x_{n-k+j}, j = 2, \ldots, k - 1$. Therefore, the existence of the curve $c$ is proved.

In order to show that a curve $c$ is unique up to an isotropic motion, we can see at first that $y_1(s) = 1, y_2(s) = x_1(s), \ldots, y_{n-k+j} = x_{n-k+j-1}(s)$ form the fundamental solutions for the corresponding homogeneous differential equation of the equation (9). If $x_{n-k+j}^p(s)$ is a particular solution of (9), then the general solution of (9) is given by

$$
x_{n-k+j}(s) = C \cdot 1 + C_1 x_1(s) + \ldots + C_{n-k+j-1} x_{n-k+j-1}(s) + x_{n-k+j}^p(s).
$$

Therefore, every curve which is obtained by an isotropic motion from the curve $x(s) = (x_1(s), \ldots, x_{n-k}(s), x_{n-k+1}^p(s), \ldots, x_n^p(s))$ satisfies the conditions of the theorem.

**7. Geometrical interpretations of the curvatures**

Using explicit expressions of the curvatures obtained in the previous section we can show that the following propositions hold.

**Proposition 4.** Let $c$ be an admissible $C^n$-curve. Then

$$
|\kappa_{n-1}(s_0)| = \lim_{s \to 0} \left| \frac{\theta}{s} \right|
$$

where $\theta$ is the angle between the tangent vectors at $s_0$.
where $\theta$ denotes the angle between the osculating hyperplanes at the points $x(s_0)$ and $x(s + s_0)$ and $s$ is the parameter of the arc length.

**Proof:** Since the osculating hyperplanes of an admissible curve $c$ at the points $x(s_0)$ and $x(s + s_0)$ are non-isotropic, their angle is given by

$$|\theta| = \frac{1}{s_0} \left| \begin{array}{ccc} x'_1(s + s_0) & \cdots & x'_{n-2}(s + s_0) & x'_n(s + s_0) \\ \vdots & \ddots & \vdots & \vdots \\ x'^{(n-1)}_1(s + s_0) & \cdots & x'^{(n-1)}_{n-2}(s + s_0) & x'^{(n-1)}_n(s + s_0) \\ x'_1(s + s_0) & \cdots & x'_{n-2}(s + s_0) & x'_n(s + s_0) \\ \vdots & \ddots & \vdots & \vdots \\ x'^{(n-1)}_1(s + s_0) & \cdots & x'^{(n-1)}_{n-2}(s + s_0) & x'^{(n-1)}_n(s + s_0) \\ x'_1(s_0) & \cdots & x'_{n-2}(s_0) & x'_n(s_0) \\ \vdots & \ddots & \vdots & \vdots \\ x'^{(n-1)}_1(s_0) & \cdots & x'^{(n-1)}_{n-2}(s_0) & x'^{(n-1)}_n(s_0) \end{array} \right|.$$

Using the Taylor expansion of $x^{(k)}(s + s_0) = x^{(k)}(s_0) + x^{(k+1)}(s_0)s + \cdots$, $k = 1, \ldots, n - 1$, $i = 1, \ldots, n$, we get that

$$\lim_{s \to 0} \frac{\theta}{s} = \left| \begin{array}{ccc} x'_1(s_0) & \cdots & x'_{n-2}(s_0) & x'_n(s_0) \\ \vdots & \ddots & \vdots & \vdots \\ x'^{(n-2)}_1(s_0) & \cdots & x'^{(n-2)}_{n-2}(s_0) & x'^{(n-2)}_n(s_0) \\ x^{(n)}_1(s_0) & \cdots & x^{(n)}_{n-2}(s_0) & x^{(n)}_n(s_0) \\ x'_1(s_0) & \cdots & x'_{n-2}(s_0) & x'_n(s_0) \\ \vdots & \ddots & \vdots & \vdots \\ x'^{(n-1)}_1(s_0) & \cdots & x'^{(n-1)}_{n-2}(s_0) & x'^{(n-1)}_n(s_0) \end{array} \right|^2.$$
Some calculation shows that the numerator of this expression is equal to

\[
\text{det}(x', \ldots, x^{(n)}) = \begin{vmatrix}
  x'_1(s_0) & \cdots & x'_{n-2}(s_0) \\
  \vdots & \ddots & \vdots \\
  x^{(n-2)}_1(s_0) & \cdots & x^{(n-2)}_{n-1}(s_0)
\end{vmatrix}
\]

which, comparing by (11) for \( j = k - 1 \), implies the statement of the proposition.

For the curvatures \( \kappa_{n-k}, \ldots, \kappa_{n-2} \) we have the following interpretation.

**PROPOSITION 5.** Let \( c \) be an admissible \( C^{(n)} \)-curve. Then

\[
|\kappa_{n-k+j}(s_0)| = \lim_{s \to 0} \left| \frac{\omega}{s} \right|, \quad j = 0, \ldots, k - 2
\]

where \( \omega \) denotes the angle between the \((k-j-1)\)-isotropic hyperplanes at the points \( x(s_0) \) and \( x(s + s_0) \) spanned by the vectors \( t_1, \ldots, t_{n-k+j}, b_{n-k+j+2}, \ldots, b_n \), \( s \) is the parameter of the arc length, and \( b_{n-k+j+2}, \ldots, b_n \) are the vectors of the orthonormal basis for \( U_2 \).

**Proof.** For the curvatures \( \kappa_{n-k+1}, \ldots, \kappa_{n-2} \) the proof is analogues to the proof of the previous proposition, if we consider the projection of the curve \( c \) to the \((n-k+j+1)\)-dimensional space spanned by the first \((n-k+j+1)\) coordinate vectors.

For the curvature \( \kappa_{n-k} \) we consider \((k-1)\)-isotropic hyperplanes spanned by \( t_1, \ldots, t_{n-k}, b_{n-k+2}, \ldots, b_n \) at the points \( x(s_0) \) and \( x(s + s_0) \). First let us notice that for the formally introduced Euclidean normal vector \( u = (u_1, \ldots, u_n) = t_1 \wedge \ldots \wedge t_{n-k} \wedge b_{n-k+2} \ldots \wedge b_n \) of such a hyperplane we have \( \pi_1(u') = \kappa_{n-k} \pi_1(t_{n-k}) \) and therefore \( ||u'|| = |\kappa_{n-k}| \). Now we have

\[
\lim_{s \to 0} \frac{\omega^2}{s^2} = \lim_{s \to 0} \left[ \frac{u_1(s + s_0) - u_1(s_0)}{s} \right]^2 + \cdots + \left[ \frac{u_{n-k}(s + s_0) - u_{n-k}(s_0)}{s} \right]^2 = ||u'||^2 = |\kappa_{n-k}|^2
\]

which completes the proof.

Furthermore, by using the explicit expressions for the curvatures, we can show that the following propositions hold.

**PROPOSITION 6.** The only admissible \( C^n \)-curves for which \( \kappa_{n-1} \equiv 0 \) holds are the non-degenerated \( C^n \)-curves in non-isotropic hyperplanes.

**Proof.** Let us first remark that \( \kappa_{n-1} \equiv 0 \) if and only if

\[
\text{det}(x', \ldots, x^{(n)}) = 0, \quad \begin{vmatrix}
  x'_1 & \cdots & x'_{n-1} \\
  \vdots & \ddots & \vdots \\
  x^{(n-1)}_1 & \cdots & x^{(n-1)}_{n-1}
\end{vmatrix} \neq 0.
\]
Now let $c$ be a curve in a non-isotropic hyperplane. Then by an isotropic motion we obtain that $c$ lies in a hyperplane $x_n = 0$. Therefore $c$ is given by

$$x(s) = (x_1(s), \ldots, x_{n-1}(s), 0)$$

from which (14) follows.

Conversely, let us show that $c$ lies in its osculating hyperplane at an arbitrary point $x(s)$ and that that hyperplane is non-isotropic. The equation of the osculating hyperplane at the point $x(s)$ is given by

$$\det(x - x(s), t_1(s), \ldots, t_{n-1}(s)) = 0.$$ 

We can formally introduce its Euclidean normal vector by $t_1(s) \wedge \cdots \wedge t_{n-1}(s)$ and by using the Frenet's equations and the assumption $\kappa_{n-1} \equiv 0$ we can show that this vector is a constant vector. Indeed, differentiation yields

$$(t_1(s) \wedge \cdots \wedge t_{n-1}(s))' = t_1(s) \wedge \cdots \wedge t_{n-2}(s) \wedge \kappa_{n-1}(s)t_n = 0.$$ 

Therefore, all the osculating hyperplanes are parallel. Let us show now that they are all equal. It is enough to show that

$$\det(x(s), t_1(s), \ldots, t_{n-1}(s))$$

is constant. This follows also by differentiating the previous determinant. So, $c$ lies in its osculating hyperplane. From the condition (14) follows that this hyperplane is non-isotropic.

Analogously, the following geometrical interpretations for the curvatures $\kappa_{n-k}, \ldots, \kappa_{n-2}$ hold.

**Proposition 7.** Let $c$ be a simple $C^{(n-k+j+1)}$-curve. Then $\kappa_{n-k+j} \equiv 0$ if and only if $c$ is a curve in an $(k - j - 1)$-isotropic hyperplane, $j = 0, \ldots, k - 2$.

**Proof.** Let us first notice that from (10) and (11) follows that $\kappa_{n-k+j} \equiv 0$ if and only if

$$\begin{vmatrix}
  x_1' & \cdots & x_{n-k+j+1}' \\
  \vdots & \vdots & \vdots \\
  x_1^{(n-k+j+1)} & \cdots & x_{n-k+j+1}^{(n-k+j+1)}
\end{vmatrix} = 0,
\begin{vmatrix}
  x_1' & \cdots & x_{n-k+j}' \\
  \vdots & \vdots & \vdots \\
  x_1^{(n-k+j)} & \cdots & x_{n-k+j}^{(n-k+j)}
\end{vmatrix} \neq 0.$$ 

Then the proof proceeds analogously to the proof of the Proposition 6 if we consider the projection of the curve $c$ onto the $(n - k + j + 1)$-dimensional subspace of $V$ spanned by the first $n - k + j + 1$ coordinate vectors. We can conclude that this projection lies in a non-isotropic $(n - k + j)$-plane which means that $c$ lies in an $(k - j - 1)$-isotropic hyperplane.

Furthermore, we know that $\kappa_m \equiv 0$, $m < n - k$, if and only if the projection $\pi_1(c)$ of $c$ is a curve in a $m$-plane in the basic subspace $U_1$. That is exactly the case.
when \( c \) lies in a \( k \)-isotropic \( (m + k) \)-plane in \( V \). By using this fact and the previous propositions we may understand better the nature of a degenerated curve \( c \). This can be described by introducing the supplementary curvatures.

We shall distinguish several cases.

**Case 1.** If \( \kappa_m \equiv 0, m < n - k \), then \( c \) is a curve in a \( k \)-isotropic \( (m + k) \)-plane spanned by vectors \( x', \ldots , x^{(m+k)} \). We construct the Frenet \( (m + k) \)-frame in the same way as we did it for non-degenerated curves. We obtain \( m \) Euclidean vectors \( t_1, \ldots , t_m \) and one \( 1 \)-isotropic vector \( t_{m+1}, \ldots , t_{m+k} \). Now, there exist functions \( \kappa_1, \ldots , \kappa_{m-1}, \kappa_m, \ldots , \kappa_{m+k-1} \): \( I \rightarrow \mathbb{R} \) such that the following Frenet's equations are satisfied

\[
\begin{align*}
t_1' &= \kappa_1 t_2, \\
t_i' &= -\kappa_{i-1} t_{i-1} + \kappa_i t_{i+1}, & i = 2, \ldots , m - 1, \\
t_m' &= -\kappa_m t_{m-1} + \kappa_m(t_{m+1}, \\
t_{m+j}' &= \kappa_{m+j} t_{m+j+1}, & j = 1, \ldots , k - 1, \\
t_{m+k}' &= 0.
\end{align*}
\]

For the supplementary curvatures \( \kappa_m^{(1)}, \ldots , \kappa_{m+k-1}^{(1)} \) we can obtain explicit expressions in the same way as we did it for non-degenerated curves. For the higher curvatures \( \kappa_{m+1}^{(1)}, \ldots , \kappa_{m+k-1}^{(1)} \) we get

\[
\kappa_{m+i+1}^{(1)} = \left( \left( x_{n-k+i+2}, x_{n-k+i+1}^{(m+1)} : x_{n-k+i}^{(m+1)} : \kappa_{m+i+1}^{(1)} : \kappa_{m+i+2}^{(1)} : \cdots : \kappa_{m+i}^{(1)} \right) \right) i = 0, \ldots , k - 2,
\]

or (by supposing that \( V \) is unitarian)

\[
\left( \kappa_{m+i+1}^{(1)} \right)^2 = \frac{\Gamma_{n-k+i+1}(x', \ldots , x^{(m+i+2)})(x', \ldots , x^{(m+i+1)})}{\Gamma_{n-k+i}^2(x', \ldots , x^{(m+i+1)})}, \tag{15}
\]

\[
i = 0, \ldots , k - 2.
\]

For the next curvature \( \kappa_m^{(1)} \) we get

\[
\left( \kappa_m^{(1)} \right)^2 = \frac{\Gamma_{n-k+1}(x', \ldots , x^{(m+1)})(x', \ldots , x^{(m-1)})}{\Gamma^2(x', \ldots , x^{(m)})}.
\]

Using Propositions 6, 7 we can conclude as follows.

**Proposition 8.** Let \( c \) be a simple \( C^{(m+k)} \)-curve such that \( \kappa_m \equiv 0, m < n - k. \) Then \( \kappa_{m+i}^{(1)} \equiv 0 \) if and only if \( c \) is a curve in a \((k - i - 1)\)-isotropic \((m + k - 1)\)-plane, \( i = 0, \ldots , k - 1. \)

Now we can proceed by supposing \( \kappa_m = \kappa_m^{(1)} \equiv 0. \) Then \( c \) lies in a \((k - 1)\)-isotropic \((m + k - 1)\)-plane spanned by \( m \) Euclidean vectors \( t_1, \ldots , t_m \), one \( 2 \)-isotropic
vector \( \mathbf{t}_{m+1}, \ldots \), one \( k \)-isotropic vector \( \mathbf{t}_{m+k-1} \). We introduce supplementary curvatures \( \kappa_m^{(2)}, \ldots, \kappa_{m+k-2}^{(2)} : I \to \mathbb{R} \) such that the following Frenet's equations hold

\[
\begin{align*}
\mathbf{t}_1' &= \kappa_1 \mathbf{t}_2, \\
\mathbf{t}_i' &= -\kappa_{i-1} \mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, & i &= 2, \ldots, m-1, \\
\mathbf{t}_m' &= -\kappa_{m-1} \mathbf{t}_{m-1} + \kappa_m^{(2)} \mathbf{t}_{m+1}, \\
\mathbf{t}_{m+j}' &= \kappa_{m+j}^{(2)} \mathbf{t}_{m+j+1}, & j &= 1, \ldots, k-2, \\
\mathbf{t}_{m+k-1}' &= 0.
\end{align*}
\]

By proceeding inductively under the assumptions \( \kappa_m \equiv \kappa_m^{(1)} \equiv \cdots \equiv \kappa_m^{(l-1)} \equiv 0 \) we obtain supplementary curvatures \( \kappa_{m+1}^{(l)}, \ldots, \kappa_{m+k-l}^{(l)} \) for which we obtain the following explicit expressions. For the higher curvatures \( \kappa_{m+1}^{(l)}, \ldots, \kappa_{m+k-l}^{(l)} \) we get

\[
\kappa_{m+i+1}^{(l)} = \left( \left( \frac{(x_{n-k+i+l+1}^{(m+1)} : x_{n-k+i+l+1}^{(m+2)}) \Gamma_{n-k+i+l+1, \ldots, l-1}(x', \ldots, x^{(m+i+2)})}{\Gamma_n^{2}(x', \ldots, x^{(m+i+1)})} \right) \right)_{i=0, \ldots, k-l},
\]

or (by supposing that \( V \) is unitarian)

\[
\left( \kappa_{m+i+1}^{(l)} \right)^2 = \left( \frac{\Gamma_{n-k+i+l+1, \ldots, l-1}(x', \ldots, x^{(m+i+2)}) \Gamma_{n-k+i+l+1, \ldots, l-1}(x', \ldots, x^{(m+i+1)})}{\Gamma_n^{2}(x', \ldots, x^{(m+i+1)})} \right)_{i=0, \ldots, k-l},
\]

and for the next curvature \( \kappa_m^{(l)} \) we obtain

\[
\left( \kappa_m^{(l)} \right)^2 = \frac{\Gamma_{n-k+l, \ldots, l-1}(x', \ldots, x^{(m+1)}) \Gamma_{n-k+l, \ldots, l-1}(x', \ldots, x^{(m-1)})}{\Gamma^2(x', \ldots, x^{(m)})},
\]

where \( \Gamma_{n-k+i, \ldots, i}(y_1, \ldots, y_m) \), \( i = 1, \ldots, k, l = 1, \ldots, k-1 \), denotes the Gram's determinant of the projections of the given vectors onto the \((n-k+i-l)\)-dimensional subspace of \( V \) spanned by the first \( n-k+i \) coordinate vectors except the first isotropic, \( \ldots, \) \( l \)-th isotropic direction.

Furthermore, the following theorem holds.

**THEOREM 4.** Let \( c \) be a simple \( C^{(m+k)} \)-curve such that \( \kappa_m \equiv \kappa_m^{(1)} \equiv \cdots \equiv \kappa_m^{(l-1)} \equiv 0 \), \( m < n-k \), \( l = 1, \ldots, k-1 \). Then \( \kappa_{m+i}^{(l)} \equiv 0 \) if and only if \( c \) is a curve in a \((k-l-i)\)-isotropic \((m+k-l)\)-plane.

**COROLLARY 3.** Let \( c \) be a simple \( C^{(m+k)} \)-curve, \( m < n-k \). Then \( c \) is a curve in a non-isotropic \( m \)-plane if and only if \( \kappa_m \equiv \kappa_m^{(1)} \equiv \cdots \equiv \kappa_m^{(k)} \equiv 0 \).
Case 2. Let us now consider the case $\kappa_{n-k} \equiv 0$. By Proposition 7 it means that $c$ lies in a $(k-1)$-isotropic hyperplane spanned by vectors $x', \ldots, x^{(n-1)}$. Constructing the Frenet $(n-1)$-frame in the same way as we did it for non-degenerated curves, we obtain $n-k$ Euclidean vectors $t_1, \ldots, t_{n-k},$ one 2-isotropic vector $t_{n-k+1}, \ldots,$ one $k$-isotropic vector $t_{n-1}$. We introduce supplementary curvatures $\kappa_{n-k}^{(1)}, \ldots, \kappa_{n-2}^{(1)}: I \to \mathbb{R}$ such that the following Frenet’s equations are true

$$
t'_i = \kappa_i t_i, \quad t'_{n-k} = -\kappa_{n-k} t_{n-k-1} + \kappa_{n-k}^{(1)} t_{n-k+1},
$$

$$
t'_{n-k+1} = \kappa_{n-k+1} t_{n-k+2}, \quad \ldots,
$$

$$
t'_{n-1} = 0.
$$

We can obtain the explicit expressions for the supplementary curvatures. For the higher curvatures $\kappa_{n-k+1}^{(1)}, \ldots, \kappa_{n-2}^{(1)}$ we have

$$
\kappa_{n-k+i}^{(1)} = \left( \left(\left( x^{(n-k+i+2)} : x^{(n-k+i+1)} : x^{(n-k+i)} : x^{(n-k+i-1)} \right) : \kappa_{n-k+i+1}^{(1)} \right) : \ldots : \kappa_{n-k+i-1}^{(1)} \right)'
$$

for $i = 1, \ldots, k-2,$

or

$$
\left( \kappa_{n-k+i}^{(1)} \right)^2 = \frac{\Gamma_{n-k+i+2,1}(x', \ldots, x^{(n-k+i+1)}) \Gamma_{n-k+1,1}(x', \ldots, x^{(n-k+i-1)})}{\Gamma^2_{n-k+i+1,1}(x', \ldots, x^{(n-k+i)})},
$$

for $i = 1, \ldots, k-2,$

and for the next curvature $\kappa_{n-k}^{(1)}$ we get

$$
\left( \kappa_{n-k}^{(1)} \right)^2 = \frac{\Gamma_{n-k+2,1}(x', \ldots, x^{(n-k+1)}) \Gamma(x', \ldots, x^{(n-k-1)})}{\Gamma^2(x', \ldots, x^{(n-k)})}.
$$

Furthermore, the following proposition holds.

**Proposition 9.** Let $c$ be a simple $C^{(n-1)}$-curve such that $\kappa_{n-k} \equiv 0$. Then $\kappa_{n-k+i}^{(1)} \equiv 0$ if and only if $c$ lies in a $(k-i-2)$-isotropic $(n-2)$-plane, $i = 0, \ldots, k-2$.

Let us suppose now that $\kappa_{n-k} \equiv \kappa_{n-k}^{(1)} \equiv 0$. Then $c$ lies in a $(k-2)$-isotropic $(n-2)$-plane spanned by $n-k$ Euclidean vectors $t_1, \ldots, t_{n-k},$ one 3-isotropic vector $t_{n-k+1}, \ldots,$ one $k$-isotropic vector $t_{n-2}$. We introduce supplementary curvatures
\( \kappa_{n-k}^{(2)}, \ldots, \kappa_{n-3}^{(2)} : I \to \mathbb{R} \) such that the following Frenet's equations hold

\[
\begin{align*}
t_1' &= \kappa_1 t_2, \\
t_i' &= -\kappa_{i-1} t_{i-1} + \kappa_i t_{i+1}, \quad i = 2, \ldots, n - k - 1, \\
t_{n-k}' &= -\kappa_{n-k-1} t_{n-k-1} + \kappa_{n-k}^2 t_{n-k+1}, \\
t_{n-k+j}' &= \kappa_{n-k+j}^2 t_{n-k+j+1}, \quad j = 1, \ldots, k - 2, \\
t_{n-2}' &= 0.
\end{align*}
\]

By proceeding inductively under the assumptions \( \kappa_{n-k}^{(1)} \equiv \kappa_{n-k}^{(l)} \equiv \ldots \equiv \kappa_{n-k}^{(l-1)} \equiv 0 \) we obtain supplementary curvatures \( \kappa_{n-k+i}^{(l)} \), \( l = 1, \ldots, k - 1, i = 0, \ldots, k - l - 1 \), for which the following explicit expressions hold. For the higher curvatures \( \kappa_{n-k+1}^{(1)}, \ldots, \kappa_{n-2}^{(1)} \) we have

\[
\kappa_{n-k+i}^{(l)} = \left( \begin{pmatrix} \kappa_{n-k+i+1}^{(l)} & \kappa_{n-k+i+2}^{(l)} & \cdots & \kappa_{n-k+i-l+1}^{(l)} \\ x_{n-k+i+1}^{(n-k+1)} & x_{n-k+i+2}^{(n-k+1)} & \cdots & x_{n-k+i-l+1}^{(n-k+1)} \end{pmatrix} \right)'^l
\]

or

\[
\left( \kappa_{n-k+i}^{(l)} \right)^2 = \frac{\Gamma_{n-k+i+l+1, \ldots, l}(x', \ldots, x^{(n-k+i)}) \Gamma_{n-k+i+l-1, \ldots, l}(x', \ldots, x^{(n-k+i-1)})}{\Gamma_{n-k+i+l-1, \ldots, l}(x', \ldots, x^{(n-k+i)})},
\]

\( i = 1, \ldots, k - l - 1 \).

For the next curvature \( \kappa_{n-k}^{(l)} \) we get

\[
\left( \kappa_{n-k}^{(l)} \right)^2 = \frac{\Gamma_{n-k+l+1, \ldots, l}(x', \ldots, x^{(n-k+1)}) \Gamma(x', \ldots, x^{(n-k)})}{\Gamma^2(x', \ldots, x^{(n-k)})}.
\]

Now the following statements hold.

**THEOREM 5.** Let \( c \) be a simple \( C^{(n-1)} \)-curve such that \( \kappa_{n-k}^{(1)} \equiv \kappa_{n-k}^{(l)} \equiv \ldots \equiv \kappa_{n-k}^{(l-1)} \equiv 0 \), \( l = 1, \ldots, k - 1 \). Then \( \kappa_{n-k+i}^{(l)} \equiv 0 \) if and only if \( c \) is a curve in a \((k - l - i - 1)\)-isotropic \((n - l - 1)\)-plane, \( i = 1, \ldots, k - l - 1 \).

**COROLLARY 4.** Let \( c \) be a simple \( C^{(n-1)} \)-curve. Then \( c \) is a curve in a non-isotropic \((n - k)\)-plane if and only if \( \kappa_{n-k}^{(1)} \equiv \kappa_{n-k}^{(l)} \equiv \ldots \equiv \kappa_{n-k}^{(k-1)} \equiv 0 \).

**Case 3.** Finally, let us consider the case when \( \kappa_{n-k+j}^{(l)} \equiv 0, j = 1, \ldots, k - 2, \) holds. By Proposition 7 it follows that \( c \) lies in a \((k - j - 1)\)-isotropic hyperplane spanned by vectors \( x', \ldots, x^{(n-1)} \). By constructing the Frenet's \((n - 1)\)-frame we get \( n - k \) Euclidean vectors \( t_1, \ldots, t_{n-k} \), one \( 1 \)-isotropic vector \( t_{n-k+1}, \ldots, \) one \( j \)-isotropic vector \( t_{n-k+j}, \) one \((j + 2)\)-isotropic vector \( t_{n-k+j+1}, \ldots, \) one \( k \)-isotropic vector \( t_{n-1} \). Since the geometry of the \((k - j - 1)\)-isotropic hyperplane,
j = 1, \ldots, k - 2 coincides with the geometry of the space \( I_{n-1}^{k-1} \), we introduce supplementary curvatures \( \kappa_{n-k+j}^{(1)}, \ldots, \kappa_{n-2}^{(1)} : I \rightarrow \mathbb{R} \) such that the following Frenet's equations hold

\[
\begin{align*}
t'_2 &= \kappa_1 t_2, \\
t'_i &= -\kappa_{i-1} t_{i-1} + \kappa_i t_{i+1}, & i = 2, \ldots, n - k, \\
t'_{n-k+i} &= \kappa_{n-k+i} t_{n-k+i+1}, & i = 1, \ldots, j - 1, \\
t'_{n-k+i} &= \kappa_{n-k+i} t_{n-k+i+1}, & l = j, \ldots, n - 2, \\
t'_{n-1} &= 0.
\end{align*}
\]

In the same way as before we obtain the explicit expressions for the supplementary curvatures. We get

\[
\kappa_{n-k+j+i}^{(1)} = 
\left( \left( \left( (x_{n-k+j+i+2}^{(n-k+j+1)}, x_{n-k+1}^{(n-k+1)})' : \kappa_{n-k+1} \right)' : \cdots : \kappa_{n-k+j-1} \right)' : \kappa_n^{(1)} \right)'
\]

\[
i = 0, \ldots, k - j - 2,
\]

or

\[
\left( \kappa_{n-k+j}^{(1)} \right)^2 = 
\frac{\Gamma_{n-k+j+2,j+1}(x', \ldots, x^{(n-k+j+1)}) \Gamma_{n-k+j}(x', \ldots, x^{(n-k+j-1)})}{\Gamma_{n-k+j}^2(x', \ldots, x^{(n-k+j)})},
\]

\[
\left( \kappa_{n-k+j+1}^{(1)} \right)^2 = 
\frac{\Gamma_{n-k+j+3,j+1}(x', \ldots, x^{(n-k+j+2)}) \Gamma_{n-k+j}(x', \ldots, x^{(n-k+j)})}{\Gamma_{n-k+j+2,j+1}^2(x', \ldots, x^{(n-k+j+1)})},
\]

\[
\left( \kappa_{n-k+j+i}^{(1)} \right)^2 = 
\frac{\Gamma_{n-k+j+i+2,j+1}(x', \ldots, x^{(n-k+j+i+1)}) \Gamma_{n-k+j+i,j+1}(x', \ldots, x^{(n-k+j-i-1)})}{\Gamma_{n-k+j+i+1,j+1}^2(x', \ldots, x^{(n-k+i)})},
\]

\[
i = 2, \ldots, k - j - 2.
\]

Furthermore, the following proposition is true.
PROPOSITION 10. Let \( c \) be a simple \( C^{(n-1)} \)-curve such that \( \kappa_{n-k+j} \equiv 0, j = 1, \ldots, k-2 \). Then \( \kappa_{n-k+j+i}^{(1)} \equiv 0 \) if and only if \( c \) lies in a \( (k-j-i-2) \)-isotropic \( (n-2) \)-plane, \( i = 0, \ldots, k-j-2 \).

Let us now suppose that \( \kappa_{n-k+j} \equiv \kappa_{n-k+j}^{(1)} \equiv 0 \). Then \( c \) lies in a \( (k-j-2) \)-isotropic \( (n-2) \)-plane spanned by \( n-k \) Euclidean vectors \( t_1, \ldots, t_{n-k} \), one \( 1 \)-isotropic vector \( t_{n-k+1} \), \( j \)-isotropic vector \( t_{n-k+j+1} \), one \( (j+3) \)-isotropic vector \( t_{n-k+j+2} \), \( k \)-isotropic vector \( t_{n-k+2} \). Again we introduce supplementary curvatures \( \kappa_{n-k}^{(2)}, \ldots, \kappa_{n-3}^{(2)} : I \to \mathbb{R} \) such that the following Frenet's equations hold

\[
\begin{align*}
t_1' &= \kappa_1 t_2, \\
t_i' &= -\kappa_{i-1} t_{i-1} + \kappa_i t_{i+1}, \quad i = 2, \ldots, n-k, \\
t_{n-k+l}' &= \kappa_{n-k+l} t_{n-k+l+1}, \quad i = 1, \ldots, j-1, \\
t_{n-k+l}' &= \kappa_{n-k+l}^{(2)} t_{n-k+l+1}, \quad l = j, \ldots, n-3, \\
t_{n-2}' &= 0.
\end{align*}
\]

By proceeding inductively under the assumptions \( \kappa_{n-k+j} \equiv \kappa_{n-k+j}^{(1)} \equiv \ldots \equiv \kappa_{n-k+j}^{(l-1)} \equiv 0 \) we obtain supplementary curvatures \( \kappa_{n-k+j+i}^{(l)} : I \to \mathbb{R} \) for which the following explicit expressions hold

\[
\left( \kappa_{n-k+j+i}^{(l)} \right)^2 = \frac{\Gamma_{n-k+j+i+l+1,j+1,\ldots,j+l}(x', \ldots, x^{(n-k+j+i+1)}) \Gamma_{n-k+j+i,j+1,\ldots,j+l+1}(x', \ldots, x^{(n-k+j)})}{\Gamma_{n-k+j+i,j+1,\ldots,j+l+1}(x', \ldots, x^{(n-k+j)})},
\]

or

\[
\left( \kappa_{n-k+j}^{(l)} \right)^2 = \frac{\Gamma_{n-k+j+i+1,j+1,\ldots,j+l}(x', \ldots, x^{(n-k+j+i+1)}) \Gamma_{n-k+j+i+1}(x', \ldots, x^{(n-k+j)})}{\Gamma_{n-k+j+i+1,j+1,\ldots,j+l+1}(x', \ldots, x^{(n-k+j+i+1)})},
\]

\[
\left( \kappa_{n-k+j+i}^{(l)} \right)^2 = \frac{\Gamma_{n-k+j+i+i+1,j+1,\ldots,j+l+1}(x', \ldots, x^{(n-k+j+i+i+1)}) \Gamma_{n-k+j+i+i+1}(x', \ldots, x^{(n-k+j+i)})}{\Gamma_{n-k+j+i+i+1,j+1,\ldots,j+l+1}(x', \ldots, x^{(n-k+j+i)})}.
\]
The following statements hold.

**THEOREM 6.** Let $c$ be a simple $C^{(n-1)}$-curve such that $\kappa_{n-k+j}^{(l)} \equiv 0$, $l = 1, \ldots, k - j - 1$. Then $\kappa_{n-k+j+l}^{(l)} \equiv 0$ if and only if $c$ is a curve in a $(k-j-l-1)$-isotropic $(n-l-1)$-plane, $i=0, \ldots, k-j-l-1$.

**COROLLARY 5.** Let $c$ be a simple $C^{(n-1)}$-curve. Then $c$ is a curve in a non-isotropic $(n-k+j)$-plane if and only if $\kappa_{n-k+j}^{(1)} \equiv \kappa_{n-k+j}^{(k-j-1)} \equiv 0$.

**REFERENCES**
