PROPER $n$-SHAPE CATEGORIES

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Abstract. In this paper, it is shown that the proper $n$-shape category of Ball-Sher type is isomorphic to a subcategory of the proper $n$-shape category defined by proper $n$-shapings. It is known that the latter is isomorphic to the shape category defined by the pair $(\mathcal{Y}_{pn}^n, \mathcal{Y}_{Ipn}^n)$, where $\mathcal{Y}_{pn}^n$ is the category whose objects are locally compact separable metrizable spaces and whose morphisms are the proper $n$-homotopy classes of proper maps, and $\mathcal{Y}_{Ipn}^n$ is the full subcategory of $\mathcal{Y}_{pn}^n$ whose objects are spaces having the proper $n$-homotopy type of polyhedra. In case $n = \infty$, this shows the relation between the original Ball-Sher’s category and the proper shape category defined by proper shapings. We also discuss the proper $n$-shape category of spaces of dimension $\leq n + 1$.

1. Introduction

In this paper, all spaces are assumed to be separable metrizable. By $[X, Y]_p^p$, we denote the set of the proper homotopy classes of proper maps from $X$ to $Y$. The proper shape category $\mathcal{S}_p$ is defined as the category whose objects are locally compact spaces and whose morphisms from $X$ to $Y$ are natural transformations from $\pi_Y = [Y, -]_p$ to $\pi_X = [X, -]_p$ (which are called proper shapings) $[3]$, where $\pi_X = [X, -]_p$ is the functor from the proper homotopy category of polyhedra to the category of sets. Originally, Ball and Sher defined in $[4]$ proper shape of locally compact spaces by modifying Borsuk’s definition of shape of compacta in $[5]$. Just like shape theory, the proper shape category $\mathcal{S}_p$ has various descriptions. In $[3]$, it was shown that they are equivalent to each other except for Ball-Sher’s category $\mathcal{S}_{BS}$. However, it is not known whether this category $\mathcal{S}_p$ is isomorphic to Ball-Sher’s category $\mathcal{S}_{BS}$. For shape theory, refer to $[12]$.

It is said that proper maps $f, g : X \to Y$ are properly $n$-homotopic to each other if $f \circ h$ and $g \circ h$ are properly homotopic to each other for any proper map $h : Z \to X$ of an arbitrary locally compact space $Z$ with $\dim Z \leq n$. In case $n = \infty$, proper $\infty$-homotopy is just proper homotopy. The proper $n$-homotopy class of $f$ is denoted by $[f]_p^n$. Let $[X, Y]_p^n$ denote the set of all proper $n$-homotopy classes of proper maps.


Key words and phrases: Proper $n$-shape category, proper $n$-homotopy, proper $n$-fundamental net, proper $n$-approximative map, proper $n$-shaping, expansion.

1A polyhedron is the underlying space of a locally finite simplicial complex.

2In $[3]$, the category $\mathcal{S}_p$ is denoted by $\mathcal{S}_p^0$ (but the separability of $X \in \text{Ob}\mathcal{S}_p^0$ is not assumed) and the category $\mathcal{S}_{BS}$ is denoted by $\mathcal{S}_{BS}^0$. 
from $X$ to $Y$. By replacing the functor $\pi_X = [X, -]_p$ by $\pi^n_X = [X, -]^n_p$, we can define a proper $n$-shaping and obtain the proper $n$-shape category $\mathcal{S}^n_p$. On the other hand, by using proper $n$-homotopy instead of proper homotopy in the definition of Ball-Sher's proper shape in [4], the notion of proper $n$-shape was introduced in [1]. Here we call this category the proper $n$-shape category of Ball-Sher type and denote it by $\mathcal{S}^n_{BS}$.

In this paper, we show that the category $\mathcal{S}^n_{BS}$ is isomorphic to a subcategory of $\mathcal{S}^n_p$ by a functor which is the identity on the class of objects. Thus, we can regard $\mathcal{S}^n_{BS} \subset \mathcal{S}^n_p$. In case $n = \infty$, the category $\mathcal{S}^n_{BS}$ is isomorphic to a subcategory of $\mathcal{S}^n_p$. This result is also obtained in [13] independently. It remains as an open problem whether $\mathcal{S}^n_p$ and $\mathcal{S}^n_{BS}$ are isomorphic to each other. However, describing $\mathcal{S}^n_p$ by using proper $n$-approximative maps, we make clear where is the problem.

Each locally compact space of dimension $\leq n+1$ can be embedded in $\mu^{n+1}\setminus \{pt\}$ as a closed set, where $\mu^{n+1}$ is the $(n+1)$-dimensional universal Menger compactum. By replacing $Q \setminus \{0\}$ ($Q = [-1, 1]^\omega$ is the Hilbert cube) by $\mu^{n+1}\setminus \{pt\}$, another proper $n$-shape category of Ball-Sher type was obtained in [2]. Here this category is denoted by $\mathcal{S}^n_{BS}(n+1)$. Let $\mathcal{S}^n_{BS}(n+1)$ (resp. $\mathcal{S}^n_p(n+1)$) be the full subcategory of $\mathcal{S}^n_{BS}$ (resp. $\mathcal{S}^n_p$) whose objects are spaces of dimension $\leq n+1$. Akaike [2] showed that the category $\mathcal{S}^n_{BS}(n+1)$ is isomorphic to a subcategory of $\mathcal{S}^n_{BS}(n+1)$. In this paper, we also show that $\mathcal{S}^n_{BS}(n+1)$ is isomorphic to a subcategory of $\mathcal{S}^n_p(n+1)$. Thus, in the case when $\dim \leq n+1$, we can regard $\mathcal{S}^n_{BS}(n+1) \subset \mathcal{S}^n_{BS}(n+1) \subset \mathcal{S}^n_p(n+1)$.

2. The proper $n$-shape theory

Let $\mathcal{H}^n_p$ be the category whose objects are locally compact spaces and whose morphisms are the proper $n$-homotopy classes of proper maps. By $\mathcal{H}^n_pPol$, we denote the full subcategory of $\mathcal{H}^n_p$ whose objects are spaces having the proper $n$-homotopy classes of polyhedra. The proper $n$-shape category $\mathcal{S}^n_p$ is defined as the category whose objects are locally compact spaces and whose morphisms from $X$ to $Y$ are natural transformations from $\pi^\alpha_X = [X, -]^\alpha_p$ to $\pi^\alpha_Y = [Y, -]^\alpha_p$ called proper $n$-shapings (cf. [3]), where $\pi_X^\alpha = [X, -]^\alpha_p$ is the functor from $\mathcal{H}^n_pPol$ to the category of sets.

In the general theory of shape [12, Ch.I, §2], the notion of expansions of spaces is fundamental. Let $p : X \to X$ be a morphism in the pro-category $pro-\mathcal{H}^n_p$ from a locally compact space $X$ to an inverse system $X$ in $\mathcal{H}^n_pPol$. We call $p$ an $\mathcal{H}^n_pPol$-expansion of $X$ if it satisfies the following:

- for any inverse system $Y$ in $\mathcal{H}^n_pPol$ and any morphism $q : X \to Y$ in $pro-\mathcal{H}^n_p$ from $X$ to $Y$, there exists a unique morphism $f : X \to Y$ in $pro-\mathcal{H}^n_p$ such that $q = fp$.

By [12, Ch. §2, Theorem 1], $p = (p_\lambda)_{\lambda \in \Lambda} : X \to X = (X_\lambda, p_\lambda, \lambda', \Lambda)$, is an $\mathcal{H}^n_pPol$-expansion of $X$ if and only if the following conditions are satisfied:

1. for any polyhedron $P$ and any proper map $f : X \to P$, there exist $\lambda \in \Lambda$ and a proper map $q : X_\lambda \to P$ such that $f \simeq q p_\lambda$;
(2) for any polyhedron \( P \) and any two proper maps \( g, h : X_\lambda \to P \) satisfying \( \Gamma_{\lambda, \lambda'} \succeq_p h \Gamma_{\lambda, \lambda'} \), there exists \( \lambda' \geqslant \lambda \) such that \( \Gamma_{\lambda, \lambda'} \succeq_p h \Gamma_{\lambda, \lambda'} \).

Due to [12, Ch.I, §2], the following theorem guarantees the existence of the shape theory for the pair \( (\mathcal{H}_p^n, \mathcal{H}_p^n \text{Pol}) \). By [12, Ch.I, §S, Theorem 7], the shape category defined by the pair \( (\mathcal{H}_p^n, \mathcal{H}_p^n \text{Pol}) \) is isomorphic to the proper n-shape category \( \mathcal{S}_n \) defined as above.

**Theorem 2.1.** Every locally compact space \( X \) admits an \( \mathcal{H}_p^n \text{Pol} \)-expansion, namely the category \( \mathcal{H}_p^n \text{Pol} \) is dense in \( \mathcal{H}_p^n \).

**Proof.** Since \( X \) can be embedded in \( Q \setminus \{0\} \) as a closed set, we can regard \( X \) as a closed set in \( Q \setminus \{0\} \). Each neighborhood of \( X \) in \( Q \setminus \{0\} \) contains some closed neighborhood of \( X \) which is an ANR ([3]). By \( \text{Nbd}(X) \), we denote the collection of all closed ANR neighborhoods of \( X \) in \( Q \setminus \{0\} \), which is directed by the order \( [U \leq V] \equiv [V \subset U] \). Then we have the inverse system \( U = \{ U, [i_{U,V}]_p, \text{Nbd}(X) \} \) in \( \mathcal{H}_p^n \), and the morphism \( i_X = ([i_{U,V}]_p)_{U \in \text{Nbd}(X)} : X \to U \) in \( \text{pro-} \mathcal{H}_p^n \), where \( i_{U,V} : V \to U \) and \( i_U : X \to U \) are inclusions. We show that \( i_X : X \to U \) is an \( \mathcal{H}_p^n \text{Pol} \)-expansion of \( X \).

To see (1), let \( f : X \to P \) be a proper map from \( X \) to a polyhedron \( P \). By [4, Lemma 3.2], \( f \) extends to a proper map \( \bar{f} : U \to P \) of some \( U \in \text{Nbd}(X) \). Then \( f \succeq_p \bar{f} i_U \) since \( f = \bar{f} i_U \). The condition (2) is a direct consequence of the following lemma.

**Lemma 2.2.** Let \( X \) be a locally compact space, \( A \) a closed set in \( X \), \( Y \) a locally compact ANR and let \( f, g : X \to Y \) be proper maps. If \( f|A \succeq_p g|A \) then \( f|U \succeq_p g|U \) for some neighborhood \( U \) of \( A \) in \( X \).

**Proof.** For every locally compact space \( X \), there exists an \( n \)-invertible\(^3\) proper map \( \alpha : Z \to X \) of an \( n \)-dimensional locally compact space \( Z \) [10]. Then the lemma can be proved similarly to [8, Proposition 1.7]. \( \square \)

3. **Proper \( n \)-fundamental nets and proper \( n \)-approximative maps**

For simplicity, we restrict objects of the category \( \mathcal{S}_n \) to closed sets \( X \) in \( Q \setminus \{0\} \). As in the proof of Theorem 2.1, let \( \text{Nbd}(X) \) denote the directed set of all closed ANR neighborhoods of \( X \) in \( Q \setminus \{0\} \).\(^4\)

Let \( X \) and \( Y \) be closed sets in \( Q \setminus \{0\} \). A proper \( n \)-fundamental net from \( X \) to \( Y \) is a net of maps \( f_\lambda : Q \setminus \{0\} \to Q \setminus \{0\} \) indexed by a directed set \( \Lambda = (\Lambda, \leq) \) satisfying the condition:

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\(^3\)A proper map \( \alpha : Z \to X \) is \( n \)-invertible if any proper map \( h : W \to X \) of an \( n \)-dimensional locally compact space \( W \) lifts to a proper map \( \bar{h} : W \to Z \), i.e., \( h = \alpha \bar{h} \).

\(^4\)In this section, each \( N \in \text{Nbd}(X) \) need not be an ANR since we can use the fact that \( \text{int}N \) is an ANR if necessary. However, this restriction is convenient and will be necessary in the next section.
for each \( N \in \text{Nbd}(Y) \), there is some \( M \in \text{Nbd}(X) \) and \( \lambda_0 \in \Lambda \) such that \( f_{\lambda}|M \simeq^n_p f_{\lambda_0}|M \) in \( N \) for all \( \lambda \geq \lambda_0 \).

Two proper \( n \)-fundamental nets \((f_{\lambda})_{\lambda \in \Lambda}\) and \((g_{\delta})_{\delta \in \Delta}\) from \( X \) to \( Y \) are properly \( n \)-homotopic to each other (denoted by \( (f_{\lambda}) \simeq^n_p (g_{\delta}) \)) provided for each \( N \in \text{Nbd}(Y) \), there exist \( M \in \text{Nbd}(X) \), \( \lambda_0 \in \Lambda \) and \( \delta_0 \in \Delta \) such that \( f_{\lambda}|M \simeq^n_p g_{\delta}|M \) in \( N \) for all \( \lambda \geq \lambda_0 \) and \( \delta \geq \delta_0 \).

The relation \( \simeq^n_p \) is an equivalence relation among proper \( n \)-fundamental nets. The composition of two proper \( n \)-fundamental nets \((f_{\lambda})_{\lambda \in \Lambda}\) from \( X \) to \( Y \) and \((g_{\delta})_{\delta \in \Delta}\) from \( Y \) to \( Z \) is defined as follows:

\[
(g_{\delta})_{\delta \in \Delta}(f_{\lambda})_{\lambda \in \Lambda} = (g_{\delta}f_{\lambda})_{(\lambda, \delta) \in \Lambda \times \Delta}.
\]

where \( \Lambda \times \Delta \) has the order \([[(\lambda, \delta) \leq (\lambda', \delta')] \equiv [\lambda \leq \lambda', \delta \leq \delta'] \). Then we have the category of closed sets in \( Q \setminus \{0\} \) and proper \( n \)-fundamental nets, which is denoted by \( \mathcal{F}_p^n \). The proper \( n \)-shape category of Ball-Sher type \( \mathcal{F}_{\text{BS}}^n \) is defined as the proper \( n \)-homotopy category of \( \mathcal{F}_p^n \), that is, \( \text{Ob}\mathcal{F}_{\text{BS}}^n = \text{Ob}\mathcal{F}_p^n \) and \( \text{Mor}\mathcal{F}_{\text{BS}}^n = \text{Mor}\mathcal{F}_p^n / \simeq^n_p \).

A proper \( n \)-approximative map of \( X \) towards \( Y \) is a net of proper maps \( f_{\lambda} : X \to Q \setminus \{0\} \) indexed by a directed set \( \Lambda = (\Lambda, \leq) \) satisfying the condition:

for each \( N \in \text{Nbd}(Y) \), there is some \( \lambda_0 \in \Lambda \) such that \( f_{\lambda} \simeq^n_p f_{\lambda_0} \) in \( N \) for all \( \lambda \geq \lambda_0 \).

A net of proper maps \( f_N : X \to Q \setminus \{0\} \) indexed by the directed set \( \text{Nbd}(Y) \) is a proper \( n \)-approximative map of \( X \) towards \( Y \) if \( f_N \simeq^n_p f_N \) in \( N \) for all \( N' \subset N \in \text{Nbd}(Y) \). Such a net is called a mutational proper \( n \)-approximative map.

Two proper \( n \)-approximative maps \((f_{\lambda})_{\lambda \in \Lambda}\) and \((g_{\delta})_{\delta \in \Delta}\) of \( X \) towards \( Y \) are properly \( n \)-homotopic to each other (denoted by \( (f_{\lambda}) \simeq^n_p (g_{\delta}) \)) provided for each \( N \in \text{Nbd}(Y) \), there exist \( \lambda_0 \in \Lambda \) and \( \delta_0 \in \Delta \) such that \( f_{\lambda} \simeq^n_p g_{\delta} \) in \( N \) for all \( \lambda \geq \lambda_0 \) and \( \delta \geq \delta_0 \).

The relation \( \simeq^n_p \) is an equivalence relation among proper \( n \)-approximative maps. The equivalence class of \( (f_{\lambda}) \) is called the proper \( n \)-homotopy class of \( (f_{\lambda}) \) and is denoted by \( [(f_{\lambda})]_p^n \). For two mutational proper \( n \)-approximative maps \((f_N)\) and \((g_N)\) of \( X \) towards \( Y \),

\[
(f_N) \simeq^n_p (g_N) \text{ if and only if } f_N \simeq^n_p g_N \text{ in } N \text{ for every } N \in \text{Nbd}(Y).
\]

**Lemma 3.1.** Every proper \( n \)-approximative map \((f_{\lambda})_{\lambda \in \Lambda}\) of \( X \) towards \( Y \) is properly \( n \)-homotopic to a mutational proper \( n \)-approximative map.

**Proof:** For each \( N \in \text{Nbd}(Y) \), choose \( \lambda_N \in \Lambda \) so that \( f_{\lambda} \simeq^n_p f_{\lambda_N} \) in \( N \) for all \( \lambda \geq \lambda_N \), and let \( f_{\lambda_N} = f_{\lambda_N'} \). For each \( N' \subset N \in \text{Nbd}(Y) \), by choosing \( \lambda_0 \geq \lambda_N, \lambda_N' \), we have \( f_{\lambda_N'} \simeq^n_p f_{\lambda_0} \simeq^n_p f_{\lambda_N} \) in \( N \). Thus, we have the result. \( \square \)

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5One should note that each \( f_{\lambda} \) need not be proper but the restrictions of almost all \( f_{\lambda} \) on some neighborhoods of \( X \) are proper. Since each \( f_{\lambda} \) is not required to be proper, we can replace \( Q \setminus \{0\} \) by any locally compact AR's containing \( X \) and \( Y \) as closed sets to define proper shape.
For a proper n-fundamental net \((f_\lambda)\) from \(X\) to \(Y\), \((f_\lambda|X)\) is clearly a proper n-approximative map of \(X\) towards \(Y\).

**Lemma 3.2.** For two proper n-fundamental nets \((f_\lambda)\) and \((g_\delta)\) from \(X\) to \(Y\), the following statements are equivalent

- \((f_\lambda) \approx_p (g_\delta)\) (as proper n-fundamental nets);
- \((f_\lambda|X) \approx_p (g_\delta|X)\) (as proper n-approximative maps).

**Proof.** The implication \((f_\lambda) \approx_p (g_\delta) \Rightarrow (f_\lambda|X) \approx_p (g_\delta|X)\) is clear. Assume that \((f_\lambda|X) \approx_p (g_\delta|X)\). Then, for each \(N \in \text{Nbd}(Y)\), we have \(\lambda_1 \in \Lambda\) and \(\delta_1 \in \Delta\) such that \(f_\lambda|X \approx_p g_\delta|X\) in \(N\) for all \(\lambda \geq \lambda_1\) and \(\delta \geq \delta_1\). On the other hand, we have \(M, M' \in \text{Nbd}(X), \lambda_2 \in \Lambda\) and \(\delta_2 \in \Delta\) so that \(f_\lambda|M \approx_p f_\lambda|M\) in \(N\) for all \(\lambda \geq \lambda_2\) and \(g_\delta|M' \approx_p g_\delta|M'\) in \(N\) for all \(\delta \geq \delta_2\). Choose \(\lambda_0 \in \Lambda\) and \(\delta_0 \in \Delta\) so that \(\lambda_0 \geq \lambda_1, \lambda_2\) and \(\delta_0 \geq \delta_1, \delta_2\). By Lemma 2.2, we have \(M_0 \in \text{Nbd}(X)\) such that \(M_0 \subseteq M \cap M'\) and \(f_{\lambda_0}|M_0 \approx_p g_\delta|M_0\) in \(N\). Then,
\[
f_\lambda|M_0 \approx_p f_{\lambda_0}|M_0 \approx_p g_\delta|M_0 \quad \text{in} \quad N
\]
for all \(\lambda \geq \lambda_0\) and \(\delta \geq \delta_0\). \(\square\)

Composition of proper n-approximative maps cannot be defined, but composition of proper n-homotopy classes of proper n-approximative maps can be. To define composition, we show the following:

**Lemma 3.3.** (1) Let \((f_N)\) be a mutational proper n-approximative map of \(X\) towards \(Y\) and let \((g_M)\) be one of \(Y\) towards \(Z\). For each \(M \in \text{Nbd}(Z)\), choose \(N_M \in \text{Nbd}(Y)\) so that \(g_M\) extends to a proper map \(\tilde{g}_M : N_M \rightarrow M\). Then \((\tilde{g}_M f_{N_M})\) is a mutational proper n-approximative map of \(X\) towards \(Z\).

(2) Let \((f_N')\) be a mutational proper n-approximative map of \(X\) towards \(Y\) and let \((g_M)\) be one of \(Y\) towards \(Z\) such that \((f_N) \approx_p (f_N')\) and \((g_M) \approx_p (g_M')\). For each \(M \in \text{Nbd}(Z)\), choose \(N_M' \in \text{Nbd}(Y)\) so that \(g_M\) extends to a proper map \(\tilde{g}_M' : N_M' \rightarrow M\). Then \((\tilde{g}_M f_{N_M'})\) \(\approx_p (\tilde{g}_M' f_{N_M'})\).

**Proof.** (1): For \(M' \subseteq M \in \text{Nbd}(Z)\), \(\tilde{g}_M|Y = g_M' \approx_p g_M = \tilde{g}_M|Y\) in \(M\). Then we can choose \(N \in \text{Nbd}(Y)\) so that \(N \subseteq N_M \cap N_M'\) and \(\tilde{g}_M'|N \approx_p \tilde{g}_M|N\) in \(M\) (Lemma 2.2). Since \(f_{N_M'} \approx_p f_N\) in \(N_M'\) and \(f_{N_M} \approx_p f_N\) in \(N_M\), it follows that
\[
\tilde{g}_M' f_{N_M'} \approx_p \tilde{g}_M f_{N_M} = (\tilde{g}_M'|N) f_N \approx_p (\tilde{g}_M|N) f_N \approx_p \tilde{g}_M f_N \approx_p \tilde{g}_M f_{N_M} \quad \text{in} \quad M.
\]

(2): For each \(M \in \text{Nbd}(Z)\), \(\tilde{g}_M|Y = g_M \approx_p g_M' = \tilde{g}_M'|Y\) in \(M\). Then we can choose \(N \in \text{Nbd}(Y)\) so that \(N_M \cap N_M'\) and \(\tilde{g}_M'|N \approx_p \tilde{g}_M|N\) in \(M\) (Lemma 2.2). Since \(f_{N'} \approx_p f_N\) in \(N\), \(f_{N'} \approx_p f_{N_M}\) in \(N_M\) and \(f_N \approx_p f_{N_M'}\) in \(N_M'\), we have
\[
\tilde{g}_M f_{N_M'} \approx_p \tilde{g}_M f_{N_M} \approx_p \tilde{g}_M f_{N'} \approx_p \tilde{g}_M' f_{N_M'}\quad \text{in} \quad M.
\]
Hence, \((\tilde{g}_M f_{N_M})_p \cong_p (\tilde{g}_M' f_{N_M'})_p\). \(\square\)

By this lemma, we can define \(\left[(g_M)\right]_p \left[(f_N)\right]_p = \left[(\tilde{g}_M f_{N_M})\right]_p\) for two mutational proper \(n\)-approximative maps \((f_N)\) of \(X\) towards \(Y\) and \((g_M)\) of \(Y\) towards \(Z\). By Lemma 3.1, the composition of proper \(n\)-homotopy classes of proper \(n\)-approximative maps can be defined. Thus we obtain the category of closed sets in \(Q \setminus \{0\}\) with the proper \(n\)-homotopy classes of proper \(n\)-approximative maps, which is denoted by \(\mathcal{A}_p^n\).

We have a natural functor \(R : \mathcal{F}_p^n \rightarrow \mathcal{A}_p^n\) defined by \(R(X) = X\) for all \(X \in \text{Ob}\mathcal{F}_p^n\) and \(R((f_A)) = \left[(f_A|X)\right]_p\) for all \((f_A) \in \text{Mor}\mathcal{F}_p^n\). By Lemma 3.2, \(R\) induces a categorical embedding \(\tilde{R} : \mathcal{F}_{BS}^n \rightarrow \mathcal{A}_p^n\), that is, \(\tilde{R}(X) = X\) for all \(X \in \text{Ob}\mathcal{F}_{BS}^n\), and \(\tilde{R}((f_A)) = \left[(f_A|X)\right]_p\) for all \((f_A) \in \text{Mor}\mathcal{F}_{BS}^n\). Thus we have the following:

**Theorem 3.4.** The category \(\mathcal{F}_{BS}^n\) is isomorphic to a subcategory of \(\mathcal{A}_p^n\) by a natural functor. \(\square\)

It is unknown whether \(\tilde{R} : \text{Mor}\mathcal{F}_{BS}^n \rightarrow \text{Mor}\mathcal{A}_p^n\) is surjective or not. The answer would be positive if every proper approximative map \((f_A)\) of \(X\) towards \(Y\) would extend to some proper fundamental net \((\tilde{f}_A)\) from \(X\) to \(Y\) (i.e., \(\tilde{f}_A|X = f_A\)). However, the following example shows that this is not true even in the case when \(X\) is compact:

**Example 3.5.** There exists a proper approximative map \((f_n)\) of a compactum \(X\) towards \(Y\) which cannot be extended to a proper fundamental net (sequence) \((\tilde{f}_n)\) from \(X\) to \(Y\).

**Proof.** We define

\[
\begin{align*}
X &= \{2^{-k} \mid k \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}, \\
Y &= \mathbb{R} \times \{2^{-k} \mid k \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} (2n - 1, 2n) \times \{0\} \subset \mathbb{R}^2 \quad \text{and} \\
E &= \mathbb{R} \times (0, 1) \cup \bigcup_{n \in \mathbb{N}} (2n - 1, 2n) \times \{0\} \subset \mathbb{R}^2.
\end{align*}
\]

Then \(X\) and \(Y\) are closed sets in locally compact AR's \([0, 1]\) and \(E\), respectively. We may replace \(Q \setminus \{0\}\) by \([0, 1]\) and \(E\). For each \(n \in \mathbb{N}\), let \(f_n : X \rightarrow Y\) be the map defined by \(f_n(x) = (2n - \frac{1}{2}, x)\). To see that \((f_n)\) is a proper approximative map of \(X\) towards \(Y\) in \(E\), let \(V\) be a neighborhood of \(Y\) in \(E\). Choose \(k_1 < k_2 < k_3 \cdots \in \mathbb{N}\) so that \(\bigcup_{n \in \mathbb{N}} [2n - \frac{2}{3}, 2n - \frac{1}{3}] \subset [0, 2^{-k_n}] \subset V\). For each \(n \in \mathbb{N}\), we define \(g_n : X \rightarrow Y\) (resp. \(g'_n : X \rightarrow Y\)) by \(g_n(x) = (2n - \frac{1}{2}, x)\) (resp. \(g'_n(x) = (1 + \frac{1}{2}, x)\)) if \(x > 2^{-k_n}\), and \(g_n(x) = (2n - \frac{1}{2}, 2^{-k_n})\) (resp. \(g'_n(x) = (1 + \frac{1}{2}, 2^{-k_n})\)) if \(x \leq 2^{-k_n}\). As is easily observed, \(f_n \simeq_p g_n \simeq_p g'_n \simeq_p f_1\) in \(V\) for each \(n \in \mathbb{N}\). Hence \((f_n)\) is a proper approximative map of \(X\) towards \(Y\) in \(E\).

Assume that \((f_n)\) extends to a proper fundamental net (sequence) \((\tilde{f}_n)\) from \(X\) to \(Y\) in \(([0, 1], E)\). Now, choose \(k_1 < k_2 < k_3 \cdots \in \mathbb{N}\) so that \(\tilde{f}_n([0, 2^{-k_n}] \subset \mathbb{R}) \subset \mathbb{R}\).
Then \( V_0 \) is a closed neighborhood of \( Y \) in \( E \). We have a closed neighborhood \( U_0 \) of \( X \) in \([0, 1]\) and \( n_0 \in \mathbb{N} \) such that \( f_n|U_0 \simeq_p f_{n_0}|U_0 \) in \( V_0 \) for all \( n > n_0 \). In particular, \( f_n(U_0) \subset V_0 \) for all \( n > n_0 \). Choose \( n \in \mathbb{N} \) so that \([0, 1) \subset U_0\). Then

\[
\bar{f}_n([0, 2^{-kn}]) \subset V_0 \cap [2n - \frac{2}{3}, 2n - \frac{1}{3}] \times [0, 1]
\]

\[
= [2n - \frac{2}{3}, 2n - \frac{1}{3}] \times \left( \bigcup_{k=1}^{n} \left[ \frac{3}{2}2^{-kn-1}, \frac{3}{2}2^{-k} \right] \right).
\]

On the other hand,

\[
\bar{f}_n(0) \in [2n - \frac{2}{3}, 2n - \frac{1}{3}] \times [0, \frac{3}{4}2^{-kn-1}]
\]

and

\[
\bar{f}_n(2^{-kn}) \in [2n - \frac{2}{3}, 2n - \frac{1}{3}] \times [\frac{3}{2}2^{-kn-1}, \frac{3}{2}2^{-k}].
\]

Since \( \bar{f}_n([0, 2^{-kn}]) \) is connected, this is a contradiction. \( \square \)

In the above, one should remark that \( (f_n)_{n \in \mathbb{N}} \simeq_p f_i \) (as proper approximative maps). Let \( f_i : [0, 1] \to E \) be an extension of \( f_i \). Then \( f_i \) is a proper fundamental net from \( X \) to \( Y \) in \(([0, 1), E)\) indexed by a singleton.

4. A categorical isomorphism between \( \mathcal{S}_p^n \) and \( \mathcal{A}_p^n \)

In this section, we show that the categories \( \mathcal{S}_p^n \) and \( \mathcal{A}_p^n \) are isomorphic to each other.

**Theorem 4.1.** There exists a categorical isomorphism \( \Lambda_p^n : \mathcal{S}_p^n \to \mathcal{A}_p^n \) which is the identity on the class of objects, hence the categories \( \mathcal{S}_p^n \) and \( \mathcal{A}_p^n \) are isomorphic to each other.

**Proof.** Without loss of generality, the category \( \mathcal{S}_p^n \) is restricted to closed sets in \( Q \setminus \{0\} \). (cf. [12, Appendix 2]).

We will define a functor \( \Lambda_p^n : \mathcal{S}_p^n \to \mathcal{A}_p^n \) as follows: \( \Lambda_p^n(X) = X \) for all \( X \in \text{Ob} \mathcal{S}_p^n \). Let \( F : X \to Y \) be a proper \( n \)-shaping (i.e., \( F \in \text{Mor} \mathcal{S}_p^n \)). For each \( N \in \text{Nbd}(Y) \), choose \( f_N \in F_N([i_Y^n]_p) \), where \( i_Y^n : Y \subset N \) is the inclusion. Then \( (f_N) \) is a mutational proper \( n \)-approximative map of \( X \) towards \( Y \). In fact, for \( N' \subset N \in \text{Nbd}(Y) \),

\[
[i_{N,N'}^n]_p^p[f_{N'}]_p^n = \pi_Y^n([i_{N,N'}^n]_p^n)(F_{N'}([i_Y^n]_p^n))
\]

\[
= F_N(\pi_Y^n([i_{N,N'}^n]_p^n)([i_Y^n]_p^n))
\]

\[
= F_N([i_{N,N'}^n]_p^n[i_Y^n]_p^n) = F_N([i_Y^n]_p^n) = [f_N]_p^n,
\]

\( \square \)
that is, $f_{N'} \simeq_p f_N$ in $N$. We call $(f_N)$ a proper $n$-approximative map associated with $F$.

If $(f_N)$ is another proper $n$-approximative map associated with $F$, then $(f_N) \simeq_p (f_N)$ because $f_N \simeq_p f_N'$ in $N$ for all $N \in \text{Nbd}(Y)$. Then we define $A_p^n(F) = [(f_N)]^n_p$, where $(f_N)$ is a proper $n$-approximative map associated with $F$.

Let $G : Y \to Z$ be a proper $n$-shaping and $(g_M)$ a proper $n$-approximative map associated with $G$. The proper $n$-shaping $GF : X \to Z$ is defined as the functor $F \circ G : \pi^X_n \to \pi^Z_n$. For each $M \in \text{Nbd}(Z)$, $g_M \in G_M([i_M^n])$ extends to a proper map $\tilde{g}_M : N_M \to M$ from some $N_M \in \text{Nbd}(Y)$. Then $[(\tilde{g}_M f_{N_M})]_p^n = [(g_M)]_p^n [(f_N)]_p^n$. On the other hand,

$$[[\tilde{g}_M f_{N_M}]]_p^n = [\tilde{g}_M]_p^n [f_{N_M}]_p^n = \pi_X^n([-\tilde{g}_M]_p^n) F_{N_M}([i_{N_M}^n]) = F_M(\pi^n_X([-\tilde{g}_M]_p^n) [i_{N_M}^n])$$

$$= F_M([-\tilde{g}_M]_p^n) F_M([i_{N_M}^n]) = F_M([g_M]_p^n)$$

$$= F_M(G_M([i_M^n])) = (GF)_M([i_M^n])$$

that is, $\tilde{g}_M f_{N_M} \in (GF)_M([i_M^n])$. Therefore, $(\tilde{g}_M f_{N_M})$ is a proper $n$-approximative map associated with the proper $n$-shaping $GF$. Thus we have

$$A_p^n(GF) = [(\tilde{g}_M f_{N_M})]_p^n = [(g_M)]_p^n [(f_N)]_p^n = A_p^n(G) A_p^n(F).$$

We show that the function $A_p^n : \text{Mor}_\mathscr{P}^n \to \text{Mor}_\mathscr{P}^n$ is bijective. Let $(f_N)$ be a mutational proper $n$-approximative map of $X$ towards $Y$. For each polyhedron $P$, any proper map $h : Y \to P$ extends to a proper map $\tilde{h} : N_h \to P$ for some $N_h \in \text{Nbd}(Y)$. Then we have $\tilde{h} f_{N_h} : X \to P$. Let $P'$ be another polyhedron, let $k : P \to P'$, $h' : Y \to P'$ and $\tilde{h}' : N_h' \to P'$ be proper maps such that $N_h' \in \text{Nbd}(Y)$, $h' \simeq_p kh$ and $\tilde{h}'|_Y = h$, and let $(g_N)$ be another mutational proper $n$-approximative map of $X$ towards $Y$ such that $(f_N) \simeq_p (g_N)$, whence $f_N \simeq_p g_N$ in $N$ for all $N \in \text{Nbd}(Y)$. Then, by choosing $N \in \text{Nbd}(Y)$ so that $N \subset N_h \cap N_h'$ and $\tilde{h}'|_N \simeq_p k \tilde{h} f_{N_h}$, we have

$$k \tilde{h} f_{N_h} \simeq_p k \tilde{h} f_{N_h} = (k \tilde{h}|_N) f_N \simeq_p (\tilde{h}'|_N) f_N = \tilde{h}' f_N \simeq_p \tilde{h}' g_N \simeq_p \tilde{h}' g_N,$$

that is, $[k]^n_p [\tilde{h} f_{N_h}]_p^n = [\tilde{h}' g_N]_p^n$. In the case when $P' = P$ and $k = \text{id}_P$, this shows that $[\tilde{h} f_{N_h}]_p^n$ depends only on $[f_N]_p^n$ and $[h]_p^n$. By replacing $(g_N)$ by $(f_N)$, the above shows that $\pi_Y([k]^n_p ([h]^n_p]) = [k h]^n_p = [h']^n_p$ implies $\pi_X([k]^n_p ([\tilde{h} f_{N_h}]_p^n]) = [\tilde{h} f_{N_h}]_p^n$. Therefore, we can define the natural transformation $S_p^n([f_N]_p^n) = F : \pi_Y \to \pi_X$ (i.e., the proper $n$-shaping $F : X \to Y$) by $F_p([h]^n_p) = [\tilde{h} f_{N_h}]_p^n$ for each $[h]^n_p \in [Y, P]_p^n = \pi^n_Y(P)$, where $\tilde{h} : N_h \to P$ is an extension of $h$ over some $N_h \in \text{Nbd}(Y)$. Thus, we have a function $S_p^n : \text{Mor}_\mathscr{P}^n \to \text{Mor}_\mathscr{P}^n$.

In the above, $F_N([i_M^n]) = [f_N]_p^n$ for all $N \in \text{Nbd}(Y)$, because $\text{id}_N$ is an extension of $i_M^n$. This means that $(f_N)$ is associated with the proper $n$-shaping $F$, that is, $A_p^n(S_p^n([f_N])_p^n) = A_p^n(F) = ([f_N])_p^n$. Therefore $A_p^n \circ S_p^n = \text{id}$. To see that $S_p^n \circ A_p^n = \text{id},$
it suffices to show that $F = G$ in the case when $(f_N)$ is associated with a proper $n$-shaping $G : X \to Y$. In fact, for each polyhedron $P$ and each proper map $h : Y \to P$, since

$$[h]_p^n = [\tilde{h}]_p^n = [\tilde{h}]_p^n(i_{N_h})_p = \pi_X([\tilde{h}]_p^n)([i_{N_h}]_p^n)$$

and $f_{N_h} \in G_{N_h}([i_{N_h}]_p^n)$, we have

$$G_p([h]_p^n) = \pi_X([\tilde{h}]_p^n)G_{N_h}([i_{N_h}]_p^n) = [\tilde{h}]_p^n(f_{N_h})_p = \tilde{h}f_{N_h})_p = f_P([h]_p^n).$$

Hence $G_P = F_P$ for each polyhedron $P$. This completes the proof. \(\square\)

Combining Theorems 3.4 and 4.1, we have

**Corollary 4.2.** The category $\mathcal{S}_n^n$ is isomorphic to a subcategory of $\mathcal{S}_p^n$ by a functor which is the identity on the class of objects. \(\square\)

### 5. Proper $n$-shapes of $(n + 1)$-dimensional spaces

For a category $\mathcal{C}$ of spaces, let $\mathcal{C}(k)$ denote the subcategory of $\mathcal{C}$ whose objects are spaces of dimension $\leq k$. It follows from Corollary 4.2 that the category $\mathcal{S}_n^n(\mathcal{B}_S) \subset \mathcal{S}_p^n$ is isomorphic to a subcategory of $\mathcal{S}_p^n(\mathcal{B}_S)$.

In the case when $\dim X \leq n + 1$, any proper map $f : X \to P$ of $X$ to a polyhedron $P$ is properly homotopic to a proper map $f' : X \to P^{(n+1)}$ to the $(n+1)$-skeleton of $P$, and if two proper maps $f, g : X \to P^{(n+1)}$ are properly $n$-homotopic in $P$ then they are properly $n$-homotopic in $P^{(n+1)}$. Then, objects of $\mathcal{H}_n^n\mathcal{P}(n+1)$ are spaces having the proper $n$-homotopy type of polyhedra of dimension $\leq n + 1$. Each $\mathcal{L}C^n$ locally compact space of dimension $\leq n + 1$ is properly $n$-homotopic to some polyhedron $P$ with $\dim P \leq n + 1$ [7, Proposition 1.5] (cf. [9, Proposition 4.1.10]). Each locally compact space of dimension $\leq n + 1$ can be embedded in $\mu^{n+1} \setminus \{pt\}$ as a closed set, where $\mu^{n+1}$ is the $(n+1)$-dimensional universal Menger compactum. Each neighborhood of a closed set $X$ in $Q \setminus \{0\}$ contains some closed neighborhood of $X$ which is $\mathcal{L}C^n$. Similarly to Theorem 2.1, we have the following

**Theorem 5.1.** The category $\mathcal{H}_n^n\mathcal{P}(n+1)$ is dense in $\mathcal{H}_n^n(n+1)$.

Then the category $\mathcal{S}_n^n\mathcal{P}(n+1)$ is none other but the shape category defined by the pair $(\mathcal{H}_n^n(n+1), \mathcal{H}_n^n\mathcal{P}(n+1))$.

By replacing $Q \setminus \{0\}$ by $\mu^{n+1} \setminus \{pt\}$ and letting $\text{Nbd}(X)$ be the directed set of all closed $\mathcal{L}C^n$ neighborhoods of $X$ in $\mu^{n+1} \setminus \{pt\}$, we can define the proper $n$-shape category of Ball-Sher type whose objects are locally compact spaces of dimension $\leq n + 1$. Here this category is denoted by $\mathcal{S}_n^n\mathcal{B}(n+1)$. The following is due to Akaike [2]:

**Theorem 5.2.** The category $\mathcal{S}_n^n(\mathcal{B}_S)(n+1)$ is isomorphic to a subcategory of $\mathcal{S}_n^n\mathcal{B}(n+1)$.

\(\text{6}\) The $k$-skeleton of a polyhedron $P$ is the underlying space of the $k$-skeleton of the simplicial complex triangulating $P$.  

In the above, it is unknown whether $\mathcal{S}_{BS}^{n}(n+1)$ is isomorphic to $\mathcal{S}_{BS}^{n}(n+1)$ itself. Note that every $n$-dimensional locally compact space can be embedded in an $(n+1)$-dimensional locally compact AR as a closed set ([11]). Then it is easy to prove that $\mathcal{S}_{BS}^{n}(n)$ is naturally isomorphic to $\mathcal{S}_{BS}^{n}(n)$.

Since Lemma 2.2 is valid in the case when $\dim X \leq n+1$ and $Y$ is LC$^n$, we can define the category $\mathcal{S}_{BS}^{n}(n+1)$ as in §2. Similarly to Theorem 3.4, we can show the following

**THEOREM 5.3.** The category $\mathcal{S}_{BS}^{n}(n+1)$ is isomorphic to a subcategory of $\mathcal{S}_{BS}^{n}(n+1)$.

In the above, it is unknown whether $\mathcal{S}_{BS}^{n}(n+1)$ is isomorphic to $\mathcal{S}_{BS}^{n}(n+1)$ itself. By the above remark on $\mathcal{S}_{BS}^{n}(n+1)$, the proof of Theorem 4.1 is valid for $\mathcal{S}_{BS}^{n}(n+1)$ and $\mathcal{S}_{BS}^{n}(n+1)$. Hence we have

**THEOREM 5.4.** The category $\mathcal{S}_{BS}^{n}(n+1)$ is isomorphic to $\mathcal{S}_{BS}^{n}(n+1)$.

Summarizing the above, we have the following relationship:

$$\mathcal{S}_{BS}^{n}(n+1) \subset \mathcal{S}_{BS}^{n}(n+1) \subset \mathcal{S}^{n}(n+1) = \mathcal{S}_{BS}^{n}(n+1).$$

**Acknowledgment.** The author would like to thank Y. Akaike for his helpful comments and Sibe Mardešić for his information about the paper [13].

**REFERENCES**