Variational inequality and complementarity problem in locally convex Hausdorff topological vector space

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Abstract. The purpose of this paper is to study variational inequality and complementarity problem in a locally convex Hausdorff topological vector space.

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Let $X$ be a real locally convex Hausdorff topological vector space (lc Htvs) with a continuous seminorm $p$ and let $X^*$ be its dual. Let $K$ be a closed convex subset of $X$ and $T : X \to X^*$ a mapping. Let $\eta : K \times K \to X$.

Definition 1. $T$ is said to be

(i) $\eta$-monotone if $(Tx - Ty, \eta(x, y)) \geq 0$, $\forall x, y \in K$,

(ii) strictly $\eta$-monotone if $(Tx - Ty, \eta(x, y)) > 0$, $\forall x, y \in K, x \neq y$,

(iii) strongly $\eta$-monotone if there exists a constant $C > 0$ such that

$$(Tx - Ty, \eta(x, y)) \geq C[p(\eta(x, y))]^2,$$

(iv) $\eta$-coercive if $(Tx, Ty)/p(\eta(x, x)) \to \infty$ as $p(\eta(x, x)) \to \infty$.

We consider the nonlinear variational inequality (NVI) which is defined as follows:

$$x \in K : (Tx, \eta(y, x)) \geq 0 \forall y \in K. \tag{1}$$

Another NVI can be stated as follows:

$$x \in K : (Ty, \eta(y, x)) \geq 0 \forall y \in K. \tag{2}$$

Let $S_1$ and $S_2$ denote the solutions of (1) and (2) respectively. These can be generalized as follows:

$$x \in K : (Tx - Sx, \eta(y, x)) \geq 0 \forall y \in K \tag{3}$$

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\[ x \in K : (Ty - Sy, \eta(y, x)) \geq 0 \forall y \in K. \] (4)

We have

**Theorem 1.** If \( \eta \) is antisymmetric and \( T \) is strictly \( \eta \)-monotone, then \( S_1 \) is empty or singleton.

**Proof.** Assume that \( x_1, x_2 \in S_1 \). Then
\[
(Tx, \eta(x_2, x_1)) \geq 0,
\]
and \((Tx_2, \eta(x_1, x_2)) \geq 0\). From (1), \( \eta \) is antisymmetric. We have \((Tx_1, \eta(x_1, x_2)) \leq 0\). Hence \((Tx_1 - Tx_2, \eta(x_1, x_2)) \leq 0\). Because \( T \) is strictly \( \eta \)-monotone, this is impossible unless \( x_1 = x_2 \) and this completes the proof. \( \square \)

**Theorem 2.** Let \( T \) be \( \eta \)-monotone and semicontinuous, \( \eta(x, x) = 0, \eta \) positive homogeneous. Then \( S_1 = S_2 \).

**Proof.** Let \( x \in S_1 \). Since \( T \) is \( \eta \)-monotone, \((Ty, \eta(y, x)) \geq (Tx, \eta(y, x)) \geq 0 \Rightarrow x \in S_2 \). Let \( y \in K \). Since \( K \) is convex, for \( 0 < t < 1 \), \( y_t = (1-t)x + ty = x - t(y - x) \in K \). Hence \((Ty_t, \eta(y_t, x)) \geq 0\). But \( \eta(y_t, x) = -\eta(x, y) \). Now letting \( t \to 0 \) we get \((Tx, -\eta(y, x)) \geq 0\). Thus \( x \in S_1 \) and this completes the proof. \( \square \)

We introduce the concept of the complementarity problem \((CP)\) in real lcHtvs. Let \( K \) be a closed convex cone in \( X \). Let \( X^* \) be the subset of \( X^* \) defined by \( K^* = \{ y \in X^* : \eta(y, x) \geq 0, x \in K \} \). Then \( x \in K, Tx \in K^*, (Tx, \eta(x, x)) = 0 \) will be called the generalized complementarity problem \((GCP)\). Let \( C \) denote the set of all solutions of \( GCP \).

**Theorem 3.** Let \( K \) be a closed convex cone, \( \eta \) is antisymmetric, then \( S_1 = C \).

**Proof.** Let \( x \in S_1 \). Take \( y = x \). Then \((Tx, \eta(x, x)) \geq 0\). Since \( \eta \) is antisymmetric \((Tx, \eta(x, x)) \leq 0 \Rightarrow (Tx, \eta(x, x)) = 0\). Thus \( x \in C \) and hence \( S_1 \subseteq C \). Clearly \( C \subseteq S_1 \) and this completes the proof. \( \square \)

We shall now prove the existence theorem for variational inequality in lcHtvs. For this purpose we need the following results which are due to Tarafdar [2].

**Lemma 1.** Let \( K \) be a nonempty compact and convex subset of a Hausdorff tvs \( X \) and \( S : K \to P(K) \) be a multivalued mapping such that

(i) for each \( x \in K \), \( Sx \) is a nonempty convex subset of \( K \),
(ii) for each \( y \in K \), \( S_y^{-1} \{ s \in K : y \in Sx \} \) contains an open subset \( U_y \) of \( K \) where \( U_y \) may be empty.

(iii) \( \bigcup \{ U_y : y \in K \} = K \).

Then there exists an element \( x_0 \in K \) such that \( x_0 \) belongs to \( Sx_0 \).

**Theorem 4.** Let \( K \) be a nonempty convex subset of \( X \) and let \( T : K \to X^* \) be strongly \( \eta \)-monotone. Let \( \eta \) be continuous. Suppose \( \eta \) satisfies \( \eta(y, x) = \eta(y, z) + \eta(z, x) \). Then NVI(1) has a solution in \( K \).

**Proof.** Suppose NVI has no solution in \( K \). Then for each \( x \in K \), there exists a \( y \in K \) such that \((Tx, \eta(y, x)) < 0\). Define a multivalued map \( F : K \to P(K) \) by \( F(x) = \{ y \in K : (Tx, \eta(y, x)) < 0 \} \). Clearly \( F(x) \) is nonempty and convex for each \( x \in K \). It follows that \( F^{-1}(y) = x \in K : (Tx, \eta(y, x)) < 0 \). Since \( T \) is strongly \( \eta \)-monotone, for each \( y \in K \), the complement of \( F^{-1}(y) \) is in \( K \), i.e.

\[
(F^{-1}(y))^c = K - F^{-1}(y) = \{ x \in K : (Tx, \eta(y, x)) \geq 0 \} \subseteq \{ x \in K : (Ty, \eta(y, x)) \geq C[(p(\eta)(y, x))^2] \} = H(y).
\]

x \in K : (Ty - Sy, \eta(y, x)) \geq 0 \forall y \in K. (4)
It is easy to show that \( H(y) \) is convex. We now show that \( H(y) \) is relatively closed in \( K \). For this purpose, let \( \{x_\alpha\} \) be a Moore-Smith sequence in \( H(y) \). Then \((Ty, \eta(y, x_\alpha)) \geq C[p(\eta(y, x_\alpha))]^2\). Let \( x_\alpha \to x \in K \). We claim that \( x \in H(y) \).

Let \( x_\alpha \to x \in K \). We claim that \( x \in H(y) \). Since \( \eta \) is continuous, \( \eta(X \times X) \) is dense in \( X \), \( p \) is a continuous seminorm. We have

\[
(Ty, \eta(y, x_\alpha)) = (Ty, \eta(y, x_\alpha)) + (Ty, \eta(x_\alpha, x)) \\
\geq C[p(\eta(y, x_\alpha))]^2 + (Ty, \eta(x_\alpha, x)) \\
\geq C[p(\eta(y, x_\alpha))]^2 
\]

\[\implies x \in H(y).\]

Now

\[K - H(y) = \{x \in K : (Ty, \eta(y, x)) < C(p(\eta(y, x)))^2\} \subseteq \{x \in K : (Tx, \eta(y, x)) < 0\} = F^{-1}(y).\]

This implies for each \( y \in K \) there is an element \( x \in K \) such that \( \bigcup(K-H(y)) = K \). But by Lemma 1, there exists an element \( x \in K \) such that \( x \in F(x) \), which means \( 0 > (Tx, \eta(x, x)) = 0 \). This contradiction completes the proof.

Let \( D \) be a nonempty compact, convex subset of \( X \) and \( F : D \to Y = X^* \). The following existence theorem on variational inequality was established by Karamardian [1].

**Proposition 1.** Let the mapping \((u, v) \to (u, F(v))\) be continuous on \( D \times D \). Then there exists a point \( \bar{x} \in D \) such that for all \( x \in D \), \( (x - \bar{x}, F(\bar{x})) \geq 0 \).

We now obtain the following theorem on the complementarity problem, by using the results of Karamardian stated above.

**Theorem 5.** Let \( K \) be a closed and convex cone in \( X \) and let \( F : K \to Y = X^* \) be such that

(i) the mapping \((u, v) \to (u, F(v))\) is continuous on \( K \times K \),

(ii) there exists \( \bar{x} \in K \) such that \( F(\bar{x}) \in \text{int}K^* \).

Then there exists \( x \in X \) such that \( \bar{x} \in K \), \( F(\bar{x}) \in K^* \) and \( (\bar{x}, F(\bar{x})) = 0 \).

**Proof.** For any \( u \in K \) define

\[
D_u = \{x \in D : <x, Fx> \leq <u, Fx>\} \\
D_u^0 = \{x \in D : <x, Fx> < <u, Fx>\} \\
S_u = \{x \in D : <x, Fx> = <u, Fx>\}.
\]

For each \( u \in K \), \( D_u \) is convex. From the continuity assumption it follows that \( D_u \) is a closed subset of the compact convex set \( D \) for each \( u \in K \) and hence is compact. Thus for each \( u \in K \), \( D_u \) is a nonempty, compact, convex set in \( X \), therefore by Proposition 1 it follows that for each \( u \in K \), there is \( x_u \in D_u \) such that

\[
<y - x_u, Fx_u \geq 0, \text{ for all } y \in D_u.\]
Since $0 \in D_u$, $<x_u, Fx_u> \leq 0$.

**Case 1:** Let $x_u \in D_u^0$. Then there is a $\lambda > 1$ such that $\lambda x_u \in S_u \subset D_u$. Then we have $<x_u, Fx_u> \leq <\lambda x_u, Fx_u> = \lambda <x_u, Fx_u>$. Since $<x_u, Fx_u> \leq 0$, it is impossible unless $<x_u, Fx_u> = 0$. Thus (6) holds.

**Case 2.** Let $x_u \in S_u$ for all $u \in K$. Let $u \in K$ be such that $Fx_u \in \text{int}K^*$. Then $<u, Fx_u> > 0$. By the hypothesis there is $x \in K$ such that $Fx \in \text{int}K^*$. Thus for this $x$ we have $<x, Fx> > 0$. Choose $u$ such that $<u, Fx> < x, Fx> > 0$. Thus $x \in D_u^0$. Now $x_u \in S_u$. Hence $<x_u, Fx_u> < u, Fx_u> > 0$. This contradicts $<x_u, Fx_u> \leq 0$ and thus case 2 cannot occur and this completes the proof.

**Remark 1.** Observe that in the above theorem $D_u$ is convex and (compact) if $D$ is convex and (compact): $D_u$ need not be convex if $D$ is any compact (non-convex) set. For example, take $F : R^+ \to R$, $F(x) = \sin x$, $D = [\pi/4, 2\pi]$. Then

$$D_u = \{x \in D : (x, Fx) \leq (u, Fx)\} = \{x : x \sin x \leq u \sin x\}.$$  

For $u = \pi/2$, $D_u = [\pi/4, \pi/2] \cup [\pi, 2\pi]$ which is not convex.

**References**
