On the periodic solutions of certain fourth and fifth order vector differential equations

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Abstract. The aim of the present paper is to establish some sufficient conditions which ensure that equations (1.1) and (1.2) have no periodic solution other than the trivial solution \( X = 0 \).

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1. Introduction

The problems related to the periodic behaviour of solutions of a higher order nonlinear scalar differential equation have been treated by many investigators. The papers achieved in Ezeilo [4], Tiryaki [9], Bereketoğlu [2, 3] and Tejumola [8] can be given as good examples on this subject. However, with respect to our observations, only a few studies were carried out on the same topic for the solutions of ordinary nonlinear vector differential equations of higher orders. In this aspect studies fulfilled by Ezeilo [5] and Tunç [13] could be given as examples.

In this paper, taking into account the results obtained for the ordinary nonlinear scalar differential equations

\[
x^{(4)} + f_1(x) \dddot{x} + f_2(x) \ddddot{x} + f_3(x) + f_4(x) = 0
\]

and

\[
x^{(5)} + b_1 x^{(4)} + g_1(x, \dot{x}, \ddot{x}, x^{(4)}) \dddot{x} + g_2(x) \ddddot{x} + g_3(x, \dot{x}, \ddot{x}, x^{(4)}) + g_4(x) = 0,
\]

by Tiryaki [9], we establish two new results on the same topic for the nonlinear vector differential equations as follows:

\[
X^{(4)} + \Phi(X) \dddot{X} + \Psi(X) \ddddot{X} + F(X) + G(X) = 0 \quad (1.1)
\]

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and

\[ X^{(5)} + AX^{(4)} + \Phi(X, X, X, X^{(4)}) \dot{X} + \Psi(X) \dot{X} \]
\[ + \Omega(X, X, X, X^{(4)}) \dot{X} + \Theta(X) = 0. \]  

(1.2)

in which \( X \in \mathbb{R}^n; A \) is a constant \( n \times n \)-symmetric matrix; \( \Phi, \Psi \) and \( \Omega \) are continuous \( n \times n \)-symmetric matrices depending, in each case, on the arguments shown; \( F, G, \Theta : \mathbb{R}^n \to \mathbb{R}^n \) are continuous \( n \)-vector functions. It will be assumed

\[ F(0) = 0, \quad G(0) = 0 \]  

(1.3)

and

\[ \Omega(X, 0, Z, U, V) = 0, \quad \Theta(0) = 0. \]  

(1.4)

for an arbitrary value of \( X, Z, U \) and \( V \). Let \( J_G(X) \) denote the Jacobian matrix corresponding to the function \( G(X) \), that is, \( J_G(X) = \frac{\partial g}{\partial x} \), \( (i, j = 1, 2, ..., n) \) where \( (x_1, x_2, ..., x_n) \) and \( (g_1, g_2, ..., g_n) \) are the components of \( X \) and \( G \), respectively. Other than these, it will also be assumed that the Jacobian matrices \( J_G(X) \) exist and are symmetric and continuous. The symbol \( \langle X, Y \rangle \) is used to denote the usual scalar product in \( \mathbb{R}^n \) for given any \( X, Y \) in \( \mathbb{R}^n \), thus \( \|X\|^2 = \langle X, X \rangle \). The matrix \( A \) is said to be negative-definite, when \( \langle AX, X \rangle < 0 \) for all non-zero \( X \) in \( \mathbb{R}^n \), and \( \lambda_i(A) (i = 1, 2, ..., n) \) are the eigenvalues of the \( n \times n \)-matrix \( A \).

In what follows we use the following differential systems which are equivalent to the equations (1.1) and (1.2):

\[ \dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = U, \quad \dot{U} = -\Phi(Z)U - \Psi(Y)Z - F(Y) - G(X) \]  

(1.5)

and

\[ \dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = U, \quad \dot{U} = V, \quad \dot{V} = -AV - \Phi(X, Y, Z, U, V)U - \Psi(Y)Z - \Omega(X, Y, Z, U, V)Y - \Theta(X), \]  

(1.6)

respectively.

2. Main result

We shall establish here the following theorems.

**Theorem 1.** In addition to the basic assumptions on the \( \Phi, \Psi, F \) and \( G \), suppose that there are constants \( a_2 \) and \( a_4 \) with \( a_4 > \frac{1}{4}a_2^2 \) such that

(i) \( 0 \leq \lambda_i(\Psi(Y)) \leq a_2 \) for all \( Y \in \mathbb{R}^n, (i = 1, 2, ..., n) \)
(ii) \( \lambda_i(J_G(X)) \geq a_4 \) for all \( X \in \mathbb{R}^n, (i = 1, 2, \ldots, n) \).

Then equation (1.1) has no periodic solution whatsoever other than \( X = 0 \) for all arbitrary \( \Phi \).

**Theorem 2.** In addition to the basic assumptions on the \( A, \Phi, \Psi, \Omega \) and \( \Theta \), suppose that

(i) \( \Theta(X) \neq 0 \) for \( X \neq 0 \)

(ii) \( \lambda_i(\Omega(X, Y, Z, U, V)) \geq \frac{1}{2} [\lambda_i(\Phi(X, Y, Z, U, V))]^2 \) for arbitrary \( X, Y, Z, U, V \) then the equation (1.2) has no periodic solution whatsoever other than \( X = 0 \) for all arbitrary \( A, \Psi \).

Now, we dispose of some well known algebraic results which will be required in the proof of theorems. The first of these is a quite standard one:

**Lemma 1.** Let \( A \) be a real symmetric \( n \times n \) matrix and

\[
a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1, 2, \ldots, n), \text{ where } a', a \text{ are constants.}
\]

Then

\[
a'\langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle
\]

and

\[
a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle.
\]

**Proof.** See [7].

**Lemma 2.** Let \( Q, D \) be any two real \( n \times n \) commuting symmetric matrices.

Then

(i) The eigenvalues \( \lambda_i(QD) \) \( (1, 2, \ldots, n) \) of the product matrix \( QD \) are real and satisfy

\[
\max_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D)
\]

(ii) The eigenvalues \( \lambda_i(Q + D) \) \( (1, 2, \ldots, n) \) of the sum of matrices \( Q \) and \( D \) are real and satisfy

\[
\left\{ \max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D) \right\} \geq \lambda_i(Q + D) \geq \left\{ \min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D) \right\}.
\]

**Proof.** See [1].

**Proof of the Theorem 1.** Let \( (X, Y, Z, U) = (X(t), Y(t), Z(t), U(t)) \) be an arbitrary \( \alpha \)-periodic solution of (1.5), that is

\[
(X(t), Y(t), Z(t), U(t)) = (X(t + \alpha), Y(t + \alpha), Z(t + \alpha), U(t + \alpha)) \quad (2.1)
\]
for some $\alpha > 0$. It will be shown that, subject to the conditions in Theorem 1,

$$X = Y = Z = U = 0.$$  

Our main tool in the proof of Theorem 1 is the function $\Gamma = \Gamma(X, Y, Z, U)$ given by:

$$\Gamma = \int_0^1 \langle \sigma \Phi(\sigma Z), Z \rangle d\sigma + \int_0^1 \langle \Psi(\sigma Y), Z \rangle d\sigma + \langle U, Z \rangle$$

$$+ \langle Y, G(X) \rangle + \int_0^1 \langle F(\sigma Y), Y \rangle d\sigma.$$  

(2.2)

Consider the function

$$\psi(t) = \Gamma(X(t), Y(t), Z(t), U(t)).$$

Since $\Gamma$ is continuous and $X, Y, Z, U$ are periodic in $t$, $\psi(t)$ is clearly bounded. An elementary differentiation will show that

$$\Gamma = \frac{d}{dt} \int_0^1 \langle \sigma \Phi(\sigma Z), Z \rangle d\sigma + \frac{d}{dt} \int_0^1 \langle \Psi(\sigma Y), Z \rangle d\sigma + \langle U, Z \rangle - \langle Z, \Phi(Z) \rangle$$

$$- \langle Z, \Psi(Y) \rangle - \langle Z, F(Y) \rangle + \langle Y, G(X) \rangle + \frac{d}{dt} \int_0^1 \langle F(\sigma Y), Y \rangle d\sigma.$$  

(2.3)

But

$$\frac{d}{dt} \int_0^1 \langle F(\sigma Y), Y \rangle d\sigma = \int_0^1 \sigma \frac{d}{d\sigma} \langle F(\sigma Y), Z \rangle d\sigma + \int_0^1 \langle F(\sigma Y), Z \rangle d\sigma$$

$$= \sigma \langle F(\sigma Y), Z \rangle \bigg|_0^1 = \langle F(Y), Z \rangle,$$  

(2.4)

and similarly we have

$$\frac{d}{dt} \int_0^1 \langle \sigma \Phi(\sigma Z), Z \rangle d\sigma = \int_0^1 \langle \sigma \Phi(\sigma Z), Z \rangle d\sigma + \int_0^1 \sigma^2 \langle \Phi(\sigma Z) U, Z \rangle d\sigma$$

$$+ \int_0^1 \langle \sigma \Phi(\sigma Z), U \rangle d\sigma$$

$$= \int_0^1 \sigma \frac{d}{d\sigma} \langle \sigma \Phi(\sigma Z), U \rangle d\sigma + \int_0^1 \langle \sigma \Phi(\sigma Z), U \rangle d\sigma$$

$$= \sigma^2 \langle \Phi(\sigma Z) U, Z \rangle \bigg|_0^1 = \langle \Phi(Z) U, Z \rangle.$$  

(2.5)

and similarly we have

$$\frac{d}{dt} \int_0^1 \langle \Psi(\sigma Y), Y \rangle Z d\sigma = \langle \Psi(Y), Z \rangle + \int_0^1 \langle \Psi(\sigma Y), Y \rangle d\sigma.$$  

(2.6)
Upon gathering the estimates (2.4), (2.5) and (2.6) into (2.3) we obtain
\[
\Gamma = \langle U, U \rangle + \int_0^1 \langle \Psi(\sigma Y) Y, U \rangle d\sigma + \langle Y, J_G(X) Y \rangle
\]
\[
\geq \|U\|^2 - a_2 \|Y\| \|U\| + a_4 \|Y\|^2
\]
\[
= \left(\|U\|^2 - \frac{1}{4}a_2 \|Y\|^2\right) + a_4 \|Y\|^2 - \frac{1}{4}a_2 \|Y\|^2
\]
\[
= \left(\|U\|^2 - \frac{1}{4}a_2 \|Y\|^2\right) + \left(a_4 - \frac{1}{4}a_2\right) \|Y\|^2 \geq 0.
\]
Hence \( \dot{\psi}(t) \geq 0 \), so that \( \psi(t) \) is monotone in \( t \), and therefore, being bounded, tends to a limit, \( \psi_0 \) say, as \( t \to \infty \). It is readily checked that
\[
\psi(t) \equiv \psi_0 \text{ for all } t. \tag{2.8}
\]
From by (2.1),
\[
\psi(t) = \psi(t + m\alpha) \tag{2.9}
\]
for any arbitrary fixed \( t \) an for arbitrary integer \( m \), and then letting \( m \to \infty \) in the right-hand side of (2.9) leads to (2.8).

The result (2.8) itself implies that
\[
\dot{\psi}(t) = 0 \text{ for all } t
\]
from which, by (2.7), it follows from assumptions on \( \Psi \) and \( G \), that
\[
Y = 0 \text{ for all } t, \tag{2.10}
\]
which in turn implies that
\[
X = \xi \text{ (constant), } Y = 0 = Z = U \text{ for all } t. \tag{2.11}
\]
Since \( (X, Y, Z, U) \) is a solution of (1.5), it is evident from (2.10) and (2.11) that \( G(\xi) = 0 \), so that \( \xi = 0 \), by (1.3). Hence
\[
(X, Y, Z, U) = (0, 0, 0, 0)
\]
This completes the proof of Theorem 1.

**Proof of Theorem 2.** Let \((X, Y, Z, U, V) = (X(t), Y(t), Z(t), U(t), V(t))\) be an arbitrary \( \omega \)-periodic solution of (1.6), that is
\[
(X(t), Y(t), Z(t), U(t), V(t)) = (X(t + \omega), Y(t + \omega), Z(t + \omega), U(t + \omega), V(t + \omega))
\]
for some \( \omega > 0 \).

Consider the function \( W = W(X, Y, Z, U, V) \) defined by
\[
W = \frac{1}{2} \langle AZ, Z \rangle + \langle Z, U \rangle - \langle Y, V \rangle - \langle Y, AU \rangle
\]
\[
- \int \langle \sigma \Psi(\sigma Y) Y, Y \rangle d\sigma - \int \langle \Theta(\sigma X), X \rangle d\sigma. \tag{2.12}
\]
It is clear that $W$ is bounded. An elementary differentiation from (1.6) and (2.12) yields

$$W = \langle U, U \rangle + \langle Y, \Phi(X, Y, Z, U, V)U \rangle + \langle Y, \Omega(X, Y, Z, U, V)Y \rangle + \langle Y, \Theta(X) \rangle + \langle Y, \Psi(Y) \rangle Z - \frac{d}{dt} \int_0^1 \langle \sigma \Psi(\sigma Y)Y, Y \rangle d\sigma - \frac{d}{dt} \int_0^1 \langle \Theta(\sigma X), X \rangle d\sigma. \quad (2.13)$$

But

$$\frac{d}{dt} \int_0^1 \langle \sigma \Psi(\sigma Y)Y, Y \rangle d\sigma = \langle \Psi(Y)Z, Y \rangle \quad (2.14)$$

and

$$\frac{d}{dt} \int_0^1 \langle \Theta(\sigma X), X \rangle d\sigma = \langle \Theta(X), Y \rangle. \quad (2.15)$$

Using the estimates (2.14) and (2.15) in (2.13) we obtain

$$W = \langle U, U \rangle + \langle Y, \Phi(X, Y, Z, U, V)U \rangle + \langle Y, \Omega(X, Y, Z, U, V)Y \rangle - \frac{1}{4} \langle \Phi(X, Y, Z, U, V)Y, \Phi(X, Y, Z, U, V)Y \rangle \geq \langle Y, \Omega(X, Y, Z, U, V)Y \rangle - \frac{1}{4} \langle \Phi(X, Y, Z, U, V)Y, \Phi(X, Y, Z, U, V)Y \rangle \geq 0$$

Therefore, the rest of the proof, can be shown in the same way as the proof of Theorem 1, which gives

$$X = Y = Z = U = V = 0.$$

References


On the periodic solutions of fourth and fifth order


