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ABSTRACT

In this study, we generalize the concept of $\alpha-$ distance which contains both of Taxicab distance and Chinese Checker distance as special cases to $n-$dimensional space.

Key words: Taxicab distance, CC-distance, α-distance, metric, non-Euclidean geometry

MSC 2000: 51K05, 51K99

Poopćenje $\alpha-$udaljenosti u $n-$dimenzionalnom prostoru

SAŽETAK

U članku se poopćuje pojam $\alpha-$ udaljenosti koji taxi udaljenost i CC-udaljenost sadrži kao posebne slučajeve u $n-$dimenzionalnom prostoru.

Ključne riječi: Taxi udaljenost, CC-udaljenost, α-udaljenost, metrika, neeuklidska geometrija

Tian [9] gave a generalization of both Taxicab and Chinese Checker distances in the plane, and named it as $\alpha-$distance. In [6], $\alpha-$distance have been extended to three dimensional space. In this work the concept of $\alpha-$distance is generalized to $n-$dimensional space.

In the following definition, we introduce a family of distances in $\mathbb{R}^n$, which include Taxicab and Chinese Checker distances as special cases.

Definition:

Let $P_1 = (x_1, x_2, \ldots, x_n)$ and $P_2 = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$. If

$$\Delta_{P_1P_2} = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\} = |x_j - y_j|$$

and

$$\delta_{P_1P_2} = \sum_{i\in I} |x_i - y_i|, \quad I = \{1, 2, \ldots, n\} \setminus \{j\},$$

then the function $d_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$d_\alpha(P_1, P_2) = \Delta_{P_1P_2} + (\sec\alpha - \tan\alpha)\delta_{P_1P_2}, \quad \alpha \in [0, \pi/4],$$

is called generalized $\alpha-$distance between points P_1 and P_2.

Generalized Taxicab and Chinese Checker distances between points P_1 and P_2 in $\mathbb{R}^n$ are $d_T(P_1, P_2) = \Delta_{P_1P_2} + \delta_{P_1P_2}$ and $d_c(P_1, P_2) = \Delta_{P_1P_2} + (\sqrt{2} - 1)\delta_{P_1P_2}$, respectively. (See [1], [2], [3], [4], [5], [8]).

Notice that

$d_0(P_1, P_2) = d_T(P_1, P_2)$ and $d_{\frac{\pi}{4}}(P_1, P_2) = d_c(P_1, P_2)$.

Also, if $\delta_{P_1P_2} > 0$, then for all $\alpha \in (0, \pi/4)$,

$d_E(P_1, P_2) < d_c(P_1, P_2) < d_\alpha(P_1, P_2) < d_T(P_1, P_2)$,

where d_E, d_c and d_T stand for the Euclidean, Chinese Checker and Taxicab distances, respectively.

Further, if $\delta_{P_1P_2} = 0$, then P_1 and P_2 lie on a line which is parallel to one of coordinate axes, and for all $\alpha \in [0, \pi/4]$,

$d_c(P_1, P_2) = d_\alpha(P_1, P_2) = d_T(P_1, P_2) = d_E(P_1, P_2)$.

Let l be a line through P_1 and parallel to jth-coordinate axis and $l_1, \ldots, l_n$ denote lines each of which is parallel to a coordinate axis distinct from jth-axis. Geometrically, the shortest way between the points P_1 and P_2 is the union of a line segment parallel to l_j and line segments each making α angle with one of $l_1, \ldots, l_n$, as shown in Figure 1. Thus, the shortest distance d_α from P_1 to P_2 is sum of the Euclidean lengths of such n line segments.

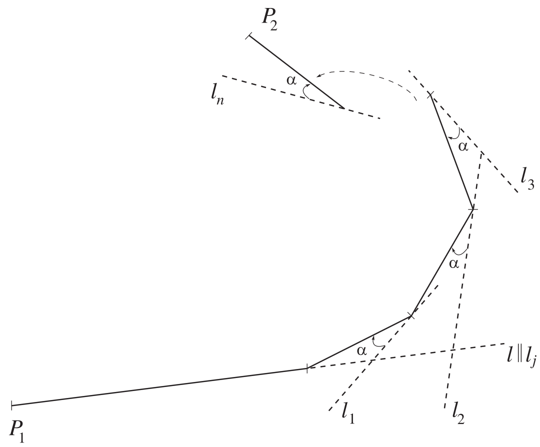

Figure 1

The next two propositions follow directly from the definition of the generalized $\alpha-$distance.

Proposition 1.

The generalized $\alpha-$distance is invariant under all translation in $\mathbb{R}^n$. That is, $T:\mathbb{R}^n \to \mathbb{R}^n$, where $T(x_1,x_2,\ldots,x_n)=(x_1+a_1,x_2+a_2,\ldots,x_n+a_n)$, $a_i\in\mathbb{R}$, does not change the distance between any two points in $\mathbb{R}^n$.

Proposition 2.

Let $P_1=(x_1,x_2,\ldots,x_n)$ and $P_2=(y_1,y_2,\ldots,y_n)$ be two points in $\mathbb{R}^n$. If $\Delta_{P_1P_2}=\left|x_{j_1}-y_{j_1}\right|$ for $j_1\in\{1,2,\ldots,n\}$, then

$$\left|x_{j_1}-y_{j_1}\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-y_i| \geq$$
$$\geq \left|x_{j_2}-y_{j_2}\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I'}|x_i-y_i|,$$

for $I=\{1,2,\ldots,n\}\setminus\{j_1\}$, $I'=\{1,2,\ldots,n\}\setminus\{j_2\}$, $j_2\in I$ and $\alpha\in[0,\pi/4]$.

Proof: Let P_1=$(x_1,x_2,\ldots,x_n)$ and P_2=$(y_1,y_2,\ldots,y_n)$.
If $\Delta_{P_1P_2}$=$\left|x_{j_1}-y_{j_1}\right|$,
then $a=\left|x_{j_1}-y_{j_1}\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-y_i|$.
Let $b=\left|x_{j_2}-y_{j_2}\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I'}|x_i-y_i|$, for $j_2\in I$.

$$\begin{aligned}
a-b &= \left|x_{j_1}-y_{j_1}\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-y_i|- \\
&\quad -\left|x_{j_2}-y_{j_2}\right|-(\sec\alpha\text{-}\tan\alpha)\sum_{i\in I'}|x_i-y_i| \\
&= \left|x_{j_1}-y_{j_1}\right|+(\sec\alpha-\tan\alpha)\left|x_{j_2}-y_{j_2}\right|- \\
&\quad -\left|x_{j_2}-y_{j_2}\right|-(\sec\alpha-\tan\alpha)\left|x_{j_1}-y_{j_1}\right| \\
&= (1-(\sec\alpha-\tan\alpha))(\left|x_{j_1}-y_{j_1}\right|-\left|x_{j_2}-y_{j_2}\right|).
\end{aligned}$$

Notice that $(1-(\sec\alpha-\tan\alpha))\geq 0$ for all $\alpha\in[0,\pi/4]$ and $(\left|x_{j_1}-y_{j_1}\right|-\left|x_{j_2}-y_{j_2}\right|)\geq 0$. Thus $a-b\geq 0$.
That is, $\left|x_{j_1}-y_{j_1}\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-y_i|\geq$

$$\geq \left|x_{j_2}-y_{j_2}\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I'}|x_i-y_i|$$

for $I=\{1,2,\ldots,n\}\setminus\{j_1\}$, $I'=\{1,2,\ldots,n\}\setminus\{j_2\}$, $j_2\in I$ and $\alpha\in[0,\pi/4]$.

The following theorem shows that generalized $\alpha-$distance is a metric.

Theorem 3.

For each $\alpha\in[0,\pi/4]$, generalized $\alpha-$distance determines a metric for $\mathbb{R}^n$.

Proof: We have to show that d_α is positive definite and symmetric, and d_α holds triangle inequality. Let $P_1=(x_1,x_2,\ldots,x_n)$, $P_2=(y_1,y_2,\ldots,y_n)$ and $P_3=(z_1,z_2,\ldots,z_n)$ be three points in $\mathbb{R}^n$. Generalized $\alpha-$distance between points P_1 and P_2 is $d_\alpha(P_1,P_2)=\Delta_{P_1P_2}+(\sec\alpha-\tan\alpha)\delta_{P_1P_2}$, $\alpha\in[0,\pi/4]$.

$d_\alpha(P_1,P_2)\geq 0$ since $|x_i-y_i|\geq 0$ and $(\sec\alpha-\tan\alpha)\geq 0$ for each $\alpha\in[0,\pi/4]$. Obviously, $d_\alpha(P_1,P_2)=0$ if and only if $P_1=P_2$. So d_α is positive definite.

Clearly $d_\alpha(P_1,P_2)=d_\alpha(P_2,P_1)$ follows from $|x_i-y_i|=|y_i-x_i|$. That is, d_α is symmetric.

Now, we try to prove that $d_\alpha(P_1,P_2)\leq d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2)$ for all $P_1,P_2,P_3\in\mathbb{R}^n$ and $\alpha\in[0,\pi/4]$. For each $\alpha\in[0,\pi/4]$, and I=$\{1,2,\ldots,n\}\setminus\{j\}$,

$$\begin{aligned}
d_\alpha(P_1,P_2) &= \left|x_j-y_j\right|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-y_i| \\
&= \left|x_j-z_j+z_j-y_j\right|+ \\
&\quad +(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-z_i+z_i-y_i| \\
&\leq \left|x_j-z_j\right|+\left|z_j-y_j\right|+ \\
&\quad +(\sec\alpha-\tan\alpha)\sum_{i\in I}(|x_i-z_i|+|z_i-y_i|) \\
&= k.
\end{aligned}$$

One can easily see that d_α satisfies the triangle inequality by examining the following cases:

Case I: If $\left|x_j-z_j\right|\geq|x_i-z_i|$ and $\left|z_j-y_j\right|\geq|z_i-y_i|$, $i,j\in\{1,2,\ldots,n\}$, $i\neq j$, then for each $\alpha\in[0,\pi/4]$, and

$I=\{1,2,\ldots,n\}\setminus\{j\}$,

$$\begin{aligned} d_\alpha(P_1,P_2) &\le k \\ &= |x_j-z_j|+|z_j-y_j|+ \\ &\quad +(\sec\alpha-\tan\alpha)\sum_{i\in I}(|x_i-z_i|+|z_i-y_i|) \\ &= |x_j-z_j|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-z_i|+ \\ &\quad +|z_j-y_j|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|z_i-y_i| \\ &= d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2). \end{aligned}$$

Case II: If $|x_j-z_j|\ge|x_i-z_i|$ and $|z_j-y_j|\le|z_i-y_i|$, $i,j\in\{1,2,\ldots,n\}$, $i\ne j$, then there are two possible situations:

(i) Let $|x_j-z_j|+|z_j-y_j|\ge|x_i-z_i|+|z_i-y_i|$. Then for each $\alpha\in[0,\pi/4]$, and $I=\{1,2,\ldots,n\}\setminus\{j\}$,

$$\begin{aligned} d_\alpha(P_1,P_2) &\le k \\ &= |x_j-z_j|+|z_j-y_j|+ \\ &\quad +(\sec\alpha-\tan\alpha)\sum_{i\in I}(|x_i-z_i|+|z_i-y_i|) \\ &= |x_j-z_j|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|x_i-z_i|+ \\ &\quad +|z_j-y_j|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|z_i-y_i| \\ &= d_\alpha(P_1,P_3)+|z_j-y_j|+ \\ &\quad +(\sec\alpha-\tan\alpha)\sum_{i\in I}|z_i-y_i| \\ &\le d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2), \end{aligned}$$

where $|z_j-y_j|+(\sec\alpha-\tan\alpha)\sum_{i\in I}|z_i-y_i|\le d_\alpha(P_3,P_2)$ because of Proposition 2.

(ii) Let $|x_j-z_j|+|z_j-y_j|\le|x_i-z_i|+|z_i-y_i|$. One can easily give a proof for the situation (ii) as in situation (i).

Case III: If $|x_j-z_j|\le|x_i-z_i|$ and $|z_j-y_j|\ge|z_i-y_i|$, $i,j\in\{1,2,\ldots,n\}$, $i\ne j$, then there are two possible situations:

(i) Let $|x_j-z_j|+|z_j-y_j|\ge|x_i-z_i|+|z_i-y_i|$.

(ii) Let $|x_j-z_j|+|z_j-y_j|\le|x_i-z_i|+|z_i-y_i|$.

One can easily give a proof for the Case III as in the Case II.

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