Generalization of $\alpha-$distance to $n-$dimensional Space

**ABSTRACT**

In this study, we generalize the concept of $\alpha-$distance which contains both of Taxicab distance and Chinese Checker distance as special cases to $n-$dimensional space.

**Key words:** Taxicab distance, CC-distance, $\alpha-$distance, metric, non-Euclidean geometry

**MSC 2000:** 51K05, 51K99

In the following definition, we introduce a family of distances in $\mathbb{R}^n$, which include Taxicab and Chinese Checker distances as special cases.

**Definition:**

Let $P_1 = (x_1, x_2, \ldots, x_n)$ and $P_2 = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$. If

\[
\Delta_{P_1 P_2} = \max \{ |x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n| \} = |x_j - y_j|
\]

and

\[
\delta_{P_1 P_2} = \sum_{i \in J} |x_i - y_i|, \quad J = \{1, 2, \ldots, n\} \setminus \{j\},
\]

then the function $d_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

\[
d_\alpha(P_1, P_2) = \Delta_{P_1 P_2} + (\sec \alpha - \tan \alpha) \delta_{P_1 P_2}, \quad \alpha \in [0, \pi/4],
\]

is called generalized $\alpha-$distance between points $P_1$ and $P_2$.

Generalized Taxicab and Chinese Checker distances between points $P_1$ and $P_2$ in $\mathbb{R}^n$ are $d_T(P_1, P_2) = \Delta_{P_1 P_2} + \delta_{P_1 P_2}$ and $d_c(P_1, P_2) = \Delta_{P_1 P_2} + (\sqrt{2} - 1) \delta_{P_1 P_2}$, respectively.

(See [1], [2], [3], [4], [5], [8]).

Notice that

\[
d_0(P_1, P_2) = d_T(P_1, P_2) \quad \text{and} \quad d_\pi(P_1, P_2) = d_c(P_1, P_2).
\]

Also, if $\delta_{P_1 P_2} > 0$, then for all $\alpha \in (0, \pi/4)$,

\[
d_\alpha(P_1, P_2) \leq d_c(P_1, P_2),
\]

where $d_E$, $d_c$ and $d_T$ stand for the Euclidean, Chinese Checker and Taxicab distances, respectively.

Further, if $\delta_{P_1 P_2} = 0$, then $P_1$ and $P_2$ lie on a line which is parallel to one of coordinate axes, and for all $\alpha \in [0, \pi/4]$, $d_\alpha(P_1, P_2) = d_\alpha(P_1, P_2) = d_T(P_1, P_2) = d_E(P_1, P_2)$.

Let $l$ be a line through $P_1$ and parallel to $j$th-coordinate axis and $l_1, \ldots, l_n$ denote lines each of which is parallel to a coordinate axis distinct from $j$th-axis. Geometrically, the shortest way between the points $P_1$ and $P_2$ is the union of a line segment parallel to $l_j$ and line segments each making $\alpha$ angle with one of $l_1, \ldots, l_n$, as shown in Figure 1.

Thus, the shortest distance $d_\alpha$ from $P_1$ to $P_2$ is sum of the Euclidean lengths of such $n$ line segments.
Let \( P \) be two points in \( \mathbb{R} \). Let \( P_1 = (x_1, x_2, \ldots, x_n) \) and \( P_2 = (y_1, y_2, \ldots, y_n) \) be two points in \( \mathbb{R}^n \).

The following theorems show that generalized \( \alpha \)-distance is a metric.

**Theorem 3.**

For each \( \alpha \in [0, \pi/4] \), generalized \( \alpha \)-distance determines a metric for \( \mathbb{R}^n \).

**Proof:** We have to show that \( d_\alpha \) is positive definite and symmetric, and \( d_\alpha \) holds triangle inequality. Let \( P_1 = (x_1, x_2, \ldots, x_n) \), \( P_2 = (y_1, y_2, \ldots, y_n) \) and \( P_3 = (z_1, z_2, \ldots, z_n) \) be three points in \( \mathbb{R}^n \). Generalized \( \alpha \)-distance between points \( P_1 \) and \( P_2 \) is \( d_\alpha(P_1, P_2) = d_\alpha(P_2, P_1) \) follows from \( |x_i - y_i| = |y_i - x_i| \). That is, \( d_\alpha \) is symmetric.

Now, we try to prove that \( d_\alpha(P_1, P_2) \leq d_\alpha(P_1, P_3) + d_\alpha(P_3, P_2) \) for all \( P_1, P_2, P_3 \in \mathbb{R}^n \) and \( \alpha \in [0, \pi/4] \). For each \( \alpha \in [0, \pi/4] \), and \( I = \{1, 2, \ldots, n\} \setminus \{j\} \).

\[
\begin{align*}
\quad d_\alpha(P_1, P_2) &= |x_j - y_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} |x_i - y_i| \\
&= |x_j - z_j + z_j - y_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} |x_i - z_i + z_i - y_i| \\
&\leq |x_j - z_j| + |z_j - y_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} (|x_i - z_i| + |z_i - y_i|) \\
&= k.
\end{align*}
\]

One can easily see that \( d_\alpha \) satisfies the triangle inequality by examining the following cases:

**Case I:** If \( |x_j - z_j| \geq |x_i - z_i| \) and \( |z_j - y_j| \geq |z_i - y_i| \), \( i, j \in \{1, 2, \ldots, n\}, i \neq j \), then for each \( \alpha \in [0, \pi/4] \), and
\(I = \{1, 2, \ldots, n\} \setminus \{j\},\)
\[
d_\alpha(P_1, P_2) \leq k
\]
\[
= |x_j - z_j| + |z_j - y_j| + \\
+ (\sec \alpha - \tan \alpha) \sum_{i \in I} (|x_i - z_i| + |z_i - y_i|)
\]
\[
= |x_j - z_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} |x_i - z_i| + \\
+ |z_j - y_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} |z_i - y_i|
\]
\[
= d_\alpha(P_1, P_3) + d_\alpha(P_3, P_2).
\]

Case II: If \(|x_j - z_j| \geq |x_i - z_i|\) and \(|z_j - y_j| \leq |z_i - y_i|\), \(i, j \in \{1, 2, \ldots, n\}, i \neq j\), then there are two possible situations:

(i) Let \(|x_j - z_j| + |z_j - y_j| \geq |x_i - z_i| + |z_i - y_i|\). Then for each \(\alpha \in [0, \pi/4]\), and \(I = \{1, 2, \ldots, n\} \setminus \{j\}\),
\[
d_\alpha(P_1, P_2) \leq k
\]
\[
= |x_j - z_j| + |z_j - y_j| + \\
+ (\sec \alpha - \tan \alpha) \sum_{i \in I} (|x_i - z_i| + |z_i - y_i|)
\]
\[
= |x_j - z_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} |x_i - z_i| + \\
+ |z_j - y_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} |z_i - y_i|
\]
\[
= d_\alpha(P_1, P_3) + |z_j - y_j| + \\
+ (\sec \alpha - \tan \alpha) \sum_{i \in I} |z_i - y_i|
\]
\[
\leq d_\alpha(P_1, P_3) + d_\alpha(P_3, P_2),
\]
where \(|z_j - y_j| + (\sec \alpha - \tan \alpha) \sum_{i \in I} |z_i - y_i| \leq d_\alpha(P_3, P_2)\) because of Proposition 2.

(ii) Let \(|x_j - z_j| + |z_j - y_j| \leq |x_i - z_i| + |z_i - y_i|\). One can easily give a proof for the situation (ii) as in situation (i).

Case III: If \(|x_j - z_j| \leq |x_i - z_i|\) and \(|z_j - y_j| \geq |z_i - y_i|\), \(i, j \in \{1, 2, \ldots, n\}, i \neq j\), then there are two possible situations:

(i) Let \(|x_j - z_j| + |z_j - y_j| \geq |x_i - z_i| + |z_i - y_i|\).

(ii) Let \(|x_j - z_j| + |z_j - y_j| \leq |x_i - z_i| + |z_i - y_i|\).

One can easily give a proof for the Case III as in the Case II.

References


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