The fixed point property for arc component preserving mappings of non-metric tree-like continua

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Abstract. The main purpose of this paper is to study the fixed point property of non-metric tree-like continua. Using the inverse systems method, it is proved that if $X$ is a non-metric tree-like continuum and if $f : X \to X$ is a mapping which sends each arc component into itself, then $f$ has the fixed point property.

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1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space $X$ is denoted by $w(X)$. The cardinality of a set $A$ is denoted by $\text{card}(A)$. We shall use the notion of an inverse system as in [2, pp. 135-142]. An inverse system is denoted by $X = \{X_a, p_{ab}, A\}$.

Let $A$ be a partially ordered directed set. We say that a subset $A_1 \subseteq A$ majorates [1, p. 9] another subset $A_2 \subseteq A$ if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates $A$ is called cofinal in $A$. A subset of $A$ is said to be a chain if its every two elements are comparable. The symbol $\text{sup}B$, where $B \subseteq A$, denotes the lower upper bound of $B$ (if such an element exists in $A$). Let $\tau \geq \aleph_0$ be a cardinal number. A subset $B$ of $A$ is said to be $\tau$-closed in $A$ if for each chain $C \subseteq B$ with $\text{card}(B) \leq \tau$, we have $\text{sup}C \in B$, whenever the element $\text{sup}C$ exists in $A$. Finally, a directed set $A$ is said to be $\tau$-complete if for each chain $C$ of elements of $A$ with $\text{card}(C) \leq \tau$, there exists an element $\text{sup}C$ in $A$.

Suppose that we have two inverse systems $X = \{X_a, p_{ab}, A\}$ and $Y = \{Y_b, q_{bc}, B\}$. A morphism of the system $X$ into the system $Y$ [1, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$.

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consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in $A$, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following diagram commutes. Any morphism \{\varphi, \{f_b : b \in B\}\} : X \to Y induces a map, called the limit map of the morphism

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim X \to \lim Y$$

In the present paper we deal with the inverse systems defined on the same indexing set $A$. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation \{\varphi, \{f_a : X_a \to Y_a; a \in A\}\} : X \to Y.

We say that an inverse system $X = \{X_a, p_{ab}, A\}$ is factorizing [1, p. 17] if for each real-valued mapping $f : \lim X \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $X = \{X_a, p_{ab}, A\}$ is said to be $\sigma$-directed if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of $A$ there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

**Lemma 1.** [1, Corollary 1.3.2, p. 18]. If $X = \{X_a, p_{ab}, A\}$ is a $\sigma$-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.

An inverse system $X = \{X_a, p_{ab}, A\}$ is said to be $\tau$-continuous [1, p. 19] if for each chain $B$ in $A$ with $\text{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta \{p_{ab} : a \in B\}$ maps the space $X_b$ homeomorphically into the space $\lim \{X_a, p_{ab}, B\}$.

An inverse system $X = \{X_a, p_{ab}, A\}$ is said to be a $\tau$-system [1, p. 19] if:

a) $\omega(X_a) \leq \tau$ for every $a \in A$,

b) The system $X = \{X_a, p_{ab}, A\}$ is $\tau$-continuous,

c) The indexing set $A$ is $\tau$-complete.

A $\sigma$-system is a $\tau$-system, where $\tau = 8\omega$. The following theorem is called the **Spectral Theorem** [1, p. 19].

**Theorem 1.** [1, Theorem 1.3.4, p. 19]. If a $\tau$-system $X = \{X_a, p_{ab}, A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another $\tau$-system $Y = \{Y_a, q_{ab}, A\}$ is induced by a morphism of cofinal and $\tau$-closed subsystems. If two factorizing $\tau$-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and $\tau$-closed subsystems.

Let us remark that the requirement of surjectivity of the limit projections of systems in Theorem 1 is essential [1, p. 21].
2. The fixed point property of non-metric continua

A fixed point of a function \( f : X \to X \) is a point \( p \in X \) such that \( f(p) = p \). A space \( X \) is said to have the fixed point property provided that every mapping \( f : X \to X \) has a fixed point.

The key theorem is the following.

**Theorem 2.** Let \( X = \{X_a, p_{ab}, A\} \) be a \( \sigma \)-system of compact spaces with limit \( X \) and onto projections \( p_a : X \to X_a \). Let \( \{f_a : X_a \to X_a\} : X \to X \) be a morphism. Then the induced mapping \( f = \lim \{f_a\} : X \to X \) has a fixed point if and only if each mapping \( f_a : X_a \to X_a, a \in A, \) has a fixed point.

**Proof.** The if part. Let \( F_a, a \in A, \) be the set of fixed points of the mapping \( f_a \).

**Claim 1.** Every set \( F_a \) is closed. This is a consequence of the following theorem [2, Theorem 1.5.4., p. 59]: For any pair \( f, g \) of mappings of a space \( X \) into a Hausdorff space \( Y \), the set

\[
\{x \in X : f(x) = g(x)\}
\]

is closed in \( X \).

It suffices to set \( g(x) = x \) and \( Y = X \).

**Claim 2.** If \( a \leq b \), then \( p_{ab}(F_b) \subset F_a \). Let \( x_b \) be any point of \( F_b \). From the commutativity of diagram (1) it follows \( p_{ab}(f_b(x)) = f_a(p_{ab}(x_b)) \). We have \( p_{ab}(x_b) = f_a(p_{ab}(x_b)) \) since \( f_b(x) = x_b \). This means that for the point \( y = p_{ab}(x_b) \in X_a \) we have \( y = f_a(y) \), i.e., \( y \in F_a \). We infer that \( p_{ab}(x_b) \in F_a \) and \( p_{ab}(F_b) \subset F_a \).

**Claim 3.** \( F = \{F_a, p_{ab}[F_b, A]\} \) is an inverse system of compact spaces with a non-empty limit \( F \).

**Claim 4.** The set \( F \subset X \) is the set of fixed points of the mapping \( f \). Let \( x \in F \) and let \( x_a = p_a(x), a \in A \). Now, \( f_a(x_a) = x_a \) since \( x_a \in F_a \). We infer that \( f(x) = x \) since the morphism \( \{f_a : a \in A\} \) induces \( f \). The proof of the "if" part is complete.

The only if part. Suppose that the induced mapping \( f \) has a fixed point \( x \). Let us prove that every mapping \( f_a, a \in A, \) has a fixed point. Now we have \( f_a p_a(x) = p_a f(x) \). From \( f(x) = x \) it follows \( f_a p_a(x) = p_a(x) \). We infer that \( p_a(x) \) is a fixed point for \( f_a \).

As an immediate consequence of this theorem and the Spectral theorem 1 we have the following result.

**Theorem 3.** Let a non-metric continuum \( X \) be the inverse limit of an inverse \( \sigma \)-system \( X = \{X_a, p_{ab}, A\} \) such that each \( X_a \) has the fixed point property and each bonding mapping \( p_{ab} \) is onto. Then \( X \) has the fixed point property.

3. The fixed point property of the inverse limit space of tree-like continua

A continuum \( X \) with precisely two non-separating points is called a generalized arc.

A simple \( n \)-od is the union of \( n \) generalized arcs \( A_1O, A_2O, ..., A_nO, \) each two of which have only the point \( O \) in common. The point \( O \) is called the vertex or the top of the \( n \)-od.

By a branch point of a compact space \( X \) we mean a point \( p \) of \( X \) which is the vertex of a simple triod lying in \( X \). A point \( x \in X \) is said to be the end point of
There exists a triangulation \( \triangledown \) of a continuum \( X \). Let \( pq \) be a free arc in \( X \) if \( pq \cap S = \{p, q\} \).

A continuum is a graph if it is the union of a finite number of metric free arcs. A tree is an acyclic graph.

A continuum \( X \) is tree-like (arc-like) if for each open cover \( \mathcal{U} \) of \( X \), there is a tree \( (arc) \) \( X_\mathcal{U} \) and a \( \mathcal{U} \)-mapping \( f_\mathcal{U} : X \to X_\mathcal{U} \) (the inverse image of each point is contained in a member of \( \mathcal{U} \)).

Every tree-like continuum is hereditarily unicoherent. Every arc-like continuum is tree-like.

Let \( Y^X \) be the set of all mappings of \( X \) to \( Y \). If \( Y \) is a metric space with a metric \( d \), then on the set \( Y^X \) one can define a metric \( \hat{d} \) by letting

\[
\hat{d}(f, g) = \sup_{x \in X} d(f(x), g(x)).
\]

\[ \text{(4)} \]

**Proposition 1.** Let \( X \) be any tree-like continuum, let \( P \) be a polyhedron with a given metric \( d \), \( r > 0 \) a real number and \( f : X \to P \) a mapping. Then there exists a tree \( Q \), a mapping \( g : X \to Q \) and a mapping \( p : Q \to P \) such that \( g(X) = Q \) and

\[
\hat{d}(f, pg) \leq r.
\]

**Proof.** Let \( K \) be a triangulation of \( P \) of mesh not greater than \( r/2 \). Let \( a_i \) be the vertices of \( K \), and let \( St a_i \) be the open star of \( K \) around the vertex \( a_i \). Since \( \{St a_i\} \) is an open covering for \( P \), and so is \( \mathcal{U} = \{f^{-1}(St a_i)\} \) for \( X \). There exist a tree \( Q \) and a mapping \( g : X \to Q \) such that \( g \) is \( \mathcal{U} \)-mapping and \( g(X) = Q \).

There exists a triangulation \( L \) of \( Q \) with vertices \( b_j \) such that the cover \( \mathcal{V} = \{g^{-1}(St b_j)\} \) refines the cover \( \mathcal{U} \). Let \( x \) be a point of \( X \) and let \( s \) be a simplex of \( Q \) with vertices \( b_{j_1}, ..., b_{j_k} \) containing \( g(x) \). This means that \( \{g^{-1}(St b_{j_1}), ..., g^{-1}(St b_{j_k})\} \) is a collection of some \( g^{-1}(St b_j) \) containing \( x \). It follows that \( g^{-1}(St b_{j_1}) \cap ... \cap g^{-1}(St b_{j_k}) \neq \emptyset \). We infer that \( St b_{j_1} \cap ... \cap St b_{j_k} \neq \emptyset \). Let \( p : Q \to P \) be a simplicial mapping sending each vertex \( b_j \) of \( Q \) into a vertex \( a_i \) having the property that \( g^{-1}(St b_j) \subset f^{-1}(St a_i) \). It remains to prove that \( \hat{d}(f, pg) \leq r \). Now, for each \( g^{-1}(St b_{j_i}) \) we have some \( f^{-1}(St a_{i_1}) \) with \( g^{-1}(St b_{j_i}) \subset f^{-1}(St a_{i_1}) \). From \( g^{-1}(St b_{j_{i_1}}) \cap ... \cap g^{-1}(St b_{j_{i_k}}) \neq \emptyset \) it follows that \( f^{-1}(St b_{j_{i_1}}) \cap ... \cap f^{-1}(St b_{j_{i_k}}) \neq \emptyset \), i.e., there exists a simplex \( \sigma \) of \( K \) with vertices \( b_{j_1}, ..., b_{j_k} \) such that \( f(x) \in St \sigma \).

Clearly, \( pg(x) \in St \sigma \). Finally, \( \hat{d}(f, pg) \leq r \). \( \square \)

**Proposition 2.** If \( X = \{X_a, p_{ab}, A\} \) is an inverse system of tree-like continua and if \( p_{ab} \) are onto mappings, then the limit \( X = \lim X \) is a tree-like continuum.

**Proof.** Let \( \mathcal{U} = \{U_1, ..., U_n\} \) be an open covering of \( X \). There exist \( a \in A \) and an open covering \( U_a = \{U_{1a}, ..., U_{ka}\} \) such that \( \{p_{a}^{-1}(U_{1a}), ..., p_{a}^{-1}(U_{ka})\} \) refines the covering \( \mathcal{U} \). There exists a tree \( T_a \) and a \( \mathcal{U}_a \)-mapping \( f_{\mathcal{U}_a} : X_a \to T_a \) since \( X_a \) is tree-like. It is clear that \( f_{\mathcal{U}_a} : X_0 \to T_a \) is a \( \mathcal{U} \)-mapping. Hence, \( X \) is tree-like.

**Proposition 3.** If \( X \) is a tree-like continuum, \( Q \) a tree and \( f : X \to Q \) a mapping, then \( f(X) \) is also a tree.

**Proof.** This follows from the fact that a subcontinuum of a tree is a tree. \( \square \)

Now we shall prove an expanding theorem of tree-like continua into inverse \( \sigma \)-systems of metric tree-like continua.
Theorem 4. If $X$ is a non-metric tree-like continuum, then there exists a $\sigma$-system $X_\sigma = \{X_\Delta, P_{\Delta}, A_\sigma\}$ of metric tree-like continua $X_\Delta$ and onto mappings $P_{\Delta}$ such that $X$ is homeomorphic to $\lim X_\sigma$.

Proof. Let us observe that Propositions 1-3 are conditions (A)-(C) in [4, p. 220]. Then from Mardešić’s General Expansion Theorem [4, Theorem 2] it follows that there exists an inverse system $X = \{X_\alpha, p_{ab}, A\}$ of metric tree-like continua $X_\alpha$ and onto bonding mappings $p_{ab}$ such that $X$ is homeomorphic to $\lim X$. It remains to prove that there exists such $\sigma$-system. The proof is broken down into several steps.

Step 1. For each subset $\Delta_0$ of $\langle A, \subseteq \rangle$ we define sets $\Delta_n, n = 0, 1, \ldots$, by the inductive rule $\Delta_{n+1} = \Delta_n \cup \{m(x, y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of $A$ such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that $\operatorname{card}(\Delta) = \operatorname{card}(\Delta_0)$. Moreover, $\Delta$ is directed by $\subseteq$. For each directed set $\langle A, \subseteq \rangle$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \operatorname{card}(\Delta) \leq \aleph_0 \text{ and } \Delta \text{ is directed by } \subseteq\}. \tag{5}$$

Step 2. If $A$ is a directed set, then $A_\sigma$ is $\sigma$-directed and $\sigma$-complete. Let $\{\Delta^1, \Delta^2, \ldots, \Delta^n, \ldots\}$ be a countable subset of $A_\sigma$. Then $\Delta_0 = \bigcup\{\Delta^1, \Delta^2, \ldots, \Delta^n, \ldots\}$ is a countable subset of $A_\sigma$. Define sets $\Delta_n, n = 0, 1, \ldots$, by the inductive rule $\Delta_{n+1} = \Delta_n \cup \{m(x, y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of $A$ such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that $\operatorname{card}(\Delta) = \operatorname{card}(\Delta_0)$. This means that $\Delta$ is countable. Moreover, $\Delta \supseteq \Delta^i, i \in \mathbb{N}$. Hence, $A_\sigma$ is $\sigma$-directed. Let us prove that $A_\sigma$ is $\sigma$-complete. Let $\Delta^1 \subset \Delta^2 \subset \ldots \subset \Delta^n \subset \ldots$ be a countable chain in $A_\sigma$. Then $\Delta = \bigcup\{\Delta^i : i \in \mathbb{N}\}$ is a countable and directed subset of $A$, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i, i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i, i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup\{\Delta^i : i \in \mathbb{N}\}$. This means that $A_\sigma$ is $\sigma$-complete.

Step 3. If $\Delta \in A_\sigma$, let $X^\Delta = \{X_b, p_{ab}, \Delta\}$ and $X_\Delta = \lim X^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $P_{\Delta \Gamma}: X_\Gamma \to X_\Delta$ denote the map induced by the projections $p^\delta_\Gamma: X_\Gamma \to X_\delta, \delta \in \Delta$, of the inverse system $X^\Gamma$.

Step 4. If $X = \{X_\alpha, p_{ab}, A\}$ is an inverse system, then $X_\sigma = \{X_\Delta, P_{\Delta}, A_\sigma\}$ is a $\sigma$-directed and $\sigma$-complete inverse system such that $\lim X$ and $\lim X_\sigma$ are homeomorphic. Each thread $x = (x_\alpha : a \in A)$ induces a thread $(x_\Delta : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $x_\Delta \in X_\Delta$. This means that we have a mapping $H : \lim X \to \lim X_\sigma$ such that $H(x) = (x_\Delta : \Delta \in A_\sigma)$. It is obvious that $H$ is continuous and 1-1. The mapping $H$ is onto since the collection of threads $\{x_\Delta : \Delta \in A_\sigma\}$ induces the thread in $X$. We infer that $H$ is a homeomorphism since $\lim X$ is compact.

Step 5. Every $X_\Delta$ is a metric tree-like continuum. Apply Proposition 2.

Step 6. Every projection $P_{\Delta} : \lim X_\sigma \to X_\Delta$ is onto. This follows from the assumption that the bonding mappings $p_{ab}$ are surjective.

Finally, $X_\sigma = \{X_\Delta, P_{\Delta}, A_\sigma\}$ is the desired $\sigma$-system.

By a similar proof we obtain the following theorem.

Theorem 5. If $X$ is a non-metric arc-like continuum, then there exists a $\sigma$-system $X_\sigma = \{X_\Delta, P_{\Delta}, A_\sigma\}$ of metric arc-like continua $X_\Delta$ and onto mappings $P_{\Delta}$ such that $X$ is homeomorphic to $\lim X_\sigma$.

From [4, Theorem 1], [4, Theorem 2] and [4, Corollary 1] we obtain the following well known result [5, Theorem 2.13, p. 24].
Theorem 6. Each metrizable tree-like (arc-like) continuum is homeomorphic with the inverse limit of an inverse sequence of trees (arcs).

Now we shall investigate the fixed point property of non-metric tree-like continua. Let us recall the following known result.

Theorem 7. [3, Theorem 1.2]. Suppose f is a map of a tree-like metric continuum M that sends each arc component of M into itself. Then f has a fixed point.

A map f : X → X is a deformation if there exists a map H : X × [0, 1] → X onto X such that H(p, 0) = p and H(p, 1) = f(p) for each point p ∈ X. We say that a map f : X → X is a generalized deformation if there exist a generalized arc L (with end points 0 and 1) and a map H : X × L → X onto X such that H(p, 0) = p and H(p, 1) = f(p) for each point p ∈ X. Since (generalized) deformations send arc-components into themselves, we have [3, Corollary 1.3].

Corollary 1. Every metric tree-like continuum has the fixed point property for deformations.

For non-metric tree-like continua we shall prove the following theorem.

Theorem 8. Let X = {Xa, pab, A} be an inverse σ-system of metric tree-like continua with onto bonding mappings and with limit X. If f : X → X is a mapping which sends each arc component into itself, then f has a fixed point.

Proof. By virtue of Theorem 1 there exist a cofinal subset B(f) of A and mappings f_b : X_b → Y_b, where b ∈ B(f), such that the mapping f is induced by a collection \{f_b : b ∈ B(f)\}. From Theorems 2 and 7 it follows that it suffices to prove that each f_b sends each arc component into itself. Let x_b ∈ X_b. We have to prove that x_b and f_b(x_b) lie in some arc component of X_b, i.e., there is an arc in X_b with end points x_b and f_b(x_b). There exists a point x ∈ X such that x_b = p_b(x). There exists a generalized arc L in X with end points x and f(x) since f sends each arc component into itself. This means that f_b(L) contains the points p_b(x) = x_b and p_b(f(x)) = f_b(p_b(x)) = f_b(x_b). We infer that there is an arc with end points x_b and f_b(x_b) since f_b(L) is arcwise connected [6, p. 201, Theorem 9]. The proof is complete.

The non-metric analogue of Theorem 7 is the following result.

Theorem 9. Let X be a non-metric continuum tree-like. If f : X → X is a mapping which sends each arc component into itself, then f has a fixed point.

Proof. Apply Theorems 4, 7 and 8.

Corollary 2. Every non-metric tree-like continuum has the fixed point property for generalized deformations.

Corollary 3. Let X be a non-metric tree-like arcwise connected continuum. If f : X → X, then f has a fixed point.

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References


