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Note

Relations Between the Product- and Sum-connectivity Indices

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Abstract. Several relations between the product- and sum-connectivity indices are established. (doi: 10.5562/cca2052)

Keywords: Randić connectivity index, product-connectivity index, sum-connectivity index, vertex degree

INTRODUCTION

Let G be a simple graph with vertex-set $V(G)$ and edge-set $E(G)$.[1] For $v \in V(G)$, $\Gamma(v)$ denotes the set of its (first) neighbors in G and the degree of v is $d_v = |\Gamma(v)|$. The product-connectivity index or the Randić connectivity index $R = R(G)$ of G is defined as[2]

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

The product-connectivity index is used successfully in QSPR and QSAR modeling[3–10] and its mathematical properties have also been studied extensively, as summarized in two monographs.[11,12]

The sum-connectivity index $\chi = \chi(G)$ of G is defined as[13]

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}.$$

The use of sum-connectivity index in QSPR and QSAR modeling has been investigated.[14–16] As already pointed by Balaban *et al.*,[17] the sum-connectivity index may be considered as a proper topological index[1] (molecular descriptor[6,7]). Mathematical properties of the sum-connectivity index have also been established.[18–21] Related to the sum-connectivity index, sum-connectivity matrix and sum-connectivity energy have been proposed.[22]

In this note, we establish several relations between the product- and the sum-connectivity indices.

RELATIONS BETWEEN THE PRODUCT- AND SUM-CONNECTIVITY INDICES

Let G be a graph. If G has no pendant vertices (vertices of degree one), then $\chi(G) \geq R(G)$ with equality if and only if all non-isolated vertices have degree two.[13] Obviously, $d_u + d_v \leq 2d_u d_v$ with equality if and only if $d_u = d_v = 1$. Thus

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}} \geq \sum_{uv \in E(G)} \frac{1}{\sqrt{2d_u d_v}} = \frac{1}{\sqrt{2}} R(G)$$

with equality if and only if all non-isolated vertices have degree one. If $d_u \geq d_v \geq 1$ and $d_u \geq 2$, then $d_u + d_v \leq \frac{3}{2} d_u d_v$ with equality if and only if $d_u = 2$ and $d_v = 1$. Thus, if G has no components on two vertices, then

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}} \geq \sum_{uv \in E(G)} \frac{1}{\sqrt{\frac{3}{2} d_u d_v}} = \sqrt{\frac{2}{3}} R(G)$$

with equality if and only if all non-trivial components of G are paths on three vertices. Thus we have

Proposition 1. Let G be a graph. Then $\chi(G) \geq \frac{1}{\sqrt{2}} R(G)$ with equality if and only if all non-isolated vertices have degree one. Moreover, if G has no components on two vertices, then $\chi(G) \geq \sqrt{\frac{2}{3}} R(G)$ with equality

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if and only if all non-trivial components of G are paths on three vertices, and if no pendant vertices, then $\chi(G) \geq R(G)$ with equality if and only if all non-isolated vertices have degree two.

Let G be a graph with n vertices and m edges. By Caychy-Schwarz inequality,

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}} \leq \sqrt{m \sum_{uv \in E(G)} \frac{1}{d_u + d_v}}$$

with equality if and only if the sum of degrees of the two end vertices is a constant for all edges. On the other hand, by the arithmetic-geometric mean inequality,

$$\sum_{uv \in E(G)} \frac{1}{d_u + d_v} \leq \sum_{uv \in E(G)} \frac{1}{2\sqrt{d_u d_v}} = \frac{R(G)}{2}$$

with equality if and only if every component is regular. Combining the above two inequalities, we have

Proposition 2. Let G be a graph with m edges. Then

$$\chi(G) \leq \sqrt{\frac{mR(G)}{2}}$$

with equality if and only if G is regular.

From Theorem 1, upper bounds for the product-connectivity index will result in upper bounds for the sum-connectivity index. We give several such examples.

Recall that $R(G) \leq \frac{n}{2}$ with equality if and only if every component is nontrivial and regular.[11] Thus we have

Corollary 3.[23] Let G be a graph with n vertices and m edges. Then

$$\chi(G) \leq \frac{\sqrt{nm}}{2}$$

with equality if and only if G is regular.

By Corollary 3, among n-vertex graphs the complete graph is the unique graph with maximum sum-connectivity index, equal to $\frac{n\sqrt{n-1}}{2\sqrt{2}}$,[13] and among n-vertex bipartite graphs the complete bipartite graph with one partite set with $\left\lfloor \frac{n}{2} \right\rfloor$ vertices is the unique graph with maximum sum-connectivity index, equal to $\sqrt{\frac{n}{4}\left\lfloor \frac{n^2}{4} \right\rfloor}$.

Let $\overline{G}$ be the complement of the graph G. Let $\overline{m} = \left|E(\overline{G})\right|$. From Corollary 3, we have

$$\begin{aligned}\chi(G) + \chi(\overline{G}) &\leq \frac{\sqrt{n}}{2}(\sqrt{m} + \sqrt{\overline{m}}) \\ &\leq \frac{\sqrt{n}}{2} \cdot \sqrt{2(m+\overline{m})} \\ &= \frac{\sqrt{n}}{2} \cdot \sqrt{n(n-1)} \\ &= \frac{n\sqrt{n-1}}{2}\end{aligned}$$

with equalities if and only if G and $\overline{G}$ are regular and $m = \overline{m}$, *i.e.*, G is regular and $2m = \frac{n(n-1)}{2}$, *i.e.*, G is regular of degree $\frac{n-1}{2}$. Thus we have

Corollary 4. Let G be a graph with n vertices. Then

$$\chi(G) + \chi(\overline{G}) \leq \frac{n\sqrt{n-1}}{2}$$

with equality if and only if G is regular of degree $\frac{n-1}{2}$.

Let G be a graph with n vertices. Recall that $\chi(G) + \chi(\overline{G}) \geq \frac{n\sqrt{n-1}}{2\sqrt{2}}$ with equality if and only if G or $\overline{G}$ is the complete graph.[13]

Let G be a graph with no isolated vertices and clique number ω. A particular result in Reference 24 says $R \leq \frac{\omega - 1}{2\omega}\left(\sum_{u \in V(G)} \frac{1}{\sqrt{d_u}}\right)^2$ with equality if and only if G is regular complete ω-partite graph. Thus we have

Corollary 5. Let G be a graph with no isolated vertices, m edges and clique number ω. Then

$$\chi(G) \leq \frac{1}{2}\sqrt{\frac{\omega - 1}{\omega} m \sum_{u \in V(G)} \frac{1}{\sqrt{d_u}}}$$

with equality if and only if G is regular complete ω-partite graph.

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