Existence and approximation of solutions of a system of differential equations of Volterra type*

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**Abstract.** The present paper deals with the nonlinear systems of differential equations of Volterra type regarding the existence, behaviour, approximation and stability of their definite solutions, all solutions in a corresponding region or parametric classes of solutions on an unbounded interval. The approximate solutions with precise error estimates are determined. The theory of qualitative analysis of differential equations and the topological retraction method are used.

**Key words:** Volterra system, existence and behaviour of solutions, approximation of solutions

**AMS subject classifications:** 34C05, 34D05

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1. Introduction

We shall consider the systems of differential equations of Volterra type in the form

$$x_i = f_i(x, t)x_i, \quad i = 1, \cdots, n,$$

or in the special forms

$$x_i = [p_i(t) + h_i(x, t)]x_i, \quad i = 1, \cdots, n,$$  \hspace{1cm} (2)

$$x_i = [q_i + g_i(x)]x_i, \quad i = 1, \cdots, n,$$  \hspace{1cm} (3)

where $x(t) = (x_1(t), \cdots, x_n(t))^T$, $f_i, h_i \in C(\Omega, \mathbb{R})$, $g_i \in C(D, \mathbb{R})$, $p_i \in C(I, \mathbb{R})$, $q_i \in \mathbb{R}$, $i = 1, \cdots, n$, $D \subset \mathbb{R}^n$ is an open set, $\Omega = D \times I$, $I = (a, \infty)$, $a \in \mathbb{R}$. Functions $f_i, h_i, g_i$ satisfy the Lipschitz condition with respect to the variable $x$ on $D$.

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consider the behaviour of integral curves (chemistry, biochemistry and economy. temsoftype(3)areconsideredveryoften. Thismodel isusedinphysics, biophysics, for example, K. Sigmund, Y. Takeuchi, N. Adachi, A. Tineo ([5], [6], [7]). The sys-

The vectors

be a curve in \(\Omega\)

\[
\begin{align*}
\frac{dx_1}{dt} &= (\rho - \eta x_1 - \theta x_2) x_1, \\
\frac{dx_2}{dt} &= (-\mu + \nu x_1) x_2, \quad \rho, \eta, \theta, \mu, \nu \in \mathbb{R}^+.
\end{align*}
\]

Many authors considered the systems of differential equations of Volterra type, for example, K. Sigmund, Y. Takeuchi, N. Adachi, A. Tineo ([5], [6], [7]). The sys-

tems of type (3) are considered very often. This model is used in physics, biophysics, chemistry, biochemistry and economy.

Let

\(\Gamma = \{(x, t) \in \Omega : x = \varphi(t), t \in I\}\)

be a curve in \(\Omega\), for some \(\varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t)), \varphi_i \in C^1(I, \mathbb{R})\). We shall consider the behaviour of integral curves \((x(t), t), t \in I, \) of systems (1), (2) and (3), with respect to the sets \(\omega, \sigma \subset \Omega\), which are the appropriate neighbourhoods of curve \(\Gamma\), in the forms

\[
\omega = \{(x, t) \in \Omega : \|x - \varphi(t)\| < r(t)\},
\]

\[
\sigma = \{(x, t) \in \Omega : |x_i - \varphi_i(t)| < r_i(t), i = 1, \cdots, n\}
\]

(\(|\cdot|\) is a Euclidian norm on \(\mathbb{R}^n\)), where \(r, r_i \in C^1(I, \mathbb{R}^+), i = 1, \cdots, n, \mathbb{R}^+ = \{0, \infty\}\).

The boundary surfaces of \(\omega\) and \(\sigma\) are, respectively,

\[
W = \left\{(x, t) \in \text{Cl}\omega \cap \Omega : B(x, t) := \sum_{i=1}^n (x_i - \varphi_i(t))^2 - r^2(t) = 0\right\},
\]

\[
W^k_i = \left\{(x, t) \in \text{Cl}\sigma \cap \Omega : B_i^k(x, t) := (-1)^k (x_i - \varphi_i(t)) - r_i(t) = 0\right\},
\]

\(k = 1, 2, \quad i = 1, \cdots, n\).

Let us denote the tangent vector field to an integral curve \((x(t), t), t \in I,\) of (1), (2) and (3) by \(T\). For example, for system (1) we have

\[
T(x, t) = (f_1(x, t)x_1, \cdots, f_i(x, t)x_i, \cdots, f_n(x, t)x_n, 1).
\]

The vectors \(\nabla B\) and \(\nabla B^k_i\) are the external normals on surfaces \(W\) and \(W^k_i\), respectively,

\[
\frac{1}{2} \nabla B(x, t) = \begin{pmatrix} x_1 - \varphi_1(t), & \cdots, & x_i - \varphi_i(t), & \cdots, & x_n - \varphi_n(t) \\ -\sum_{i=1}^n (x_i - \varphi_i(t)) \varphi_i'(t) - r(t) r'(t) \end{pmatrix}
\]

\[
\nabla B^k_i(x, t) = (-1)^k \begin{pmatrix} \delta_{1i}, & \cdots, & \delta_{ii}, & \cdots, & \delta_{ni}, -\varphi_i'(t) - (-1)^k r_i'(t) \end{pmatrix}
\]
where \( \delta_{mi} \) is the Kronecker delta symbol.

Considering the sign of the scalar products

\[
P(x,t) = \left( \frac{1}{2} \nabla B(x,t), T(x,t) \right)
\]

and

\[
P_i^k(x,t) = \left( \nabla B_i^k(x,t), T(x,t) \right) \quad \text{on} \; W_i^k, \; k = 1, 2, \; i = 1, \ldots, n,
\]

we shall establish the behaviour of integral curves of (1), (2) and (3) with respect to the sets \( \omega \) and \( \sigma \), respectively.

The results of this paper are based on the following Lemmas 1 and 2 (see [11]) and Lemma 3 (see [3]). In the following \( (n_1, \ldots, n_n) \), denote a permutation of indices \( (1, \ldots, n) \).

**Lemma 1.** If, for system (1), the scalar product

\[
P(x,t) = \left( \frac{1}{2} \nabla B(x,t), T(x,t) \right) < 0 \quad \text{on} \; W,
\]

then system (1) has an \( n \)-parameter class of solutions belonging to the set \( \omega \) (graphs of solutions belong to \( \omega \) for all \( t \in I \)).

According to this Lemma, for any point \( P_0 = (x^0, t_0) \in \omega \), the integral curve passing through \( P_0 \) belongs to \( \omega \) for all \( t \geq t_0 \).

**Lemma 2.** If, for system (1), the scalar product

\[
P(x,t) = \left( \frac{1}{2} \nabla B(x,t), T(x,t) \right) > 0 \quad \text{on} \; W,
\]

then system (1) has at least one solution on \( I \) whose graph belongs to the set \( \omega \) for all \( t \in I \).

**Lemma 3.** If, for the system (1), the scalar products

\[
P_i^k = \left( \nabla B_i^k, T \right) < 0 \quad \text{on} \; W_i^k, \; k = 1, 2, \; i = n_1, \ldots, n_p,
\]

and

\[
P_i^k = \left( \nabla B_i^k, T \right) > 0 \quad \text{on} \; W_i^k, \; k = 1, 2, \; i = n_{p+1}, \ldots, n_n,
\]

where \( p \in \{0, 1, \ldots, n\} \), then system (1) has a \( p \)-parameter class of solutions which belongs to the set \( \sigma \) (graphs of solutions belong to \( \sigma \) for all \( t \in I \)).

The case \( p = 0 \) means that system (1) has at least one solution belonging to the set \( \sigma \) for all \( t \in I \).

2. The \( n \)-parameter classes of solutions

First, let us consider the behaviour of integral curves of systems (1), (2) and (3) with respect to the set \( \omega \).

**Theorem 1.** If

\[
\sum_{i=1}^{n} f_i(x,t) x_i^2 < r(t) v'(t) \quad \text{or}
\]

\[
\sum_{i=1}^{n} f_i(x,t) x_i^2 < r(t) v'(t)
\]

then system (1) has at least one solution belonging to the set \( \sigma \) for all \( t \in I \).
Theorem 1. Let $f_i (x, t) < \frac{r' (t)}{r (t)}$ for $i = 1, \ldots, n$, on $W$, \hspace{1cm} (15)

then system (1) has an $n$-parameter class of solutions $x(t)$ satisfying the condition

$$
\|x(t)\| < r(t), \quad t \in I,
$$

i.e. every solution $x(t)$ of system (1) which satisfies the condition

$$
\|x(t_0)\| < r(t_0), \quad t_0 \in I,
$$

satisfies (16) for every $t \geq t_0$.

Proof. Here we have that the curve $\Gamma$ is a $t$-axis ($\varphi = 0$). For the scalar product

$$
P(x, t) = \sum_{i=1}^{n} f_i (x, t) x_i^2 - r(t) r' (t).
$$

According to (14), obviously $P(x, t) < 0$ on $W$, and according to (15), we have

$$
P(x, t) < \frac{r' (t)}{r (t)} \sum_{i=1}^{n} x_i^2 - r(t) r' (t) = \frac{r' (t)}{r (t)} r^2 (t) - r(t) r' (t) = 0 \quad \text{on } W.
$$

According to Lemma 1, the above estimates confirm the statement of the Theorem.

Using Theorem 1 we can give special results. For example, the following

Corollary 1. If

$$
f_i (x, t) < 0, \quad i = 1, \ldots, n, \quad \text{on } W,
$$

then system (1) has an $n$-parameter class of solutions $x(t)$ satisfying condition (16), where function $r$ satisfies conditions (15) and

$$
r' (t) \leq 0 \quad \text{on } I.
$$

Obviously, in the general case for function $r$ we can take an arbitrary positive constant.

Theorem 2. Let $\Gamma$ be any integral curve of system (1), $M \in C(\Omega, \mathbb{R})$ and

$$
f_i (x, t) \leq M(x, t), \quad i = 1, \ldots, n, \quad \text{on } W.
$$

If, on $W$,

$$
|f_i (x, t) - f_i (y, t)| \leq L_i \|x - y\|, \quad i = 1, \ldots, n, \quad x, y \in D
$$

and

$$
\sum_{i=1}^{n} L_i |\varphi_i(t)| \leq -M(x, t) + \frac{r'(t)}{r(t)}
$$

then system (1) has an $n$-parameter class of solutions $x(t)$ satisfying condition (16), where function $r$ satisfies conditions (15) and

$$
r' (t) \leq 0 \quad \text{on } I.
$$
or
\[ \sum_{i=1}^{n} |(f_i(x, t) - f_i(\varphi, t))\varphi_i(t)| < -M(x, t)r(t) + r'(t), \]
(22)

then system (1) has an \( n \)-parameter class of solutions \( x(t) \) satisfying the condition
\[ \|x(t) - \varphi(t)\| < r(t), \quad t \in I, \]
(23)
i.e. every solution \( x(t) \) of system (1) which satisfies the condition
\[ \|x(t_0) - \varphi(t_0)\| < r(t_0), \quad t_0 \in I, \]
(24)
satisfies (23) for every \( t \geq t_0 \).

**Proof.** For the scalar product \( P(x, t) \) on \( W \) we have
\[
P(x, t) = \sum_{i=1}^{n} f_i(x, t) [x_i - \varphi_i(t)] x_i - \sum_{i=1}^{n} x_i - \varphi_i(t) \varphi_i'(t) - r(t)r'(t)
\]
\[
= \sum_{i=1}^{n} f_i(x, t) [x_i - \varphi_i(t)]^2 + \sum_{i=1}^{n} f_i(x, t) [x_i - \varphi_i(t)] \varphi_i'(t) - r(t)r'(t)
\]
\[
= \sum_{i=1}^{n} f_i(x, t) [x_i - \varphi_i(t)]^2
\]
\[
+ \sum_{i=1}^{n} [f_i(x, t) \varphi_i(t) - \varphi_i'(t)] [x_i - \varphi_i(t)] - r(t)r'(t)
\]
\[
= \sum_{i=1}^{n} f_i(x, t) [x_i - \varphi_i(t)]^2
\]
\[
+ \sum_{i=1}^{n} [f_i(x, t) - f_i(\varphi, t) + f_i(\varphi, t) \varphi_i(t) - \varphi_i'(t)] [x_i - \varphi_i(t)]
\]
\[
- r(t)r'(t)
\]
\[
= \sum_{i=1}^{n} f_i(x, t) [x_i - \varphi_i(t)]^2 + \sum_{i=1}^{n} [f_i(x, t) - f_i(\varphi, t)] [x_i - \varphi_i(t)] \varphi_i(t)
\]
\[
- r(t)r'(t).
\]
(25)

Now, we have on \( W \), by (21)
\[
P(x, t) \leq M(x, t)r^2(t) + \sum_{i=1}^{n} L_i |\varphi_i(t)| r^2(t) - r(t)r'(t)
\]
\[
< M(x, t)r^2(t) + \left(-M(x, t) + \frac{r'(t)}{r(t)}\right) r^2(t) - r(t)r'(t) = 0,
\]
and by (22)
\[ P(x, t) \leq M(x, t) r^2(t) + \sum_{i=1}^{n} |f_i(x, t) - f_i(\varphi, t)| \varphi_i(t) r(t) - r(t) r'(t) < M(x, t) r^2(t) + (-M(x, t) r(t) + r'(t)) r(t) - r(t) r'(t) = 0. \]

The above estimates for \( P(x, t) \) on \( W \), according to Lemma 1, imply the statement of the Theorem.

**Theorem 3.** Let \( \Gamma \) be any smooth curve in \( \Omega \) and let \( m, M \in C(\Omega, \mathbb{R}) \), \( \xi \in C(\Omega, \mathbb{R}^+) \), \( \mathbb{R}_0^+ = [0, \infty) \), such that
\[
\sum_{i=1}^{n} |f_i(x, t) \varphi_i(t) - \varphi'_i(t)| \leq \xi(x, t) \quad \text{on } W. \tag{26}
\]

(i) If
\[
f_i(x, t) \leq M(x, t), \quad i = 1, \cdots, n, \quad \text{and} \quad \xi(x, t) < -M(x, t) r(t) + r'(t) \quad \text{on } W, \tag{27}
\]
then system (1) has an \( n \)-parameter class of solutions \( x(t) \) satisfying condition (23).

(ii) If
\[
m(x, t) \leq f_i(x, t), \quad i = 1, \cdots, n, \quad \text{and} \quad \xi(x, t) < m(x, t) r(t) - r'(t) \quad \text{on } W, \tag{29}
\]
then system (1) has at least one solution \( x(t) \) which satisfies condition (23).

**Proof.** In view of (25), for the scalar product \( P(x, t) \) on \( W \) we have:
\[
P = \sum_{i=1}^{n} f_i(x_i - \varphi_i)^2 + \sum_{i=1}^{n} (f_i \varphi_i - \varphi'_i) (x_i - \varphi_i) - rr' \quad \text{and}
\]

(i) \( P \leq Mr^2 + \xi r - rr' < 0, \)

(ii) \( P \geq mr^2 - \xi r - rr' > 0. \)

According to Lemmas 1 and 2, in cases (i) and (ii), respectively, the above estimates for \( P(x, t) \) on \( W \) imply the statements of the Theorem.

**Theorem 4.** Let \( (\varphi(t), t), \quad t \in I, \quad \varphi \neq 0, \) be a curve of stationary points of system (1) \( (f_i(\varphi(t), t) = 0, \quad t \in I, \quad i = 1, \cdots, n), \) let (20) hold and let \( m, M \in C(I, \mathbb{R}) \).

(i) If
\[
f_i(x, t) \leq M(t), \quad i = 1, \cdots, n, \quad (x, t) \in W \quad \text{and} \quad r(t) \sum_{i=1}^{n} L_i |\varphi_i(t)| + \sum_{i=1}^{n} |\varphi'_i(t)| < -M(t) r(t) + r'(t), \quad t \in I,
\]
then system (1) has an \( n \)-parameter class of solutions \( x(t) \) satisfying condition (23).
The conditions of the Theorem imply the estimates in case (ii) according to The above estimates for

\[\text{Lemma 2 in case (ii)},\]

(ii) If

\[m(t) \leq f_i(x,t), \quad i = 1, \cdots, n, \quad (x,t) \in W \quad \text{and} \quad r(t) \sum_{i=1}^{n} L_i |\varphi_i(t)| + \sum_{i=1}^{n} |\varphi'_i(t)| < m(t) r(t) - r'(t), \quad t \in I,\]

then system (1) has at least one solution \(x(t)\) which satisfies condition (23).

**Proof.** From (25) we have on \(W\)

\[P(x,t) = \sum_{i=1}^{n} f_i(x,t) [x_i - \varphi_i(t)]^2 \]

\[+ \sum_{i=1}^{n} \{ [f_i(x,t) - f_i(\varphi,t)] \varphi_i(t) - \varphi'_i(t) \} [x_i - \varphi_i(t)] - r(t) r'(t) \]

\[= \sum_{i=1}^{n} f_i(x,t) [x_i - \varphi_i(t)]^2 + \sum_{i=1}^{n} [f_i(x,t) - f_i(\varphi,t)] [x_i - \varphi_i(t)] \varphi_i(t) \]

\[- \sum_{i=1}^{n} \varphi'_i(t) [x_i - \varphi_i(t)] - r(t) r'(t).\]

The conditions of the Theorem imply the estimates in case (i) and (ii), respectively:

\[P(x,t) \leq M(t) r^2(t) + \sum_{i=1}^{n} L_i |\varphi_i(t)| r^2(t) + \sum_{i=1}^{n} |\varphi'_i(t)| r(t) - r(t) r'(t) < 0,\]

\[P(x,t) \geq m(t) r^2(t) - \sum_{i=1}^{n} L_i |\varphi_i(t)| r^2(t) - \sum_{i=1}^{n} |\varphi'_i(t)| r(t) - r(t) r'(t) > 0.\]

The above estimates for \(P(x,t)\) on \(W\), according to Lemma 1 in case (i) and according to Lemma 2 in case (ii), imply the statements of the Theorem. \(\Box\)

Now let us consider system (2).

**Theorem 5.** Let \(m, M \in C(\Omega, R)\) and

\[\varphi_i(t) = C_i \exp \left[ \int p_i(t) dt \right], \quad i = 1, \cdots, n, \quad C_i \in \mathbb{R}\]  \hspace{1cm} (31)

(i) If

\[p_i(t) + h_i(x,t) \leq M(x,t), \quad i = 1, \cdots, n \quad \text{and} \quad \sum_{i=1}^{n} |h_i(x,t) \varphi_i(t)| \leq -M(x,t) r(t) + r'(t) \quad \text{on} \ W,\]

then system (2) has an \(n\)-parameter class of solutions \(x(t)\) satisfying condition (23).

(ii) If

\[m(x,t) \leq p_i(x,t) + h_i(x,t), \quad i = 1, 2, \cdots, n \quad \text{and} \]

\[p_i(t) + h_i(x,t) \leq M(x,t), \quad i = 1, \cdots, n \quad \text{and} \]

\[\sum_{i=1}^{n} |h_i(x,t) \varphi_i(t)| \leq -M(x,t) r(t) + r'(t) \quad \text{on} \ W,\]

then system (2) has an \(n\)-parameter class of solutions \(x(t)\) satisfying condition (23).
\[
\sum_{i=1}^{n} |h_i(x,t)\varphi_i(t)| \leq m(x,t) r(t) + r'(t) \quad \text{on } W,
\]

then system (2) has at least one solution \( x(t) \) which satisfies condition (23).

**Proof.** Firstly, we can note that \( p_i(t)\varphi_i(t) - \varphi_i'(t) = 0, \quad i = 1, \ldots, n, \quad t \in I. \)

Now, according to (25) and conditions of this theorem, we have on \( W \):

\[
P = \sum_{i=1}^{n} (p_i + h_i)(x_i - \varphi_i)^2 + \sum_{i=1}^{n} [h_i\varphi_i - \varphi_i'](x_i - \varphi_i) - rr'
\]

and

\[
\begin{align*}
(i) & \quad P \leq r^2 \sum_{i=1}^{n} (p_i + h_i) + r \sum_{i=1}^{n} |h_i\varphi_i| - rr' \\
& \leq Mr^2 + r \sum_{i=1}^{n} |h_i\varphi_i| - rr' < 0, \quad (32) \\
(ii) & \quad P \geq Mr^2 - r \sum_{i=1}^{n} |h_i\varphi_i| - rr' > 0. \quad (33)
\end{align*}
\]

According to Lemma 1, estimate (32) implies that system (2) has the \( n \)-parameter class of solutions \( x(t) \) belonging to the corresponding set \( \omega \) for all \( t \in I \), and estimate (33) implies that system (2) has at least one solution \( x(t) \) which satisfies that condition. This confirms the statements of the Theorem. \( \square \)

Now let us consider system (3) in neighbourhood of \((x^0, t)\), \( t \in I \), where \( x^0 = (x^0_1, \ldots, x^0_n) \in \mathbb{R}^n \) and

\[
q_i + g_i(x^0) = 0, \quad i = 1, \ldots, n. \quad (34)
\]

**Theorem 6.** Let (34) and

\[
|g_i(x) - g_i(y)| \leq L_i \|x - y\|, \quad i = 1, \ldots, n, \quad x, y \in D \quad (35)
\]

\[
q_i + g_i(x) \leq Q \in \mathbb{R}, \quad i = 1, \ldots, n, \quad \text{on } D. \quad (36)
\]

If

\[
\sum_{i=1}^{n} L_i |x^0_i| < -Q + \frac{r'(t)}{r(t)}, \quad t \in I, \quad (36)
\]

then system (3) has an \( n \)-parameter class of solutions \( x(t) \) satisfying the condition

\[
\|x(t) - x^0\| < r(t), \quad t \in I. \quad (37)
\]
Proof. For the scalar product $P(x,t)$ on $W$, using (25) with $f_i(x,t) = q_i + g_i(x)$, $\varphi_i = x_0^i$, $i = 1, \ldots, n$, we have

$$P(x,t) = \sum_{i=1}^{n} (q_i + g_i(x)) (x_i - x_0^i)^2 + \sum_{i=1}^{n} (q_i + g_i(x)) x_0^i (x_i - x_0^i) - r(t) r'(t)$$

$$= \sum_{i=1}^{n} (q_i + g_i(x)) (x_i - x_0^i)^2 + \sum_{i=1}^{n} (g_i(x) - g_i(x_0^i)) x_0^i (x_i - x_0^i) - r(t) r'(t)$$

$$\leq Qr^2(t) + \sum_{i=1}^{n} L_i |x_0^i| r^2(t) - r(t) r'(t) < 0.$$

Hence, according to Lemma 1, the statement of the Theorem is valid. □

**Corollary 2.** If in Theorem 6 condition (36) is replaced by

$$\sum_{i=1}^{n} L_i |x_0^i| + Q < 0,$$

the statement of Theorem 6 holds with the function

$$r(t) = \alpha e^{-\beta t}, \quad \alpha, \beta \in \mathbb{R}^+ \text{ and } 0 < \beta \leq s = -\left(\sum_{i=1}^{n} L_i |x_0^i| + Q\right).$$

For $\beta = s$ condition (37) should be replaced by

$$\|x(t) - x_0^i\| \leq r(t), \quad t \in I.$$
Theorem 8. If, on $W^k_i$, $k = 1, 2$, $i = 1, \ldots, n$, we have

\[ P^k_i(x, t) = (-1)^k f_i(x, t) x_i - r'_i(t) \]

and

\[ P^k_i(x, t) < 0 \quad \text{for} \quad i = n_1, \ldots, n_p, \]

\[ P^k_i(x, t) > 0 \quad \text{for} \quad i = n_{p+1}, \ldots, n_n. \]

According to Lemma 3, the above estimates imply the statement of the Theorem. □

Using Theorem 7 we can give special results. For example,

Corollary 3. If, on $W^k_i$, $k = 1, 2$,

\[ f_i(x, t) < 0, \quad i = n_1, \ldots, n_p, \]

\[ f_i(x, t) > 0, \quad i = n_{p+1}, \ldots, n_n, \]

then system (1) has a $p$-parameter class of solutions $x(t)$ which satisfy condition

\[ |x_i(t)| < r(t), \quad i = 1, \ldots, n, \quad t \in I, \]

where function $r$ satisfies conditions (18) and, on $W^k_i$, $k = 1, 2$,

\[ f_i(x, t) - \frac{r'(t)}{r(t)} < 0, \quad i = n_1, \ldots, n_p. \]

For function $r$ we can take, for example, some positive constant.

Theorem 8. If, on $W^k_i$, $k = 1, 2$,

\[ |f_i(x, t) x_i - \varphi'_i(t)| < r'_i(t) \quad \text{or} \]

\[ |f_i(x, t) \varphi_i(t) - \varphi'_i(t)| < -f_i(x, t) r_i(t) + r'_i(t) \]

for $i = n_1, \ldots, n_p$, and

\[ |f_i(x, t) x_i - \varphi'_i(t)| < -r'_i(t) \quad \text{or} \]

\[ |f_i(x, t) \varphi_i(t) - \varphi'_i(t)| < f_i(x, t) r_i(t) - r'_i(t) \]

for $i = n_{p+1}, \ldots, n_n$, then system (1) has a $p$-parameter class of solutions $x(t)$ satisfying the condition

\[ |x_i(t) - \varphi_i(t)| < r_i(t), \quad i = 1, \ldots, n, \quad t \in I. \]

Proof. Here for $P^k_i(x, t)$ on $W^k_i$, $k = 1, 2, i = 1, \ldots, n$, we have

\[ P^k_i(x, t) = (-1)^k f_i(x, t) x_i - (-1)^k \varphi'_i(t) - r'_i(t) \]

\[ = (-1)^k [f_i(x, t) x_i - \varphi'_i(t)] - r'_i(t) \]
or

$$P_i^k (x,t) = (-1)^k [x_i - \varphi_i (t)] f_i (x,t) + (-1)^k [f_i (x,t) \varphi_i (t) - \varphi_i' (t)] - r_i' (t)$$

$$= f_i (x,t) r_i (t) + (-1)^k [f_i (x,t) \varphi_i (t) - \varphi_i' (t)] - r_i' (t).$$

(47)

Now (41) with (46) or (42) with (47) imply, respectively,

$$P_i^k \leq |f_i x_i - \varphi_i' | - r_i' < 0,$$

$$P_i^k \leq f_i r_i + |f_i \varphi_i - \varphi_i' | - r_i' < 0$$

on $W_i^k$, $k = 1, 2$, for $i = n_1, \cdots, n_p$. The conditions (43) with (46) or (44) with (47) imply, respectively,

$$P_i^k \geq - |f_i x_i - \varphi_i' | - r_i' > 0,$$

$$P_i^k \geq f_i r_i - |f_i \varphi_i - \varphi_i' | - r_i' > 0$$

on $W_i^k$, $k = 1, 2$, for $i = n_{p+1}, \cdots, n_n$. These estimates, according to the Lemma 3, confirm the statement of this Theorem.

**Theorem 9.** Let $\Gamma$ be any integral curve of system (1), let (20) hold and let

$$\rho (t) = \sqrt{r_1^2 (t) + \cdots + r_n^2 (t)}. \quad (48)$$

If, on $W_i^k$, $k = 1, 2$,

$$L_i |\varphi_i (t)| \rho (t) < - f_i (x,t) r_i (t) + r_i' (t)$$

for $i = n_1, \cdots, n_p$, and

$$L_i |\varphi_i (t)| \rho (t) < f_i (x,t) r_i (t) - r_i' (t)$$

for $i = n_{p+1}, \cdots, n_n$, then system (1) has a $p$-parameter class of solutions $x(t)$ satisfying condition (45).

**Proof.** In view of (47) for $P_i^k (x,t)$ on $W_i^k$, $k = 1, 2$, we have

$$P_i^k (x,t) = f_i (x,t) r_i (t) + (-1)^k [(f_i (x,t) - f_i (\varphi, t) + f_i (\varphi, t)) \varphi_i (t) - \varphi_i' (t)] - r_i' (t)$$

$$= f_i (x,t) r_i (t) + (-1)^k [f_i (x,t) - f_i (\varphi, t)] \varphi_i (t) - r_i' (t).$$

Now, it is enough to note that, on $W_i^k$, $k = 1, 2$,

$$P_i^k (x,t) \leq f_i (x,t) r_i (t) + |f_i (x,t) - f_i (\varphi, t)||\varphi_i (t)| - r_i' (t)$$

$$\leq f_i (x,t) r_i (t) + L_i \|x - \varphi\| \|\varphi_i (t)| - r_i' (t)$$

$$= f_i (x,t) r_i (t) + L_i \rho(t) |\varphi_i (t)| - r_i' (t) < 0$$

for $i = n_1, \cdots, n_p$, and

$$P_i^k (x,t) \geq f_i (x,t) r_i (t) - L_i \rho(t) |\varphi_i (t)| - r_i' (t) > 0$$

for $i = n_{p+1}, \cdots, n_n$. \[\square\]
Theorem 10. Let $\Gamma$ be a curve of stationary points of system (1) different from the $t-$axis, i.e. $f_i(\varphi(t), t) = 0$, $i = 1, \ldots, n$, $t \in I$, $\varphi \neq 0$, and let (20) and (48) be valid. If, on $W^k_i$, $k = 1, 2$,

$$L_i |\varphi_i(t)| \rho(t) + |\varphi_i'(t)| < -f_i(x, t) r_i(t) + r_i'(t), \quad i = n_1, \ldots, n_p,$$

$$L_i |\varphi_i(t)| \rho(t) + |\varphi_i'(t)| < f_i(x, t) r_i(t) - r_i'(t), \quad i = n_p + 1, \ldots, n_n,$$

then system (1) has a $p$-parameter class of solutions $x(t)$ satisfying condition (45).

Proof. In view of (47) we have, on $W^k_i$, $k = 1, 2$,

$$P^k_i(x, t) = f_i(x, t) r_i(t) + (-1)^k |(f_i(x, t) - f_i(\varphi, t)) \varphi_i(t) - \varphi_i'(t)| - r_i'(t),$$

$$P^k_i(x, t) \leq f_i(x, t) r_i(t) + L_i |x - \varphi| |\varphi_i(t)| + |\varphi_i'(t)| - r_i'(t)$$

$$= f_i(x, t) r_i(t) + L_i |\varphi_i(t)| \rho(t) + |\varphi_i'(t)| - r_i'(t) < 0, \quad i = n_1, \ldots, n_p,$$

$$P^k_i(x, t) \geq f_i(x, t) r_i(t) - L_i |\varphi_i(t)| \rho(t) - |\varphi_i'(t)| - r_i'(t) > 0, \quad i = n_p + 1, \ldots, n_n.$$

These estimates confirm the statement of this Theorem.

Let us now consider the behaviour of integral curves of system (2) with respect to the set $\sigma$, where $\varphi$ is defined by (31).

Theorem 11. If, on $W^k_i$, $k = 1, 2$,

$$|h_i(x, t) \varphi_i(t)| < -(p_i(t) + h_i(x, t)) r_i(t) + r_i'(t)$$

for $i = n_1, \ldots, n_p$, and

$$|h_i(x, t) \varphi_i(t)| < (p_i(t) + h_i(x, t)) r_i(t) - r_i'(t)$$

for $i = n_p + 1, \ldots, n_n$, then system (2) has a $p$-parameter class of solutions $x(t)$ satisfying condition (45), where $\varphi$ is defined by (31).

Proof. According to (47), we have, on $W^k_i$, $k = 1, 2$,

$$P^k_i = (p_i + h_i) r_i + (-1)^k [(p_i + h_i) \varphi_i - \varphi_i'] - r_i'$$

$$= (p_i + h_i) r_i + (-1)^k h_i \varphi_i - r_i'.$$

Now, it is enough to note that, on $W^k_i$, $k = 1, 2$,

$$P^k_i \leq (p_i + h_i) r_i + |h_i \varphi_i| - r_i' < 0$$

for $i = n_1, \ldots, n_p$, and

$$P^k_i \geq (p_i + h_i) r_i - |h_i \varphi_i| - r_i' > 0$$

for $i = n_p + 1, \ldots, n_n$. \qed

Now let us consider the behaviour of integral curves of system (3) in neighbourhood of $(x^0, t), t \in I$, where $x^0 \in \mathbb{R}^n$.

Theorem 12. Let (34) and (35) be valid and let function $\rho(t)$ be defined by (48). If, on $W^k_i$, $k = 1, 2$,

$$L_i |x^0| \rho(t) < -(q_i + g_i(x)) r_i(t) + r_i'(t)$$

(49)
for \( i = n_1, \ldots, n_p, \) and
\[
L_i |x_i^0| g(t) < (q_i + g_i(x)) r_i(t) - r'_i(t) \tag{50}
\]
for \( i = n_{p+1}, \ldots, n_n, \) then system (3) has a \( p \)-parameter class of solutions \( x(t) \) satisfying the condition
\[
|x_i(t) - x_i^0| < r_i(t), \quad i = 1, \ldots, n, \quad t \in I.
\]

**Proof.** Using (47) we have, on \( W^k, k = 1, 2, \)
\[
P^k_i(x, t) = (q_i + g_i(x)) r_i(t) + (-1)^k (q_i + g_i(x)) x_i^0 - r'_i(t)
\]
\[
= (q_i + g_i(x)) r_i(t) + (-1)^k [g_i + g_i(x^0) + g_i(x) - g_i(x^0)] x_i^0 - r'_i(t)
\]
\[
= (q_i + g_i(x)) r_i(t) + (-1)^k [g_i(x) - g_i(x^0)] x_i^0 - r'_i(t)
\]
and
\[
P^k_i(x, t) \leq (q_i + g_i(x)) r_i(t) + L_i |x_i^0| g(t) - r'_i(t) < 0
\]
for \( i = n_1, \ldots, n_p, \) and
\[
P^k_i(x, t) \geq (q_i + g_i(x)) r_i(t) - L_i |x_i^0| g(t) - r'_i(t) > 0
\]
for \( i = n_{p+1}, \ldots, n_n. \) These estimates confirm this Theorem.

We can note that in case \( x^0 = 0 \) Theorem 12 holds without assumption (35).

4. **Examples**

Let us consider two known examples.

**Example 1.** *The Lotka-Volterra model (4)*
\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1 x_2, \\
\dot{x}_2 &= -x_2 + x_1 x_2.
\end{align*}
\tag{52}
\]

**Corollary 4.** Let functions \( r_1, r_2 \in C^1(I, \mathbb{R}^+) \) satisfy the conditions
\[
r_1(t) < 1 + \frac{r'_2(t)}{r_2(t)}, \quad r_2(t) < 1 - \frac{r'_1(t)}{r_1(t)}, \quad t \in I.
\]
System (52) has a one-parameter class of solutions \((x_1(t), x_2(t))\) satisfying the condition
\[
|x_1(t)| < r_1(t), \quad |x_2(t)| < r_2(t), \quad t \in I.
\]

This Corollary follows from Theorem 7. Conditions (38) and (39) are valid for \( i = 2 \) and \( i = 1, \) respectively. Here we have on \( W^k, (i, k = 1, 2):\)
\[
P^k_i(x_1, x_2, t) = (x_1 - 1) r_2(t) - r'_2(t) \leq (r_1(t) - 1) r_2(t) - r'_2(t) < 0,
\]
\[
P^k_i(x_1, x_2, t) = (1 - x_2) r_1(t) - r'_1(t) \geq (1 - r_2(t)) r_1(t) - r'_1(t) > 0.
\]
For functions \( r_i \) we can take, for example,
\[
r_1 (t) = r_2 (t) = a e^{-\beta t}, \quad \alpha, \beta \in \mathbb{R}^+, \quad \alpha + \beta \leq 1, \quad t > 0.
\]

**Example 2.** The Volterra model (4), applying appropriate substitution ([4]), can be written in the form
\[
\begin{align*}
\dot{x}_1 &= (1 - x_1 - ax_2) x_1, \\
\dot{x}_2 &= (-b + ax_1) x_2, \quad a, b \in \mathbb{R}^+.
\end{align*}
\]

**Corollary 5.** Let \( \alpha, \beta \in \mathbb{R}^+ \).

(i) System (53) has a one-parameter class of solutions \( x(t) \) which satisfy the condition
\[
|x_1 (t)| < a e^{-\beta t}, \quad |x_2 (t)| < a e^{-\beta t} \quad \text{for} \quad t > 0,
\]
where
\[
1 + \beta \geq \alpha (1 + \alpha), \quad b \geq a \alpha + \beta.
\]

(ii) System (53) has a one-parameter class of solutions \( x(t) \) which satisfy the condition
\[
|x_1 (t) - 1| < a e^{-\beta t}, \quad |x_2 (t)| < a e^{-\beta t} \quad \text{for} \quad t > 0,
\]
where
\[
\beta \geq (1 + a) (1 + \alpha), \quad b \geq a (1 + \alpha) + \beta.
\]

The statements of this Corollary follow from Theorem 12. Conditions (49) and (50) hold for \( i = 2 \) and \( i = 1 \), respectively, in both cases (i) and (ii). In this example we consider \( x^0 = (0, 0) \) in case (i) and \( x^0 = (1, 0) \) in case (ii). According to (51), here we have on \( W_k^t \) \( i, k = 1, 2 \):

(i)
\[
\begin{align*}
P^k_2 (x_1, x_2, t) &= (-b + ax_1) a e^{-\beta t} + \beta a e^{-\beta t} < (-b + a a + \beta) a e^{-\beta t} \leq 0, \\
P^k_1 (x_1, x_2, t) &= (1 - x_1 - ax_2) a e^{-\beta t} + \beta a e^{-\beta t} > (1 - a - a a + \beta) a e^{-\beta t} \geq 0;
\end{align*}
\]

(ii)
\[
\begin{align*}
P^k_2 (x_1, x_2, t) &= (-b + ax_1) a e^{-\beta t} + \beta a e^{-\beta t} < (-b + a (1 + \alpha) + \beta) a e^{-\beta t} \leq 0, \\
P^k_1 (x_1, x_2, t) &= (1 - x_1 - ax_2) a e^{-\beta t} + (-1)^k (1 - x_1 - ax_2) + \beta a e^{-\beta t} \\
&\geq (-a e^{-\beta t} - a a e^{-\beta t}) a e^{-\beta t} - a e^{-\beta t} - a a e^{-\beta t} + \beta a e^{-\beta t} \\
&> (\alpha - a - 1 - a + \beta) a e^{-\beta t} \geq 0.
\end{align*}
\]

**Remark.** We can note that the obtained results also contain answers to questions on approximation and stability or instability of solutions \( x(t) \) whose existence is established. The errors of the approximation and the functions of stability or instability (including autostability and stability along the coordinates) are defined by functions \( r(t) \) and \( r_i (t), \quad i = 1, \cdots, n \) (see [9], [10], [11]).
References


