The unramified subquotient of the unramified principal series

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Abstract. We prove that, in a certain induced representation of $p$–adic symplectic group, the unramified subquotient appears as a subrepresentation. This result has not only local importance, but is also very useful in calculations with automorphic representations of the corresponding group over adeles, since for an irreducible automorphic representation, almost every local component representation is unramified.

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1. Introduction and preliminaries

Let $F$ be a local non-archimedean field of characteristic zero, with ring of integers $O_F$. We are interested in the admissible representations of the symplectic group, which we realize as a matrix group in the following way:

$$Sp_{2n}(F) = \left\{ g \in GL_{2n}(F) : g \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} \right\}$$

where $J_n$ is the $n \times n$ matrix defined by $J_n = \begin{bmatrix} 1 & & \\
 & \ddots & \\
 & & 1 \end{bmatrix}$.

We fix $K = Sp_{2n}(O_F)$ as a maximal compact subgroup of $Sp_{2n}(F)$. We say that an irreducible admissible representation of $Sp_{2n}(F)$ is unramified if it has a non-zero $K$-fixed vector (also is called $K$–spherical). Then, necessarily, this vector is unique, up to a scalar. We fix a Borel subgroup $B_n$ of $Sp_{2n}(F)$ consisting of all the upper-triangular matrices in $Sp_{2n}(F)$, and according to that choice, block-upper-triangular matrices form the standard parabolic subgroups. Each such subgroup is uniquely determined by an ordered partition $(n_1, n_2, \ldots, n_k)$ of $m \leq n$; in that case the corresponding standard parabolic subgroup, denoted by $P_{(n_1, \ldots, n_k)}$, has Levi subgroup isomorphic to $GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times Sp_{2(n-m)}(F)$ (if $m = n$ the last factor in this product is not there). In this situation, for admissible representations $\rho_i$ of $GL(n_i, F)$, $i = 1, \ldots, k$ and an admissible representation $\sigma$ of $Sp_{2(n-m)}(F)$, we denote the parabolically induced (normalized) representation

$$\text{Ind}_{P_{(n_1, \ldots, n_k)}}^{Sp_{2n}(F)}(\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \sigma)$$

by $\rho_1 \times \rho_2 \times \cdots \times \rho_k \times \sigma$.

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For a unitary character $\chi$ of $GL(1, F)$, and $\alpha, \beta \in \mathbb{R}$ such that $\beta + \alpha + 1 \in \mathbb{Z}_{\geq 1}$, we denote by $\zeta(-\beta, \alpha; \chi)$ the unique irreducible subrepresentation of

$$\text{Ind}_{B_n}^{GL_{\beta + \alpha + 1}}(\nu^{-\beta} \chi \otimes \cdots \otimes \nu^\alpha \chi)$$

(which is a character $\nu^{\frac{-\beta^2}{2}} \chi \circ \det$ of $GL(\beta + \alpha + 1, F)$). Here $\nu$ denotes a non-archimedean absolute value on $F$.

In this paper we want to prove that in certain (parabolically induced) unramified principal series representations the (unique) irreducible unramified subquotient appears as a subrepresentation. This result has an immediate application in the global calculations with automorphic forms, since an automorphic representation of $Sp_{2n}(\mathbb{A})$ has, at almost every local place, an unramified representation as a local constituent (e.g., we use this result in [3]).

To be more precise, we prove the following (the notion of negative unramified representation of a symplectic $p$-adic group is defined below):

**Theorem 1.** Let $\beta > \alpha > 0$ be integers, and $\chi$ an unramified character of $F$ with $\chi^2 = 1$. Let $\pi$ be a negative representation. Then, an irreducible unramified subquotient of $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ is a subrepresentation; it is also a negative representation.

We need the following simple, but important results:

**Lemma 1.** Let $\pi$ be an irreducible representation of a reductive $p$-adic group and let $P = MN$ be a parabolic subgroup of $G$. Suppose that $M$ is a direct product of two reductive subgroups $M_1$ and $M_2$. Let $\tau_1$ be an irreducible representation of $M_1$ and let $\tau_2$ be a representation of $M_2$. Suppose that $\pi \hookrightarrow \text{Ind}_P^G(\tau_1 \otimes \tau_2)$. Then there exists an irreducible representation $\tau'_2$ such that $\pi \hookrightarrow \text{Ind}_P^G(\tau_1 \otimes \tau'_2)$.

**Proof.** This is Lemma 3.2 of [5].

**Lemma 2.** Let $G = Sp_{2n}(F)$ and $K$ as above. Assume that $\sigma$ is $K$-spherical smooth representation of $G$, and $\sigma$ is a subquotient of $\text{Ind}_{MN}(\sigma' \otimes 1_N)$, for some smooth representation $\sigma'$ of $M$. Then $\sigma'$ is $M \cap K$-spherical.

**Proof.** This is Lemma 1.1 (ii) of [7].

We also need this fundamental result (essentially due to Tadić [9]), cf. Theorem 3.1 of [7]. Here, for a smooth representation $\sigma$ of $Sp_{2n}(F)$, the expression $\mu^*(\sigma)$ denotes (a semisimplification) of the sum of the Jacquet modules of $\sigma$ with respect to all standard maximal parabolic subgroups of $Sp_{2n}(F)$.

**Theorem 2.** Let $\sigma$ be an irreducible admissible representation of $Sp_{2n}(F)$. We decompose into irreducible constituents in the appropriate Grothendieck group

$$\mu^*(\sigma) = \sum_{\xi, \sigma'} \xi \otimes \sigma'.$$
Assume that $\alpha, \beta, \in \mathbb{R}$, $\alpha + \beta + 1 \in \mathbb{Z}_{>0}$, and $\chi$ is character of $F^*$. Then the following holds

$$\mu^*(\zeta(-\beta, \alpha; \chi) \rtimes \sigma) = \sum_{\xi, \sigma'} \sum_{i=0}^{\alpha + \beta + 1} \sum_{j=0}^{i} \zeta(-\alpha, \beta - i; \chi^{-1}) \times \zeta(-\beta, j - \beta - 1; \chi) \times \xi \otimes \zeta(j - \beta, i - \beta - 1; \chi) \rtimes \sigma'.$$

For an irreducible representation $\pi$ of $Sp_{2n}(F)$, by $r_{1,\ldots,1;0}(\pi)$ we denote the Jacquet module of that representation with respect to the Borel subgroup $B_n$. An irreducible admissible unramified representation $\pi$ of $Sp_{2n}(F)$ is strictly (or strongly) negative if for every irreducible subquotient $\chi_{1}^{\mu_{1}} \otimes \cdots \otimes \chi_{n}^{\mu_{n}}$ of $r_{1,\ldots,1;0}(\pi)$ (where $\chi_{i}$ are unitary characters, $s_{i} \in \mathbb{R}; i = 1, \ldots, n$), the following holds

$$s_{1} < 0, \quad (1)$$
$$s_{1} + s_{2} < 0, \quad (2)$$
$$\vdots \quad (3)$$
$$s_{1} + s_{2} + \cdots + s_{n} < 0. \quad (4)$$

We say that an unramified representation is negative if, in the situation as above, inequalities are not necessarily strict (i.e., $\leq$ holds).

Let $\chi_0$ be the unique quadratic non–trivial unramified character of $F$.

The Jordan blocks are defined for an unramified strongly negative and negative representations of a symplectic group $Sp_{2n}(F)$ as follows. For an unramified strictly negative representation $\sigma$ of $Sp_{2n}(F)$ there exists (a unique) set of positive odd rational integers $2m_{1} + 1 < 2m_{2} + 1 < \ldots < 2m_{l} + 1$ and $2n_{1} + 1 < 2n_{2} + 1 < \ldots < 2n_{k} + 1$ such that $k$ is even and $2m_{1} + 1 + \cdots + 2m_{l} + 1 + 2n_{1} + 1 + \cdots + 2n_{k} + 1 = 2n + 1$ (so $l$ is odd) such that

$$\sigma \hookrightarrow \zeta(-n_{k}, n_{k-1}; \chi_{0}) \times \cdots \times \zeta(-n_{2}, n_{1}; \chi_{0}) \times \zeta(-m_{l}, m_{l-1}; 1) \times \cdots \times \zeta(-m_{3}, m_{2}; 1) \times \zeta(-m_{1}, -1; 1) \times 1,$$

where, if $m_{1} = 0$, there is no factor $\zeta(-m_{1}, -1; 1)$ (cf. [7], Lemma 5.5). Then, we define

$$\text{Jord}(\sigma) = \{(\chi_{0}, 2n_{1} + 1), \ldots, (\chi_{0}, 2n_{k} + 1), (1, 2m_{1} + 1), \ldots, (1, 2m_{l} + 1)\}.$$

For a negative representation $\sigma_{neg}$ there exist a unique strongly negative representation $\sigma_{sn}$ and pairs $(\chi_{1}, l_{1}), \ldots, (\chi_{j}, l_{j})$ $(l_{i} \in \mathbb{Z}_{\geq 1}, \chi_{i}$ unramified unitary characters) unique up to a permutation and taking inverses of characters, such that (cf.[8] Theorem 0-3)

$$\sigma_{neg} \hookrightarrow x_{i=1}^{j} \zeta(-\frac{l_{i} - 1}{2}; -\frac{l_{i} - 1}{2}; \chi_{i}) \rtimes \sigma_{sn}.$$  

Then we define a multiset $\text{Jord}(\sigma_{neg}) = \text{Jord}(\sigma_{sn}) + \sum_{i=1}^{k} \{(\chi_{i}, l_{i}), (\chi_{i}^{-1}, l_{i})\}$. If $\chi$ is an unramified unitary character, and $\sigma$ negative or strongly negative unramified representation, we denote $\text{Jord}_{\chi}(\sigma) = \{a : (\chi, a) \in \text{Jord}(\sigma)\}$. 
2. The proof of the main theorem

As we can see from (5), Lemma 5.5 of [7] gives an embedding of strictly negative unramified representations in principal series representations (more precisely, an embedding in a representation which is a subrepresentation of principal series representations) but the exponents in this principal series are given in a precise order. But often, especially in global calculations with automorphic forms, this is not enough. We need to recognize the position of the unramified irreducible subquotient inside a more general induced representation. In this section we prove that we can embed strictly negative and negative representations in different ways in degenerate principal series representations, i.e., we prove, at the end, the Theorem mentioned in the Introduction. We prove it as a consequence of couple of propositions.

Let \( \chi \) be an unramified character of \( F \) with \( \chi^2 = 1 \). Let \( \pi \) be a strongly negative representation of \( Sp_{2n}(F) \), and \( \alpha \) and \( \beta \) rational integers satisfying \( \beta > \alpha \geq 0 \).

**Proposition 1.** If \([2\alpha+1,2\beta+1] \cap \text{Jord}_k(\pi) = \emptyset\), then, in the appropriate Grothendieck group, we have

\[
\zeta(-\beta, \alpha; \chi) \times \pi = \pi_1 + \pi_2 + \Pi,
\]

where \( \pi_1 \) and \( \pi_2 \) are non-isomorphic irreducible subrepresentations of \( \zeta(-\beta, \alpha; \chi) \times \pi \) (now we view it as a genuine representation, not in the semisimplification), one of them is strongly negative spherical; and \( \Pi \) is the unique irreducible quotient.

**Proof.** We know that, up to a sign, Aubert duality ([1, 2]) (at the level of Grothendieck groups) applied to an irreducible representation \( \pi \) gives again an irreducible representation ([1], Corollaire 3.9). We denote this (genuine) representation by \( \hat{\pi} \). We use the fact that, under the Aubert involution, the duals of strongly negative representations are square-integrable representations. In the case of the cuspidal support we have here, this involution is equivalent to Iwahori-Matsumoto involution. This means that \( \hat{\pi} \) is a square-integrable representation with the same cuspidal support as \( \pi \). Also, if \((\chi, a) \in \text{Jord}(\hat{\pi})\), then \((\chi, a) \in \text{Jord}(\pi)\) where now \( \text{Jord}(\hat{\pi}) \) is Jordan block of square-integrable representation, as defined by Moeglin ([4, 5]). Indeed, assume that \((\chi, a) \in \text{Jord}(\hat{\pi})\). Then \( \delta(\nu^2 \chi, \nu^2 \chi) \times \hat{\pi} \) is irreducible and \( a \) is odd (since \( L(\chi, \Lambda^2 \mathbb{C}, s) = 1 \)) (cf. for example, Section 2 of [5] where Jordan block of a square-integrable representation of a symplectic group is defined).

But, according to [1], the representation \( \delta(\nu^2 \chi, \nu^2 \chi) \times \hat{\pi} \) is irreducible if and only if \( \zeta(-\frac{a-1}{2}, \frac{a+1}{2}; \chi) \times \pi \) is irreducible. So, if \( a \in \text{Jord}_k(\hat{\pi}) \), then \( \zeta(-\frac{a-1}{2}, \frac{a+1}{2}; \chi) \times \pi \) is irreducible. But then, (if we also assume \( a \geq 3 \) if \( \chi = 1 \)) by Corollary 5.1 of [7], \( a \in \text{Jord}_k(\pi) \). This means \( \text{Jord}(\hat{\pi}) \subset \text{Jord}(\pi) \), unless \( (1_{GL_1}, 1) \in \text{Jord}(\hat{\pi}) \); this case is a bit more subtle. Namely, if \( \pi \) is embedded in the induced representation as in (5) below, we want to prove that the irreducibility of \( 1_{GL_1} \times \pi \) forces that the representation \( \zeta(-m_1, -1; 1) \) does not appear in the induced representation, i.e., that \( m_1 = 0 \), and \((1_{GL_1}, 1) \in \text{Jord}(\pi) \). If we assume the opposite, the irreducible representation \( 1_{GL_1} \times \pi \) is a subrepresentation of \( \Pi_1 = \zeta(-m_k, m_{k-1}; 1) \times \cdots \times \zeta(-m_3, m_2; 1) \times \pi_0 \), where \( \pi_0 \) is a unique irreducible spherical (negative) subrepresentation of \( 1_{GL_1} \times \zeta(-m_1, -1; 1) \times 1 \). This means that \( 1_{GL_1} \times \pi \) has to appear with multiplicity at least two in the appropriate Jacquet module of \( \Pi_1 \). But, if \( \zeta(-m_1, -1; 1) \) really appears in \( \Pi_1 \), this is not the case, since then
Assume that $m \cdots \times \text{irreducible spherical subquotient of } \pi$ prove the proposition by a case by case analysis. In Proposition 3 we prove that the irreducible spherical subquotient of (6). We moreover prove in Proposition 3 that (since $\alpha, \beta$ satisfy conditions imposed above) $\pi_1$ is necessarily a strongly negative or negative representation. In the case $\pi_1$ is strongly negative, we have that $\text{Jord}(\pi_1) = \{(a_1, 1), \ldots, (a_k, 1)\} \cup \{(2a + 1, 1), (2\beta + 1, 1)\}$. We then denote

$$
\zeta(\chi_0, \ldots) = \zeta(-n_l, n_l-1; \chi_0) \times \cdots \times \zeta(-n_2, n_1; \chi_0),
$$
for the unramified representation of the appropriate general linear group appearing in the description of \( \pi \). We then have

\[
\zeta(-\beta, \alpha; 1) \times \pi \leftarrow \zeta(-\beta, \alpha; 1) \times \zeta(\chi_0, \ldots) \times \zeta(-m_k, m_{k-1}; 1) \times \cdots \times \zeta(-m_3, m_2; 1) \\
\times \zeta(-m_1, -1; 1) \times 1 \equiv \zeta(\chi_0, \ldots) \times \zeta(-\beta, \alpha; 1) \times \zeta(-m_k, m_{k-1}; 1) \times \cdots \\
\times \zeta(-m_3, m_2; 1) \times \zeta(-m_1, -1; 1) \times 1 = \Pi
\]

so that, assuming \( \pi_1 \) is a subrepresentation of \( (6) \) (by Proposition 3), \( \zeta(\chi_0, \ldots) \times \pi_1 \) is a subrepresentation of \( \Pi \). On the other hand, we can embed \( \pi_1 \) into a representation, say, \( \Pi' \) according to Jord(\( \pi_1 \)) (in the same way we embedded \( \pi \)). But then \( \zeta(\chi_0, \ldots) \times \Pi' \) has a unique (strongly negative!) unramified subquotient, say \( \pi_0 \), which is a subrepresentation there (cf. introduction of [7]). Because of the multiplicity one result for unramified representations, this means that \( \pi_0 \leftarrow \zeta(\chi_0, \ldots) \times \pi_1 \). This means that \( \pi_0 \leftarrow \Pi \), but then another multiplicity one argument forces \( \pi_0 \leftarrow \zeta(-\alpha, \beta; 1) \times \pi \), and the claim is shown.

If \( \pi_1 \) is a negative representation, and we have that \( \pi_1 \) is a subrepresentation of \( (6) \), then \( \pi_1 \leftarrow \zeta(-\alpha, \alpha; 1) \times \sigma_{sn} \), or \( \pi_1 \leftarrow \zeta(-\alpha, \alpha; 1) \times \zeta(-\beta, \beta; 1) \times \sigma_{sn} \), or \( \pi_1 \leftarrow \zeta(-\beta, \beta; 1) \times \sigma_{sn} \) for some strongly negative representation \( \sigma_{sn} \) (depending whether \( \{(2a+1, 1), (2\beta+1, 1)\} \cap \text{Jord}(\pi) = \{(2a+1, 1), (2\beta+1, 1)\} \) or \( \{(2\beta+1, 1)\} \) respectively). But then again, if, for example, the first case occurs,

\[
\zeta(\chi_0, \ldots) \times \pi_1 \leftarrow \zeta(\chi_0, \ldots) \times \zeta(-\alpha, \alpha; 1) \times \sigma_{sn} \equiv \zeta(-\alpha, \alpha; 1) \times \zeta(\chi_0, \ldots) \times \sigma_{sn}
\]

so that \( \zeta(\chi_0, \ldots) \times \sigma_{sn} \) has again a strongly negative subrepresentation, say \( \sigma'_{sn} \), then \( \zeta(-\alpha, \alpha; 1) \times \sigma'_{sn} \) has a (negative) unramified subrepresentation \( \pi_0 \) (Theorem 6.1 of [7]; note that [7] does not use the fact \( \sigma'_{sn} \) is unitary). This means (again by multiplicity one) that \( \zeta(\chi_0, \ldots) \times \pi_1 \) has \( \pi_0 \) for a subrepresentation, and, in the same way as before, we have that \( \pi_0 \leftarrow \zeta(-\beta, \alpha; 1) \times \pi \). Other cases are treated similarly.

So, to complete the proof of Proposition 2, we are left to prove the following statement. Keeping the notation from above, let \( \pi'_1 \) be an unramified strongly negative representation with Jord(\( \pi'_1 \)) = \( \{(1, 2a_1+1), \ldots, (1, 2a_k+1)\} \) and \( \beta > \alpha \geq 0 \) integers.

**Proposition 3.** The unramified subquotient of

\[
\zeta(-\beta, \alpha; 1) \times \pi'_1
\]

is a negative subrepresentation; strongly negative only if Jord(\( \pi'_1 \)) \( \cap \{1, 2a+1, (1, 2\beta+1)\} = \emptyset \).

**Proof.** First, assume that Jord(\( \pi'_1 \)) \( \cap \{2a+1, 1, (2\beta+1, 1)\} = \emptyset \). If \( 2a+1 \) is greater than every element in Jord(\( \pi'_1 \)), the statement is just the canonical description of strongly negative representations, cf. Introduction of ([7]); the same thing goes if \( 2\beta+1 \) is smaller than any element in Jord(\( \pi'_1 \)). So let \( \{|2a+1, 2\beta+1| \cap \text{Jord}(\pi'_1)| = l > 0 \), so that

\[
a_{l-1} < \alpha < a_l < \cdots < a_{l+1} < \beta < a_{l+1},
\]

where \( \{2a+1 < 2a_2+1 < \cdots < 2a_{l+1} < 2a_{l+1}+1 < \cdots < 2a_k+1\} = \text{Jord}(\pi'_1) \). Now we divide our discussion into several cases:
(i) \( l \) even and \( t \) even,

(ii) \( l \) even and \( t \) odd,

(iii) \( l \) odd and \( t \) even,

(iv) \( l \) odd and \( t \) odd.

We now discuss the first case, so \( l \) and \( t \) are even. Then

\[
\pi' \hookrightarrow \zeta(-a_k, a_k - 1; 1) \times \cdots \times \zeta(-a_t + t + 1, a_t + 1; 1) \\
\times \zeta(-a_t + t - 1, a_t + 1; 1) \times \cdots \times \zeta(-a_t + 1, a_t; 1) \\
\times \zeta(a_t - 1, a_t - 2; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1.
\]

On the other hand, for the strongly negative representation \( \pi \) with \( \text{Jord}(\pi) = \text{Jord}(\pi') \cup \{ (2a + 1, 1), (2\beta + 1, 1) \} \) the following holds:

\[
\pi \hookrightarrow \zeta(-a_k, a_k - 1; 1) \times \cdots \times \zeta(-a_t + t + 1, a_t + 1; 1) \times \zeta(-\beta, a_t + t - 1; 1) \\
\times \zeta(-a_t + t - 2, a_t + t - 3; 1) \times \cdots \times \zeta(-a_t + 2, a_t + 1; 1) \times \zeta(-a_t, \alpha; 1) \\
\times \zeta(-a_t - 1, a_t - 2; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1.
\]

Since the cuspidal support of \( \zeta(-\beta, \alpha; 1) \rtimes \pi'_1 \) coincides with the cuspidal support of \( \pi \), we have \( \pi \preceq \zeta(-\beta, \alpha; 1) \rtimes \pi'_1 \). Let \( \tau_1 = \zeta(-a_k, a_k - 1; 1) \times \cdots \times \zeta(-a_t + t + 1, a_t + 1; 1) \), and \( \tau_2 = \zeta(-a_t - 1, a_t - 2; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1 \) be unramified representations of the appropriate general linear and symplectic group, respectively. Then

\[
\zeta(-\beta, \alpha; 1) \rtimes \pi_1 \hookrightarrow \zeta(-\beta, \alpha; 1) \times \zeta(-a_t + t - 1, a_t + t - 2; 1) \times \cdots \times \zeta(-a_t + 1, a_t; 1) \times \tau_1 \rtimes \tau_2,
\]

and

\[
\pi \hookrightarrow \zeta(-\beta, a_t + t - 1; 1) \times \zeta(-a_t + t - 2, a_t + t - 3; 1) \times \cdots \times \zeta(-a_t + 2, a_t + 1; 1) \\
\times \zeta(-a_t, \alpha; 1) \times \tau_1 \rtimes \tau_2.
\]

But this means that there exists an unramified irreducible representation \( \pi' \leq \zeta(-a_t + t - 2, a_t + t - 3; 1) \times \cdots \times \zeta(-a_t + 2, a_t + 1; 1) \times \zeta(-a_t, \alpha; 1) \times \tau_1 \rtimes \tau_2 \) such that

\[
\pi \hookrightarrow \zeta(-\beta, a_t + t - 1; 1) \times \pi' \hookrightarrow \zeta(-\beta, \alpha; 1) \times \zeta(\alpha + 1, a_t + t - 1; 1) \rtimes \pi'.
\]

Examining the cuspidal support, we see that \( \pi'_1 \) is the unique irreducible unramified subquotient of \( \zeta(\alpha + 1, a_t + t - 1; 1) \rtimes \pi' \), so that by Lemma 1 and Lemma 2 we have \( \pi \hookrightarrow \zeta(-\beta, \alpha; 1) \rtimes \pi'_1 \).

We now analyze the second case: \( l \) even and \( t \) odd. Let \( \tau_1 = \zeta(-a_k, a_k - 1; 1) \times \cdots \times \zeta(-a_t + t + 1, a_t + 1; 1) \), and \( \tau_2 = \zeta(-a_t - 1, a_t - 2; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1 \). Then

\[
\pi \hookrightarrow \tau_1 \times \zeta(-a_t + t + 1, \beta; 1) \times \cdots \times \zeta(-a_t + 1, a_t; 1) \times \zeta(-\alpha, a_t - 1; 1) \times \tau_2.
\]
Let $\pi_0$ be the unique unramified (negative) subrepresentation of $\tau_1 \rtimes \tau_2$; then, obviously $\text{Jord}(\pi_0) = \{2a_k + 1, \ldots, 2a_{t+l+1}, a_{t-2}, \ldots, a_2, a_1\}$. We then have
\[
\pi \hookrightarrow \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-a_{t+l}, \beta; 1) \times \zeta(-\alpha, a_{t-1}; 1) \rtimes \pi_0.
\]

Let $\pi'_0$ be a spherical subquotient (subrepresentation) of $\zeta(-a_{t+l}; \beta; 1) \times (\zeta(-\alpha, a_{t-1}; 1) \rtimes \pi_0)$, so that $\pi \hookrightarrow \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \rtimes \pi'_0$. By Proposition 1, we know that $\pi'_0$ is also a subrepresentation of $\zeta(-\beta, \alpha; 1) \times (\zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0)$. We therefore have a sequence of intertwining induced by the intertwining in the general linear groups
\[
\pi \hookrightarrow \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times (\zeta(-a_{t+1}, a_t; 1) \times \zeta(-\beta, \alpha; 1) \times (\zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0) \rightarrow \\
\zeta(-\beta, \alpha; 1) \times (\zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times (\zeta(-a_{t+1}, a_t; 1) \times \zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0.
\]

Since the kernel of every of the above homomorphisms is not unramified, we still have
\[
\pi \hookrightarrow \zeta(-\beta, \alpha; 1) \times (\zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi'_0.
\]

Now we easily see that $\pi \hookrightarrow \zeta(-\beta, \alpha; 1) \rtimes \pi'_0$.

The third and the fourth case are similar the second, and the first, respectively, and are left to the reader.

Special cases (i.e., when $t = 1$, or $t + l - 1 = k$ or $l = 0$) are handled in the same way as these situations above, just simpler (e.g., when $l = 0$ we immediately apply Proposition 1).

We now prove our main result, which is the generalization of Proposition 2; and is also needed in the applications in the theory of automorphic forms (e.g., [3]).

**Theorem 3.** Let $\beta > \alpha > 0$ be integers, and $\chi$ an unramified quadratic character of $F$. Let $\pi$ be a negative representation. Then, an irreducible unramified subquotient of $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ is a subrepresentation; it is also a negative representation.

**Proof.** Since the representation $\pi$ is negative, there exist (by Theorem 6.1 of [7]) a unique strongly negative representation $\sigma_{sn}$ and a sequence of unitary characters $\chi_1, \ldots, \chi_k$ (also unramified), such that
\[
\pi \hookrightarrow \zeta(-a_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-a_k, \alpha_k; \chi_k) \rtimes \sigma_{sn},
\]
where $\alpha_1, \ldots, \alpha_k \in \frac{1}{2} \mathbb{Z}_{\geq 0}$. Since the cuspidal support of $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ is the same as the cuspidal support of
\[
\zeta(-a_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-a_k, \alpha_k; \chi_k) \times \zeta(-\beta, \alpha; \chi) \rtimes \sigma_{sn},
\]
and this representation has a negative subrepresentation (this follows from the proof of Theorem 6.1 (ii) of [7] and Propositions 2 and 3 here), say $\pi_2$, we have that
\[ \pi_2 \leq \zeta(-\beta, \alpha; \chi) \times \pi. \] On the other hand, by Proposition 2, there is a negative representation, say \( \pi_3 \), which is a subrepresentation of \( \zeta(-\beta, \alpha; \chi) \times \sigma_{sn} \). Now, consider the following sequence of homomorphisms induced from the appropriate homomorphisms of general linear groups:

\[
\begin{align*}
\pi_2 & \hookrightarrow \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \pi_3 \hookrightarrow \\
\zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \zeta(-\beta, \alpha; \chi) & \times \sigma_{sn} \\
\zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\beta, \alpha; \chi) & \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \sigma_{sn} \\
& \vdots \\
\zeta(-\beta, \alpha; \chi) & \times \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \sigma_{sn}.
\end{align*}
\]

Each of this homomorphisms is either isomorphism (if \( \zeta(-\alpha_1, \alpha_1; \chi_1) \times \zeta(-\beta, \alpha; \chi) \) is irreducible), or its kernel is not spherical. More precisely, if \( \zeta(-\alpha_1, \alpha_1; \chi_1) \times \zeta(-\beta, \alpha; \chi) \) reduces, then either we have \( -\alpha_i \leq -\beta - 1 \leq \alpha_i < \alpha \) or \( -\beta \leq -\alpha_i - 1 \leq \alpha < \alpha_i \) (e.g., [7], section 2). If the first possibility should occur, we would have \( \beta + 1 \leq \alpha_i < \alpha \) which contradicts our assumptions on \( \alpha \) and \( \beta \). If the second possibility occurs, an irreducible unramified representation \( \zeta(-\alpha_1, \alpha_1; \chi_1) \times \zeta(-\alpha_i, \alpha_1; \chi_1) \) is a quotient of \( \zeta(-\alpha_1, \alpha_1; \chi_1) \times \zeta(-\beta, \alpha; \chi) \), and is not in the kernel of the homomorphism

\[
\zeta(-\alpha_1, \alpha_1; \chi_1) \times \zeta(-\beta, \alpha; \chi) \to \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_1, \alpha_1; \chi_1).
\]

So we have

\[
\pi_2 \hookrightarrow \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \sigma_{sn},
\]

and also

\[
\zeta(-\beta, \alpha; \chi) \times \pi \hookrightarrow \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \sigma_{sn},
\]

and the claim follows. 

**Remark 1.** Although all the details are not checked, the author believes that similar results hold for other (quasi-split) classical \( p \)-adic groups. There is no real mathematical obstacle to the simultaneous treatment of all the classical groups, but the notational awkwardness—namely, for other classical groups the (degenerate) principal series representations in question might have half-integer exponents (which is the consequence of the situation with the rank-one reducibility) and this would somewhat notationally complicate the exposition.

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References


