FINITE $p$-GROUPS ALL OF WHOSE MAXIMAL SUBGROUPS, EXCEPT ONE, HAVE ITS DERIVED SUBGROUP OF ORDER $\leq p$

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Abstract. Let $G$ be a finite $p$-group which has exactly one maximal subgroup $H$ such that $|H'| > p$. Then we have $d(G) = 2$, $p = 2$, $H'$ is a four-group, $G'$ is abelian of order 8 and type $(4, 2)$, $G$ is of class 3 and the structure of $G$ is completely determined. This solves the problem Nr. 1800 stated by Y. Berkovich in [3].

We consider here only finite $p$-groups and our notation is standard (see [1]). If $G$ is a $p$-group all of whose maximal subgroups have its derived subgroups of order $\leq p$, then such groups $G$ are characterized in [3, §137]. But there is no way to determine completely the structure of such $p$-groups.

It is quite surprising that we can determine completely (in terms of generators and relations) the title groups, where exactly one maximal subgroup has the commutator subgroup of order $> p$. We shall prove our main theorem (Theorem 8) starting with some partial results about the title groups. However, Propositions 4 and 6 are also of independent interest.

Proposition 1. Let $G$ be a title $p$-group. Then we have $d(G) \leq 3$, $\text{cl}(G) \leq 3$, $p^2 \leq |G'| \leq p^3$ and $G'$ is abelian of exponent $\leq p^2$. Also, $G$ has at most one abelian maximal subgroup.

Proof. Let $H$ be the unique maximal subgroup of $G$ with $|H'| > p$. This gives $|G'| \geq p^2$. Let $K \neq L$ be maximal subgroups of $G$ which are both distinct from $H$. We have $|K'| \leq p$, $|L'| \leq p$ and so $K'L' \leq Z(G)$ and $|K'L'| \leq p^2$. By a result of A. Mann ([1, Exercise 1.69]), we get $|G' : (K'L')| \leq p$. This implies that $|G'| \leq p^3$, $G'$ is abelian and $G$ is of class $\leq 3$. Since $K'L'$ is elementary

$2010$ Mathematics Subject Classification. 20D15.

Key words and phrases. Finite $p$-groups, minimal nonabelian $p$-groups, commutator subgroups, nilpotence class of $p$-groups, Frattini subgroups, generators and relations.
abelian, we also get \( \exp(G') \leq p^2 \). If \( G \) would have more than one abelian maximal subgroup, then (by the above argument) \( |G'| \leq p \), a contradiction. Hence \( G \) has at most one abelian maximal subgroup.

Note that each nonabelian \( p \)-group \( X \) has exactly 0, 1 or \( p + 1 \) abelian maximal subgroups and in the last case \( |X'| = p \) (Exercise 1.6(a) in [1]). Suppose that \( d(G) \geq 4 \). Then \( G \) has at least 1 + \( p + p^2 + p^3 \) distinct maximal subgroups and so the set \( S \) of maximal subgroups of \( G \) with the commutator group of order \( p \) has at least \( p + p^2 + p^3 - 1 \) elements. Since \( G' \) has at most \( p^2 + p + 1 \) pairwise distinct subgroups of order \( p \) (and the maximum is achieved if \( G' \cong \mathbb{E}_{p^3} \)), it follows that there are \( K \neq L \in S \) such that \( K' = L' \). By the above argument (using a result of A. Mann), we get \( |G'| = p^2 \) and so \( G' \) has at most \( p + 1 \) pairwise distinct subgroups of order \( p \) (where the maximum is achieved if \( G' \cong \mathbb{E}_{p^3} \)). If \( M \in S \), then considering \( G/M' \), we see that there are at most \( p + 1 \) elements \( N \in S \) such that \( N' = M' \). This gives

\[
p + p^2 + p^3 - 1 \leq (p + 1)^2, \quad \text{and so } p^3 - p \leq 2 \text{ or } p(p^2 - 1) \leq 2,
\]
a contradiction. Our proposition is proved.

**Proposition 2.** Let \( G \) be a title \( p \)-group. Then the subgroup:

\[ H_0 = \langle M' \mid M \text{ is any maximal subgroup of } G \text{ with } |M'| \leq p \rangle
\]
is noncyclic and so \( H_0 \) is elementary abelian of order \( p^2 \) or \( p^3 \) and \( H_0 \leq Z(G) \).

**Proof.** Suppose that \( H_0 \) is cyclic. Then we have \( |H_0| = p \) and so \( |G'| = p^2 \) because (by [1, Exercise 1.6(a)]) \( |G' : H_0| \leq p \) and Proposition 1 implies that \( |G'| \geq p^2 \). This gives that \( H' = G' \), where \( H \) is the unique maximal subgroup of \( G \) with \( |H'| > p \). Consider the nonabelian factor group \( G/H_0 \). In this case \( G/H_0 \) has exactly one nonabelian maximal subgroup \( H/H_0 \). Since \( d(G/H_0) = 2 \) or 3, the last statement would imply that the nonabelian \( p \)-group \( G/H_0 \) would have exactly \( p \) or \( p + p^2 \) abelian maximal subgroups, a contradiction (by [1, Exercise 1.6(a)]).

**Proposition 3.** Let \( G \) be a title \( p \)-group. Then we have \( d(G) = 2 \).

**Proof.** Assume that \( d(G) = 3 \) and we use the notation from Proposition 2.

First suppose that \( H_0 = G' \) so that \( G \) is of class 2 with an elementary abelian commutator subgroup. For any \( x, y \in G \), we get \( [x^p, y] = [x, y]^p = 1 \) and this implies that \( U_1(G) \leq Z(G) \). It follows that \( \Phi(G) = \mathbf{U}_1(G)G' \leq Z(G) \) and \( G/\Phi(G) \cong \mathbb{E}_{p^2} \). Let \( X \) be any maximal subgroup of \( G \) so that \( X/\Phi(G) \cong \mathbb{E}_{p^2} \) and all \( p + 1 \) maximal subgroups of \( X \) which contain \( \Phi(G) \) are abelian. This implies \( |X'| \leq p \). But then each maximal subgroup of \( G \) has its derived subgroup of order \( \leq p \), contrary to our assumption.

Now assume \( H_0 \neq G' \). In this case \( H_0 \cong \mathbb{E}_{p^2} \), \( H_0 \leq Z(G) \) and \( |G'| = p^3 \). There are exactly \( p + p^2 \) maximal subgroups \( M_i \) of \( G \) such that \( |M_i'| \leq p \),
i = 1, 2, ..., p^2. Since $H_0$ has exactly $p + 1$ subgroups of order $p$, it follows that there exist the indices \(i \neq j \in \{1, 2, ..., p^2\}\) such that $M'_i = M'_j$ is of order $p$. Again by \cite[Exercise 1.69]{1} we have \(|G' : (M'_i, M'_j)| \leq p\) and this gives \(|G'| \leq p^2\), a contradiction. Our proposition is proved. \(\square\)

**Proposition 4.** Let $G$ be a two-generator $p$-group, $p > 2$, with $G' \cong C_{p^2}$. Then each maximal subgroup of $G$ is nonabelian.

**Proof.** Assume that $G$ has an abelian maximal subgroup $M$ so that $|M/\Phi(G)| = p$. Take an element $a \in M \setminus \Phi(G)$ and an element $b \in G \setminus M$ so that we have $G = \langle a, b \rangle$ and $G' = \langle [a, b] \rangle$. Since $G'$ is cyclic, \cite[Theorem 7.1(c)]{1} implies that $G$ is regular. We have $b^p \in \Phi(G) < M$ and so $[a, b^p] = 1$. Hence

\[(a^{-1}b^{-p}a)b^p = ((b^{-1})^a)b^p = 1 \text{ and so } (b^p)^p = b^p.\]

By \cite[Theorem 7.2(a)]{1} (about regular $p$-groups), the last relation gives $((b^{-1})^a)b^p = 1$ or equivalently $[a, b]^p = 1$, a contradiction. \(\square\)

**Remark.** The assumption $p > 2$ in Proposition 4 is essential. This shows a 2-group of maximal class and order 16.

**Proposition 5.** Let $G$ be a two-generator $p$-group, $p > 2$, with $G' \cong E_{p^2}$. Then $G$ has an abelian maximal subgroup.

**Proof.** By \cite[Proposition 137.4]{3}, each proper subgroup of $G$ has its derived subgroup of order at most $p$. Then we may apply \cite[Proposition 137.5]{3} and so for each $x, y \in G$, we get $[x^p, y] = [x, y]^p = 1$. This gives that $U_1(G) \leq Z(G)$ and therefore we obtain that $\Phi(G) = U_1(G)G'$ is abelian. Let $M$ be a maximal subgroup of $G$ which centralizes $G'$. We have $|M : \Phi(G)| = p$ and $M$ centralizes $U_1(G)$ and $G'$ so that $\Phi(G) \leq Z(M)$. This implies that $M$ is abelian and we are done. \(\square\)

**Remark.** The assumption $p > 2$ in Proposition 5 is essential. Let $G$ be a faithful and splitting extension of an elementary abelian group of order 8 by a cyclic group of order 4. Then we have $d(G) = 2$ and $G' \cong E_4$ but $G$ has no abelian maximal subgroup.

**Proposition 8.** Let $G$ be a title $p$-group and $\Gamma_1 = \{H_1, H_2, ..., H_p, H\}$ be the set of all maximal subgroups of $G$, where $|H'| > p$. Then $G'$ is abelian of order $p^3$, $H' \cong E_{p^2}$, $H' \leq Z(G)$ and $H_1', H_2', ..., H_p'$ are pairwise distinct subgroups of order $p$ contained in $H'$. If $G = \langle x, y \rangle$ for some $x, y \in G$, then $[x, y] \in G' \setminus H'$ and $[x, y] \not\in Z(G)$ so that $G$ is of class 3. Finally, $G'/H'$ is nonmetacyclic minimal nonabelian and so if $a \in G \setminus G'$ is such that $a^p \in G'$, then $a^p \in H'$.

**Proof.** Let $H_0$ be the subgroup of $G'$ as defined in Proposition 2. Then $H_0 \leq Z(G)$ and $H_0$ is elementary abelian of order $p^2$ or $p^3$. Suppose for a moment that $H_0 = G'$. We have $G = \langle x, y \rangle$ for some $x, y \in G$ and $[x, y] \in H_0$.
so that \( G/\langle [x, y]\rangle \) is abelian and \( G' = \langle [x, y]\rangle \) is of order \( p \), a contradiction. It follows that \( H_0 \neq G' \) which gives that \( H_0 \cong \mathbb{E}_p^2 \), \( |G'| : H_0| = p \) and \( G' \) is abelian of order \( p^2 \). Since \( d(G/H_0) = 2 \) and \( |G'/H_0| = p \), it follows that \( G/H_0 \) is minimal nonabelian (see [2, Lemma 65.2(\(a)\))]. In particular, we have \( H' \leq H_0 \) which together with \( |H'| > p \) implies \( H' = H_0 \cong \mathbb{E}_p^2 \). If \( G/H' \) is metacyclic, then a result of N. Blackburn (see [1, Lemma 44.1] and [1, Corollary 44.6]) gives that \( G \) is also metacyclic. This is a contradiction because \( G' \) is noncyclic. Hence \( G/H' \) is nonmetacyclic minimal nonabelian so that [2, Lemma 65.1] gives that \( G'/H' \) is a maximal cyclic subgroup of \( G/H' \). Thus for each element \( a \in G \setminus G' \) such that \( a^p \in G' \), we get \( a^p \in H' \). We have \( G = \langle x, y \rangle \) for some \( x, y \in G \). It is clear that \( \langle [x, y]\rangle \) is not normal in \( G \). Indeed, if \( \langle [x, y]\rangle \leq G \), then \( G/\langle [x, y]\rangle \) is abelian and so \( \langle [x, y]\rangle = G' \) is of order \( p^2 \) (noting that \( \exp(G') \leq p^2 \)), a contradiction. We have proved that \( \langle [x, y]\rangle \) is not normal in \( G \). In particular, \( [x, y] \notin Z(G) \) and so \( [x, y] \notin G' \setminus H' \) and \( G \) is of class 3.

If \( \Gamma_1 = \{ H_1, H_2, \ldots, H_p, H \} \) is the set of all maximal subgroups of \( G \), then we have \( H_1' \leq H_0 = H' \) for all \( i = 1, 2, \ldots , p \). We claim that \( H_1', H_2', \ldots , H_p' \) are pairwise distinct subgroups of order \( p \). Indeed, if \( |H_i'H_j'| \leq p \) for some \( i \neq j \), \( i, j \in \{ 1, 2, \ldots , p \} \), then a result of A. Mann (see [1, Exercise 1.69]) implies \( |G' : (H_i'H_j')| \leq p \) and so \( |G'| \leq p^2 \), a contradiction. Our proposition is proved.

Remark 9. If \( X \) is a two-generator \( p \)-group of class 2, then it is well known that \( X' \) is cyclic. Hence if \( G \) is any two-generator \( p \)-group, then \( G'/K_3(G) \) is cyclic, where \( K_3(G) = [G', G] \).

Proposition 10. If \( G \) is a title \( p \)-group, then \( p = 2 \).

Proof. Assume that \( p > 2 \) and we use Proposition 6 together with the notation introduced there.

First suppose that \( G' \) is not elementary abelian. Then we have \( o([x, y]) = p^2 \) and \( \langle [x, y]^p \rangle \) is a subgroup of order \( p \) contained in \( H' \). Let \( H_i', i \in \{ 1, 2, \ldots , p \} \), be such that \( H_i' \neq \langle [x, y]^p \rangle \) which gives \( G' = H_i' \times \langle [x, y] \rangle \). We consider the factor group \( \tilde{G} = G/H_i' \). Since \( d(G) = 2, p > 2 \), and \( G' \cong C_{p^2} \), we may use Proposition 4 saying that each maximal subgroup of \( \tilde{G} \) is nonabelian. But \( H_i = H_i/H_i' \) is an abelian maximal subgroup of \( \tilde{G} \), a contradiction.

We have proved that \( G' \) is elementary abelian of order \( p^3 \). Let \( \{ H_1', H_2', \ldots , H_p', K \} \) be the set of all \( p + 1 \) subgroups of order \( p \) in \( H' \) and consider the factor group \( G/K \). All \( p + 1 \) maximal subgroups of \( G/K \) are nonabelian, \( d(G/K) = 2, p > 2 \), and \( G'/K \cong \mathbb{E}_p^2 \). By Proposition 5, \( G'/K \) possesses an abelian maximal subgroup, a contradiction. We have proved that we must have \( p = 2 \).

Theorem 11. Let \( G \) be a \( p \)-group with exactly one maximal subgroup \( H \) such that \( |H'| > p \). Then we have \( d(G) = 2, p = 2 \) and \( G' \) is abelian of order 8.
and type \((4,2)\). Also, \([G', G] = \Omega_1(G') \leq Z(G)\), \(\Phi(G) = C_G(G')\) is abelian and 
\(\Omega_2(G) \leq Z(G)\). Let \(\{H_1, H_2, H\}\) be the set of maximal subgroups of \(G\). Then
\(H_1' = \langle z_1 \rangle\) and \(H_2' = \langle z_2 \rangle\) are both of order 2, \(\langle z_1, z_2 \rangle = \Omega_1(G') = H' \cong E_4\),
\(d(H) = 3\) and \(\Omega_1(G') = \langle z_1 z_2 \rangle\). Finally, \(H\) is the unique maximal subgroup of 
\(G\) which contains an element acting invertingly on \(G'\). We have the following two possibilities:

(i) \(d(H_1) = d(H_2) = 2\) in which case \(H_1\) and \(H_2\) are minimal nonabelian.
In this case either \(H_1\) and \(H_2\) are both metacyclic and \(G\) is isomorphic to one of the groups of 
Theorem 100.3(a) and (b) in [3] or \(H_1\) and \(H_2\) are both nonmetacyclic and \(G\) is isomorphic to one of the groups of 
Theorem 100.3(c) in [3].

(ii) \(d(H_1) = d(H_2) = 3\) and the group \(G\) is given with:

\[
G = \langle a, b \mid [a, b] = v, v^4 = 1, [v, a] = z_1, [v, b] = z_1^2 z_2, z_1^2 = z_2^3 = 1, v^2 = z_1 z_2, \rangle
\]

\[
\begin{align*}
&[z_1, a] = [z_1, b] = [z_2, a] = [z_2, b] = 1,\ a^{2^n} = z_1^{x} z_2^{y}, b^{2^n} = z_1^{x} z_2^{y},
&\text{where } m \geq 2, n \geq 2, \text{ and } \alpha, \beta, \gamma, \delta, \epsilon \in \{0, 1\}. \ We \ have \ here \ |G| = 2^{m+n+3} \geq 2^7, \ G' = \langle v, z_1 \rangle \cong C_4 \times C_2, \ |G', G| = \langle z_1, z_2 \rangle = \Omega_1(G') \leq \Z(G) \text{ and the Frattini subgroup } \Phi(G) = \langle G', a^2, b^2 \rangle \text{ is abelian. Finally,}\n&\text{if } \epsilon = 0, \text{ then } H = \Phi(G)(ab) \text{ and if } \epsilon = 1, \text{ we have } H = \Phi(G)(b).\n\end{align*}
\]

Conversely, all groups stated in parts (i) and (ii) of this theorem are \(p\)-groups all of whose maximal subgroups, except one, have its derived subgroup of order \(\leq p\).

**Proof.** We use Proposition 6 together with the notation introduced there. By Proposition 7, we have in addition \(p = 2\).

Let \(X\) be a maximal subgroup of \(G\). By Schreier’s inequality ([2, Theorem A.25.1]), we have

\[
d(X) \leq 1 + |G : X|(d(G) - 1),
\]

and so \(d(X) \leq 3\). Since \(H' \cong E_4\) and \(H' \leq Z(H)\), the maximal subgroup \(H\) cannot be two-generator (see Remark 3). It follows that we have \(d(H) = 3\).

Since \(G\) is a nonmetacyclic two-generator 2-group, we may use [3, Theorem 107.1] saying that such a group has an even number of two-generator maximal subgroups. It follows that we have either \(d(H_1) = d(H_2) = 2\) or \(d(H_1) = d(H_2) = 3\).

Set \(H_1' = \langle z_1 \rangle, H_2' = \langle z_2 \rangle\) so that we have \(H' = \langle z_1 \rangle \times \langle z_2 \rangle \cong E_4\) and 
\(\Phi(G) = H_1 \cap H_2\). Since \((\Phi(G))' \leq \langle z_1 \rangle \cap \langle z_2 \rangle = \{1\}\), it follows that \(\Phi(G)\) is abelian and so \(\Phi(G)\) is a maximal normal abelian subgroup of \(G\) (containing \(G'\)). Take elements \(h_1 \in H_1 \setminus \Phi(G)\) and \(h_2 \in H_2 \setminus \Phi(G)\) so that we have 
\(G = \langle h_1, h_2 \rangle; [h_1, h_2] = v \in G' \setminus H'\) and \(o(v) \leq 4\). If \(v\) commutes with both \(h_1\) and \(h_2\), then we get \(v \in Z(G)\), a contradiction. Without loss of generality we may assume that \([v, h_1] \neq 1\) and so we get \([v, h_1] = z_1\).
Assume for a moment that $G' \cong E_8$ so that $v$ is an involution. We compute
\[ [h_1^2, h_2] = [h_1, h_2]h_1[h_1, h_2] = v^{h_1}v = (vz_1)v = v^2z_1 = z_1. \]
This is a contradiction since $h_1^2 \in \Phi(G)$ and $(h_1^2, h_2) \leq H_2$, where $H_2' = \langle z_2 \rangle$.
We have proved that $G'$ is abelian of type $(4, 2)$ and so $o(v) = 4$ and $1 \neq v^2 \in H'$.

We have $K_3(G) = [G', G] \geq \langle z_1 \rangle$. Since $d(G) = 2$, it follows by Remark 1 that $G'/K_3(G)$ is cyclic. Suppose that $[v, h_2] = 1$ so that in this case we have $K_3(G) = \langle z_1 \rangle$. We compute
\[ [h_1, h_2^2] = [h_1, h_2][h_1, h_2]h_2 = v^{h_2} = v^2 \neq 1. \]
We have $(h_1, h_2^2) \leq H_1$ and so $v^2 = z_1$. But then we have $G'/K_3(G) = G'/\langle z_1 \rangle \cong E_4$, a contradiction. We have proved that $[v, h_2] \neq 1$ and so $[v, h_2] = 2$. This gives
\[ K_3(G) = \langle z_1, z_2 \rangle = H' \leq Z(G) \]
and $G$ is of class 3.

We get
\[ [h_1^2, h_2] = [h_1, h_2]h_1[h_1, h_2] = v^{h_1}v = (vz_1)v = v^2z_1, \]
and since $(h_1^2, h_2) \leq H_2$, it follows that $v^2z_1 \in \langle z_2 \rangle$ and so $v^2 \in \{z_1, z_1z_2\}$. Similarly, we get
\[ [h_1, h_2^2] = [h_1, h_2][h_1, h_2]h_2 = v^{h_2} = v(vz_2) = v^2z_2, \]
and since $(h_1, h_2^2) \leq H_1$, it follows that $v^2z_2 \in \langle z_1 \rangle$ and so $v^2 \in \{z_2, z_1z_2\}$. As a result, we get $v^2 = z_1z_2$ and so $\Phi(G) = \langle z_1z_2 \rangle$. Note that $H = \Phi(G)(h_1h_2)$ and
\[ v^{h_1h_2} = (vz_1)^{h_2} = v(z_1z_2) = vv^2 = v^3 = v^{-1} \]
and so $h_1h_2$ acts invertingly on $G'$. It follows that $\Phi(G) = C_G(G')$ and $H$ is the unique maximal subgroup of $G$ which contains an element acting invertingly on $G'$.

Let $x, y \in G$. Then $\langle x^2, y \rangle$ is contained in one of the maximal subgroups $X_i$ of $G$, where $X_i$ is elementary abelian of order $\leq 4$ and $cl(X_i) = 2$ ($i = 1, 2, 3$). It follows
\[ [x^4, y] = [(x^2)^2, y] = [x^2, y]^2 = 1, \]
and so we get $\Phi(G) \leq Z(G)$.

Now suppose that $d(H_1) = d(H_2) = 2$. In this case both $H_1$ and $H_2$ are minimal nonabelian (see [2, Lemma 65.2(a)]) and $H$ is neither abelian nor minimal nonabelian. Since $d(G) = 2$ and $H_1' \neq H_2'$ such 2-groups are completely determined in [3, Theorem 100.3] which gives the groups quoted in part (i) of our theorem.

It remains to consider the case $d(H_1) = d(H_2) = 3$. By [3, Theorem 107.2(a)], a nonmetacyclic two-generator 2-group $G$ has the property that
every maximal subgroup of $G$ is not generated by two elements if and only if $G/G'$ has no cyclic subgroup of index 2. Thus $G/G'$ is abelian of type $(2^m,2^n)$, where $m \geq 2$, $n \geq 2$ and so $|G| = |G'|2^{m+n} = 2^{m+n+3} \geq 2^7$. There are normal subgroups $A$ and $B$ of $G$ such that $G = AB$, $A \cap B = G'$, $A/G' \cong C_{2^n}$, $B/G' \cong C_{2^m}$, $m \geq 2$, $n \geq 2$. Let $a \in A \setminus G'$, $b \in B \setminus G'$ be such that (a) covers $A/G'$ and (b) covers $B/G'$. Since $G/H'$ is nonmetacyclic minimal nonabelian, we know that (see [2, Lemma 65.1]) $G'/H'$ is a maximal cyclic subgroup of $G/H'$ and so we have $a^{2^m} \in H'$ and $b^{2^n} \in H'$. We have $G = \langle a, b \rangle$ and so $[a, b] = v$ is an element of order 4 contained in $G' \setminus H'$. Maximal subgroups of $G$ are $M_1 = A/(b^2)$, $M_2 = B/(a^2)$ and $M_3 = \Phi(G)/\langle ab \rangle$, where $\Phi(G) = G'/\langle a^2, b^2 \rangle$ is abelian. Since $\Phi(G) = C_2(G')$ and $\Omega_4(G') = H' \leq Z(G)$, we see that $G/\Phi(G) \cong E_4$ acts faithfully on $G'$ stabilizing the chain $G' > H' > \{1\}$. Interchanging $A$ and $B$ (if necessary), we may assume that $|M'| = 2$ and so we may set $v^a = vz_1$ which gives that $[v, a] = z_1$ and $M_1' = \langle z_1 \rangle$, where $z_1 \in H' \setminus \langle v^2 \rangle$. Set $z_2 = z_1v^{a^2}$ so that we have $v^2 = z_1z_2$. Then we have two possibilities.

(1) We assume $v^b = vz_1z_2 = v^{-1}$ or equivalently $[v, b] = z_1z_2$ so that the element $b$ inverts each element in $G'$. Since the maximal subgroup $H$ is the unique maximal subgroup of $G$ which contains an element acting invertingly on $G'$, we have in this case $M_2 = B/(a^2) = H$, where we should have $H' = \langle z_1, z_2 \rangle$. Indeed, we have

$$[a^2, b] = [a, b][a, b] = v^a v = (vz_1)v = v^2z_1 = (z_1z_2)z_1 = z_2,$$

and so we get $H' = \langle z_1, z_2 \rangle$. In this case $M_3 = \Phi(G)/\langle ab \rangle$ has the property $M_3 = \langle z_2 \rangle$. Indeed, here we have

$$[a^2, ab] = [a^2, b] = z_2,$$

$$[ab, b^2] = [a, b^2]^b = ([a, b][a, b]^b)^b = (v^b)^b = (v^{b^{-1}})^b = 1,$$

and

$$v^{ab} = (vz_1)^b = (vz_1z_2)z_1 = vz_2 \text{ and so } [v, ab] = z_2.

(2) Now we suppose $v^b = vz_2$ or equivalently $[v, b] = z_2$. In this case we get $M_2 = \langle z_2 \rangle$ since

$$[a^2, b] = [a, b][a, b] = v^a v = (vz_1)v = v^2z_1 = (z_1z_2)z_1 = z_2.$$

Also, we have here $M_3 = H$ because

$$v^{ab} = (vz_1)^b = (vz_2)z_1 = vz_1z_2 = vv^2 = v^3 = v^{-1}$$

and so $ab$ acts invertingly on $G'$. We have $[v, ab] = z_1z_2$ and

$$[a^2, ab] = [a^2, b][a, b] = v^a v = (vz_1)v = v^2z_1 = (z_1z_2)z_1 = z_2,$$

and so we have here $M_3' = \langle z_1, z_2 \rangle$. 

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In both cases (1) and (2), we may set \( v = z_1 z_2 \), where in case (1) we have \( \epsilon = 1 \) and in case (2) we have \( \epsilon = 0 \). Thus, if \( \epsilon = 0 \), then \( H = \Phi(G \langle ab \rangle) \) and if \( \epsilon = 1 \), we have \( H = \Phi(G \langle b \rangle) \).

Also, we may set
\[
\begin{align*}
    a^{2m} &= z_1^\alpha z_2^\beta, \\
    b^{2n} &= z_1^\gamma z_2^\delta,
\end{align*}
\]
where \( \alpha, \beta, \gamma, \delta \in \{0, 1\} \) since we know that \( a^{2m}, b^{2n} \in H' = \langle z_1, z_2 \rangle \).

Conversely, by inspection of groups given in parts (i) and (ii) of our theorem, we see that all these groups have the title property. Our theorem is proved.

REFERENCES


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Received: 16.6.2011.
Revised: 2.10.2011.