PROPERTY A AND ASYMPTOTIC DIMENSION

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Abstract. The purpose of this note is to characterize the asymptotic dimension \( \text{asdim}(X) \) of metric spaces \( X \) in terms similar to Property A of Guoliang Yu. We prove that for a metric space \((X, d)\) and \(n \geq 0\) the following conditions are equivalent:

a. \( \text{asdim}(X, d) \leq n \).

b. For each \( R, \varepsilon > 0 \) there exists a family \( \{A_x\}_{x \in X} \) of finite, non-empty subsets \( A_x \subset B(x, S) \times \mathbb{N}, x \in X \), such that \( \frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon \) if \( d(x, y) < R \) and the projection of \( A_x \) onto \( X \) contains at most \( n + 1 \) elements for all \( x \in X \).

c. For each \( R > 0 \) there is \( S > 0 \) and finite non-empty subsets \( A_x \subset B(x, S) \times \mathbb{N}, x \in X \), such that \( \frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{1}{n+1} \) if \( d(x, y) < R \) and the projection of \( A_x \) onto \( X \) contains at most \( n + 1 \) elements for all \( x \in X \).

1. Introduction

Property A was introduced by G.Yu in [6]. We adopt the following definition from [3] (see also [5]):

Definition 1.1. A discrete metric space \((X, d)\) has property A if for all \( R, \varepsilon > 0 \), there exists a family \( \{A_x\}_{x \in X} \) of finite, non-empty subsets of \( X \times \mathbb{N} \) such that:

- for all \( x, y \in X \) with \( d(x, y) \leq R \) we have \( \frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon \), where \( A_x \Delta A_y = (A_x \cup A_y) - (A_x \cap A_y) \) is the symmetric difference of the sets,
- there exists \( S > 0 \) such that for each \( x \in X \), if \( (y, n) \in A_x \), then \( d(x, y) \leq S \).

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Asymptotic dimension was introduced by M. Gromov in [1] (see section 1.E) as a large-scale analogue of the classical notion of topological covering dimension. It is a coarse invariant that has been extensively investigated (see chapter 9 of [4] for some results and further references).

**Definition 1.2.** A metric space \((X, d)\) is said to have finite asymptotic dimension if there exists \(k \geq 0\) such that for all \(L > 0\) there exists a uniformly bounded cover of \(X\) (i.e., there exists \(S > 0\) such that all elements of the cover are of diameter at most \(S\)) of Lebesgue number at least \(L\) (i.e., every \(L\)-ball \(B(x, L)\) is contained in some element of the cover) and multiplicity at most \(k + 1\) (i.e., each point of \(X\) belongs to at most \(k + 1\) elements of the cover). The least possible such \(k\) is the asymptotic dimension of \(X\).

One of the basic results is that spaces of finite asymptotic dimension have property A and known proofs of it use Higson-Roe characterization of Property A (see [2] and [5]). The purpose of this note is to provide a simple proof of that result and prove the following connection between Property A and asymptotic dimension.

2. The main theorem

**Theorem 2.1.** If \((X, d)\) is a metric space and \(n \geq 0\), then the following conditions are equivalent:

a. \(\text{asdim}(X, d) \leq n\).

b. For each \(R, \epsilon > 0\) there is \(S > 0\) and finite non-empty subsets \(A_x \subset B(x, S) \times \mathbb{N}\), \(x \in X\), such that \(\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon\) if \(d(x, y) < R\) and the projection of \(A_x\) onto \(X\) contains at most \(n + 1\) elements for all \(x \in X\).

c. For each \(R > 0\) there is \(S > 0\) and finite non-empty subsets \(A_x \subset B(x, S) \times \mathbb{N}\), \(x \in X\), such that \(\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{1}{n+1}\) if \(d(x, y) < R\) and the projection of \(A_x\) onto \(X\) contains at most \(n + 1\) elements for all \(x \in X\).

**Proof.** a) \(\implies\) b). Suppose \(\text{asdim}(X, d) \leq n\) and \(R, \epsilon > 0\). Pick a uniformly bounded cover \(U\) of \(X\) of multiplicity at most \(n + 1\) and Lebesgue number at least \(L = 2R + \frac{(2n + 1)R}{2}\). Let \(S\) be a number such that \(\text{diam}(U) < S\) for each \(U \in U\). For every \(U \in U\) pick an element \(a_U \in U\). We call a finite sequence \(x_0, \ldots, x_n\) of points in \(X\) an \(R\)-chain from \(x_0\) to \(x_n\) if \(d(x_i, x_{i-1}) < R\) for \(i = 1, \ldots, n\). For \(x \in X\) and \(U \in U\) let \(l_U(x)\) denote the length of the shortest \(R\)-chain joining \(x\) and a point outside of \(U\), if there is no such chain, we put \(l_U(x)\) equal to the integer part of \(\frac{S}{R}\). Then let \(A_x\) be the following union over all elements from \(U\) containing \(x\).

\[
A_x = \bigcup_{U \ni x} \{a_U\} \times \{1, \ldots, l_U(x)\}
\]

These sets are either empty (and we ignore them) or they are finite non-empty sets and \(A_x \subset B(x, S) \times \mathbb{N}\). If \(d(x, y) < R\), then \(|l_U(x) - l_U(y)| \leq 1\), and
because \( \mathcal{U} \) is a cover of Lebesque number greater than \( R \) and of multiplicity at most \( n + 1 \) the total number of elements of \( \mathcal{U} \) containing at least one of \( x \) or \( y \) is at most \( 2n + 1 \). Therefore \( |A_x \Delta A_y| \leq 2n + 1 \). There exists \( U_0 \in \mathcal{U} \) such that \( B(x, L) \subset U_0 \). Every \( R \)-chain joining \( x \) or \( y \) to \( X \setminus U \) must have at least \( \frac{L-2R}{L} \) elements. If there is no \( R \)-chain from \( x \) to \( X \setminus U \), there is also no \( R \)-chain from \( y \) to \( X \setminus U \), hence \( \{U \} \times \{1, \ldots, \lfloor \frac{R}{L} \rfloor \} \subset A_x, A_y \). In any case we have \( |A_x \cap A_y| > \frac{L-2R}{L} - 1 \). Consequently

\[
\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{(2n + 1) \cdot R}{L - 2R} = \epsilon.
\]

c) \( \Rightarrow \) a). Given \( R > 0 \) pick \( S > 0 \) and finite subsets \( A_x \subset B(x, S) \times N \), \( x \in X \), such that

\[
|A_x \Delta A_y| \leq \frac{1}{n+1}
\]

if \( d(x, y) < R \) and the projection of \( A_x \) onto \( X \) contains at most \( n + 1 \) elements for all \( x \in X \). Define sets \( U_x \) as consisting precisely of \( y \in X \) such that \( \{x\} \times N \cap A_y \neq \emptyset \). The multiplicity of the cover \( \{U_x\}_{x \in X} \) of \( X \) is at most \( n + 1 \) as \( z \in \bigcap_{i=1}^{k} U_{x_i} \) implies \( x_i \) belongs to the projection of \( A_z \), so \( k \leq n + 1 \).

Let us show that \( \{U_x\}_{x \in X} \) has Lebesgue number at least \( R \). Given \( x \in X \) choose \( z \in X \) so that \( |\{(z) \times N \cap A_x| \) maximizes all \(|\{y \times N \cap A_x| \). In particular

\[
|\{(z) \times N \cap A_x| \geq \frac{|A_x|}{n+1}.
\]

Let \( d(x, y) < R \) then \( y \notin U_z \) implies \( |A_x \Delta A_y| \geq \frac{|A_y|}{n+1} \), so \( \frac{|A_x \Delta A_y|}{|A_y|} \geq \frac{1}{n+1} \), a contradiction. Therefore \( y \in U_z \), hence \( B(x, R) \subset U_z \).

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