On a theorem of S. S. Bhatia and B. Ram

Živorad Tomovski *

Abstract. In this paper some inequalities for Dirichlet’s and Fejér’s kernels proved in [6] are refined and extended. Then we have obtained the conditions for $L^1$-convergence of the $r$-th derivatives of complex trigonometric series. These results are extensions of corresponding Bhatia’s and Ram’s results for complex trigonometric series (case $r = 0$).

Key words: Dirichlet kernel, Fejér kernel, $L^1$-convergence, complex trigonometric series, Bernstein’s inequality

AMS subject classifications: 26D15, 42A20, 45A32

Received April 14, 2003 Accepted September 22, 2003

1. Introduction and preliminaries

Let $\{c_k: k = 0, \pm 1, \pm 2, \ldots \}$ be a sequence of complex numbers and let the partial sums of the complex trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be denoted by

$$S_n(c, t) = \sum_{k=-n}^{n} c_k e^{ikt}, \quad t \in (0, \pi] .$$

(1)

If a trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$, for all $n$ and $S_n(c, t) = S_n(f, t) = S_n(f)$.

S. S. Bhatia and B. Ram [6] introduced the following class $\mathcal{R}^*$ of a complex sequence: a null sequence $\{c_n\}$ of complex numbers belongs to class $\mathcal{R}^*$ if

$$\sum_{k=1}^{\infty} \left| \Delta \left( \frac{c_{-k} - c_k}{k} \right) \right| k \log k < \infty \quad \text{and}$$

$$\sum_{k=1}^{\infty} k^2 \left| \Delta^2 \left( \frac{c_k}{k} \right) \right| < \infty .$$

*Department of Mathematics, Faculty of Mathematical and Natural Sciences, P.O. BOX 162, 1000 Skopje, Macedonia, e-mail: tomovski@iunona.pmf.ukim.edu.mk
Dirichlet’s and respectively Feier’s kernels are denoted by

\[ D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \]

\[ \tilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos \frac{t}{2} - \cos \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \]

\[ \mathcal{T}_n(t) = -\frac{1}{2} \cot \frac{t}{2} + \tilde{D}_n(t) = -\frac{\cos \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \]

\[ \tilde{K}_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} \tilde{D}_k(t) = \frac{1}{4 \sin \frac{t}{2}} \left[ \sin t - \frac{\sin(n+1)t}{n+1} \right] \]

Let \( E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt} \) and \( E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt} \). Then the \( r \)-th derivatives \( D^{(r)}_n(t) \) and \( \tilde{D}^{(r)}_n(t) \) can be written as

\[ 2 D^{(r)}_n(t) = E^{(r)}_n(t) + E^{(r)}_{-n}(t) \]

\[ 2i \tilde{D}^{(r)}_n(t) = E^{(r)}_n(t) - E^{(r)}_{-n}(t) \]

S.S. Bhatia and B. Ram [6] introduced the following modified sums

\[ g_n(c, t) = S_n(c, t) + \frac{i}{n+1} \left[ c_{n+1} E'_n(t) - c_{-n} E'_n(t) \right] \]

and proved the following result.

**Theorem 1 [6]**. Let \( \{c_n\} \in \mathbb{R}^* \). Then there exists \( f(t) \) such that

(i) \( \lim_{n \to \infty} g_n(c, t) = f(t) \) for all \( 0 < |t| \leq \pi \).

(ii) \( f(t) \in L^1(T) \) and \( \|g_n(c, t) - f(t)\|_1 = o(1) \), \( n \to \infty \).

(iii) \( \|S_n(f, t) - f(t)\|_1 = o(1) \) iff \( \hat{f}(n) \log |n| = o(1) \), \( |n| \to \infty \).

Now we define a new class \( \mathbb{R}^+(r) \), \( r = 0, 1, 2, \ldots \) of a complex sequence as follows: a null sequence \( \{c_k\} \) of complex numbers belongs to class \( \mathbb{R}^+(r) \), \( r = 0, 1, 2, \ldots \) if

\[ \sum_{k=1}^{\infty} \left| \Delta \left( \frac{c_{k+1} - c_k}{k} \right) \right| k^{r+1} \log k < \infty \]

\[ \sum_{k=1}^{\infty} k^{r+2} \left| \Delta^2 \left( \frac{c_k}{k} \right) \right| < \infty \cdot \]

If \( r = 0 \), class \( \mathbb{R}^+(r) \) reduces to \( \mathbb{R}^* \).

Č. V. Stanojević and V. B. Stanojević [7] introduced the following modified complex trigonometric sums:

\[ U_n(c, t) = S_n(c, t) - \left( c_n E_n(t) + c_{-n} E_{-n}(t) \right) \]
The complex form of the r-th derivative of this sum obtained by Sheng [5] is

\[ U_n^{(r)}(c, t) = S_n^{(r)}(c, t) - (c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)). \]

B. Ram and S. Kumari [4] introduced another set of modified cosine and sine sums as

\[ f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \frac{a_j}{j} k \cos kx \quad \text{and} \]
\[ h_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \frac{a_j}{j} k \sin kx. \]

The complex form of the r-th derivative of these modified sums obtained by Bhatia and Ram [2] is

\[ G_n^{(r)}(c, t) = S_n^{(r)}(c, t) + c_n E_n^{(r+1)}(t) - c_{-(r+1)} E_{-n}^{(r+1)}(t). \]

Using the modified complex sums \( G_n^{(r)} \) we shall prove the following theorem:

**Theorem 2.** Let \( \{c_n\} \in \mathbb{R}^r \), \( r = 0, 1, 2, \ldots \). Then

(i) \( \lim_{n \to \infty} E_n^{(r)}(c, t) = f^{(r)}(t) \) for all \( 0 < |t| \leq \pi \).

(ii) \( f^{(r)} \in L^1(T) \) and \( \|G_n^{(r)}(c, t) - f^{(r)}(t)\| = o(1), n \to \infty \).

(iii) \( \|S_n^{(r)}(f, t) - f^{(r)}(t)\| = o(1), n \to \infty \) \iff \( |n| \log |n| = o(1), |n| \to \infty \).

The case \( r = 0 \) of our Theorem yields Theorem 1.

For other criteria for \( L^1 \)-convergence of the r-th derivative of a complex trigonometric series, see [8].

The real trigonometric series version of Theorem 2 was established by Bhatia and Ram [3].

### 2. Lemmas

For the proof of our new theorem we need the following Lemmas.

**Lemma 1** [5]. For the r-th derivatives of the Dirichlet’s kernels \( D_n \) and \( \tilde{D}_n \) the following estimates hold

(i) \( \|D_n^{(r)}\| = \frac{1}{2} n^r \log n + O(n^r), r = 0, 1, 2, \ldots \)

(ii) \( \|\tilde{D}_n^{(r)}\| = O(n^r \log n), r = 0, 1, 2, \ldots \)

**Lemma 2** [5]. For each non-negative integer \( n \), \( \|c_n E_n^{(r)} + c_{-n} E_{-n}^{(r)}\| = o(1), |n| \to \infty \) holds if and only if \( |n| \log |c_n| = o(1), |n| \to \infty \), where \( \{c_n\} \) is a complex sequence. We note that this Lemma for \( r = 0 \), was obtained by Bray and Stanojević in [1].

**Lemma 3** [10]. Let \( r \) be a non-negative integer. Then for all \( 0 < |t| \leq \pi \) and all \( n \geq 1 \) the following estimates hold
(i) \(|E_{\gamma,n}^{(r)}(t)| \leq \frac{4n^r \pi}{|t|}\).

(ii) \(|\tilde{D}_{\gamma,n}^{(r)}(t)| \leq \frac{4n^r \pi}{|t|}\).

**Proof.** (i) The case \(r = 0\) is trivial. Really, since \(E_n(t) = D_n(t) + i\tilde{D}_n(t)\), we have

\[
|E_n(t)| \leq |D_n(t)| + |\tilde{D}_n(t)| \leq \frac{\pi}{2|t|} + \frac{\pi}{|t|} = \frac{3\pi}{2|t|} < \frac{4\pi}{|t|}.
\]

So \(|E_{-n}(t)| = |E_n(-t)| < \frac{4\pi}{|t|}\).

Let \(r \geq 1\). Applying Abels’s transformation, we have:

\[
E_n^{(r)}(t) = it^r \sum_{k=1}^{n} k^r e^{ikt} = it^r \left[\sum_{k=1}^{n-1} \Delta(k^r) \left(E_k(t) - \frac{1}{2}\right) + n^r \left(E_n(t) - \frac{1}{2}\right)\right]
\]

\[
|E_n^{(r)}(t)| \leq \sum_{k=1}^{n-1} \left|\left(k+1\right)^r - k^r \right| \left(\frac{1}{2} + |E_k(t)|\right) + n^r \left(|E_n(t)| + \frac{1}{2}\right)
\]

\[
\leq \left(\frac{\pi}{2|t|} + \frac{3\pi}{2|t|}\right) \left\{\sum_{k=1}^{n-1} \left|\left(k+1\right)^r - k^r \right| + n^r\right\} = \frac{4\pi n^r}{|t|}.
\]

Since \(E_{-n}^{(r)}(t) = E_n^{(r)}(-t)\), we obtain \(|E_{-n}^{(r)}(t)| \leq \frac{4\pi n^r}{|t|}\).

(ii) Applying inequality (i) and equation 2, we obtain

\[
|\tilde{D}_{\gamma,n}^{(r)}(t)| = |i\tilde{D}_n^{(r)}(t)| \leq \frac{1}{2} |E_n^{(r)}(t)| + \frac{1}{2} |E_{-n}^{(r)}(t)| \leq \frac{4n^r \pi}{|t|}.
\]

\[\square\]

**Lemma 4** [6]. \(\|\tilde{K}_n'(t)\|_1 = O(n)\).

**Lemma 5** [11]. If \(T_n(x)\) is a trigonometric polynomial of order \(n\), then

\(\|T_n^{(r)}\| \leq n^r \|T_n\|\).

This is Bernstein’s inequality in the \(L^1(0, \pi)\)-metric (see [11], vol.2, p.11).

**Lemma 6.** \(\|\tilde{K}_n^{(r)}\|_1 = O(n^r), r = 1, 2, \ldots\)

**Proof.** Since \(K_n(x) = \sum_{k=1}^{n} \frac{n+1-k}{n+1} \sin kx\), we have that

\(T_n(x) = K_n'(x) = \sum_{k=1}^{n} \frac{k(n+1-k)}{n+1} \cos kx\)

is a cosine trigonometric polynomial of order \(n\).

Applying first Bernstein’s inequality, then **Lemma 4** yields:

\(\|\tilde{K}_n^{(r)}\|_1 = \|T_n^{(r-1)}(x)\|_1 \leq n^{r-1} \|T_n(x)\|_1 = O(n^r)\).

\[\square\]
3. Proof of the main result

Applying Abel’s transformation, we have:

\[ G_n^{(r)}(c, t) = S_n^{(r)}(c, t) + \frac{i}{n + 1} \left[ c_{n+1} E_n^{(r+1)}(t) - c_{-n+1} E_{-n}^{(r+1)}(t) \right] \]

\[ = 2 \sum_{k=1}^{n} \Delta \left( \frac{c_k}{k} \right) \tilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{\infty} \Delta \left( \frac{c_{-k} - c_k}{k} \right) i E_{-k}^{(r+1)}(t). \]

By Lemma 3, we get:

\[ \sum_{k=1}^{\infty} \left| \Delta \left( \frac{c_k}{k} \right) \tilde{D}_k^{(r+1)}(t) \right| \leq \frac{4\pi}{|t|} \sum_{k=1}^{\infty} k^{r+1} \left| \Delta \left( \frac{c_k}{k} \right) \right| \leq \frac{4\pi}{|t|} \left\{ \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \left| \Delta^2 \left( \frac{c_j}{j} \right) \right| \right\} \]

\[ = \frac{4\pi}{|t|} \left\{ \sum_{j=1}^{\infty} \left( \sum_{k=1}^{j} k^{r+1} \right) \left| \Delta^2 \left( \frac{c_j}{j} \right) \right| \right\} \]

\[ = O \left( \frac{1}{|t|} \sum_{j=1}^{\infty} j^{r+2} \left| \Delta^2 \left( \frac{c_j}{j} \right) \right| \right) < \infty \]

and

\[ \sum_{k=1}^{\infty} \left| \Delta \left( \frac{c_{-k} - c_k}{k} \right) E_{-k}^{(r+1)}(t) \right| \leq \frac{4\pi}{|t|} \left\{ \sum_{k=1}^{\infty} k^{r+1} \left| \Delta \left( \frac{c_{-k} - c_k}{k} \right) \right| \right\} \]

\[ = O \left( \frac{1}{|t|} \sum_{k=1}^{\infty} k^{r+1} \log k \left| \Delta \left( \frac{c_{-k} - c_k}{k} \right) \right| \right) < \infty. \]

Consequently,

\[ f^{(r)}(t) = 2 \sum_{k=1}^{\infty} \Delta \left( \frac{c_k}{k} \right) \tilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{\infty} \Delta \left( \frac{c_{-k} - c_k}{k} \right) i E_{-k}^{(r+1)}(t) \]

exists and thus (i) follows.

Now, for \( t \neq 0 \), we have:

\[ f^{(r)}(t) - G_n^{(r)}(c, t) = 2 \sum_{k=n+1}^{\infty} \Delta \left( \frac{c_k}{k} \right) \tilde{D}_k^{(r+1)}(t) + i \sum_{k=n+1}^{\infty} \Delta \left( \frac{c_{-k} - c_k}{k} \right) E_{-k}^{(r+1)}(t) \]

\[ = 2 \sum_{k=n+1}^{\infty} (k + 1) \Delta^2 \left( \frac{c_k}{k} \right) \tilde{K}_k^{(r+1)}(t) - 2(n + 1) \Delta \left( \frac{c_{n+1}}{n + 1} \right) \tilde{K}_{n+1}^{(r+1)}(t) \]

\[ + i \sum_{k=n+1}^{\infty} \Delta \left( \frac{c_{-k} - c_k}{k} \right) E_{-k}^{(r+1)}(t). \]
Then,
\[
\|f^{(r)}(t) - G_n^{(r)}(c, t)\|_1 \leq 2 \sum_{k=n+1}^{\infty} (k + 1) \left| \frac{\Delta^2 \left( \frac{c_k}{k} \right)}{k^r} \right| \int_{-\pi}^{\pi} |K_k^{(r+1)}(t)| dt \\
+ 2(n + 1) \left| \frac{\Delta \left( \frac{c_{n+1}}{n+1} \right)}{n+1} \right| \int_{-\pi}^{\pi} |\hat{K}_{n+1}^{(r+1)}(t)| dt \\
+ \sum_{k=n+1}^{\infty} \left| \Delta \left( \frac{c_{n+1} - c_k}{k} \right) \right| \int_{-\pi}^{\pi} |\tilde{E}^{(r+1)}_k(t)| dt.
\]

Applying Lemma 6 and Lemma 1, we have:
\[
\|f^{(r)}(t) - G_n^{(r)}(c, t)\|_1 = O \left( \sum_{k=n+1}^{\infty} (k + 1)^r \left| \frac{\Delta^2 \left( \frac{c_k}{k} \right)}{k^r} \right| \right) \\
+ O \left( (n + 1)^{r+2} \left| \Delta \left( \frac{c_{n+1}}{n+1} \right) \right| \right) \\
+ O \left( \sum_{k=n+1}^{\infty} \left| \Delta \left( \frac{c_{n+1} - c_k}{k} \right) \right| k^{r+1} \log k \right).
\]

But
\[
\left| \Delta \left( \frac{c_{n+1}}{n+1} \right) \right| = \sum_{k=n+1}^{\infty} \Delta^2 \left( \frac{c_k}{k} \right) \leq \sum_{k=n+1}^{\infty} \frac{k^{r+2}}{k^{r+2}} \left| \Delta^2 \left( \frac{c_k}{k} \right) \right| \\
\leq \frac{1}{(n+1)^{r+2}} \sum_{k=n+1}^{\infty} k^{r+2} \left| \Delta^2 \left( \frac{c_k}{k} \right) \right| = o \left( \frac{1}{(n+1)^{r+2}} \right), \quad n \to \infty.
\]

Hence, \( \|f^{(r)}(t) - G_n^{(r)}(c, t)\|_1 = o(1), \ n \to \infty \) by the hypothesis of the theorem.

Since \( G_n^{(r)}(c, t) \) is a polynomial, it follows that \( f^{(r)} \in L^1 \).

The proof of (iii) follows from the estimate
\[
\left\| f^{(r)} - S_n^{(r)}(f) \right\|_1 - \left\| \frac{i}{n+1}(\hat{f}(n+1)E_n^{(r+1)} - \hat{f}(-(n+1))E_{-n}^{(r+1)}) \right\|_1 \\
\leq \left\| f^{(r)} - G_n^{(r)}(c, t) \right\|_1 = o(1), \ n \to \infty
\]

and from Lemma 2.

References


