On the number of solutions of the Diophantine equation of Frobenius – General case

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Abstract. We determine the number of solutions of the equation
\[ a_1x_1 + a_2x_2 + \cdots + a_mx_m = b \]
in non-negative integers \( x_1, x_2, \ldots, x_n \). If \( m = 2 \), then the largest \( b \) for which no solution exists is \( a_1a_2 - a_1 - a_2 \), and an explicit formula for the number of solutions is known. In this paper we give the method for computing the desired number. The method is illustrated with several examples.

Key words: Diophantine problem of Frobenius, number of solutions

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1. Introduction

Let \( a_1, a_2, \ldots, a_m \) be positive integers with \( \gcd(a_1, a_2, \ldots, a_m) = 1 \). Furthermore, let \( N(a_1, a_2, \ldots, a_m; b) \) denote the number of solutions of the equation
\[ a_1x_1 + a_2x_2 + \cdots + a_mx_m = b \tag{1} \]
in non-negative integers \( x_1, x_2, \ldots, x_m \). It is well known that \( N(1, \ldots, 1; b) = \binom{b+m-1}{m-1} \) for any non-negative integer \( b \) (see e.g. Theorem 13.1 in [11]). It is also well-known that equation (1) has a solution in non-negative integers if \( b \) is sufficiently large. Then, what is the generating function \( \sum_{b=0}^\infty N(a_1, a_2, \ldots, a_m; b) x^b \)? How can one determine the constant \( c \) as a function of \( a_1, a_2, \ldots, a_m \) such that \( N(a_1, a_2, \ldots, a_m; b) \sim cb^{m-1} \) (Problem 15C, [11])? If \( m = 2 \), the generating function can be expressed and \( N(a_1, a_2; b) \) can be given in an explicit formula (see e.g. [14], [16], [18]). But, the problem seems to be fairly difficulty if \( m \geq 3 \).

Several authors determined the greatest integer, say, \( G(a_1, a_2, \ldots, a_m) \), such that equation (1) has no such solution in non-negative integers. For \( m = 2 \) a bound \( G(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1 \) was given by Sylvester and this is the best

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possible. For \( m > 2 \) the problem has not been solved. Several bounds are given by many authors (see e.g. [3], [8], [15], [17]) and the good algorithm to calculate it is known if \( m = 3 \) ([6], [13]). There is, however, no good algorithm for its calculation if \( m \geq 4 \). The general solution of an equation (1), where each \( x_j \) can take a negative integer too, was obtained by Bond [2]. An algorithm by Djawadi and Hofmeister [7] can calculate some bound under the condition \( a_1 = 1 \). In fact, if \( m \geq 3 \), \( G(a_1, a_2, \ldots, a_m) \) cannot be given by closed formulas of a certain type ([5]) and the problem to determine \( G \) is NP-hard ([12]).

In this paper we are interested in determining the number of solutions in (1), where each \( x_i \) can take a negative integer too, was obtained by Bond [2]. An algorithm by Djawadi and Hofmeister [7] can calculate some bound under the condition \( a_1 = 1 \). In fact, if \( m \geq 3 \), \( G(a_1, a_2, \ldots, a_m) \) cannot be given by closed formulas of a certain type ([5]) and the problem to determine \( G \) is NP-hard ([12]).

2. Preliminaries

By the counting theorem one has

\[
\mathcal{N}(x) := \sum_{b=0}^{\infty} N(a_1, a_2, \ldots, a_m; b) x^b = \frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_m})} = \frac{c}{(1 - x)^m} + O((1 - x)^{-m+1}).
\]

By Schur’s theorem one has

\[
N(a_1, a_2, \ldots, a_m; b) \sim \frac{b^{m-1}}{(m-1)! a_1 a_2 \cdots a_m} (b \to \infty).
\]

In particular, there exists an integer \( N \) such that every \( b \geq N \) is so representable in at least one way ([18, pp.93–99]).

Assume that \( \gcd(a_h, a_l) = 1 \). Then we can write

\[
\mathcal{N}(x) = \sum_{b=0}^{\infty} N(a_1, a_2, \ldots, a_m; b) x^b = \frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_m})} = \frac{1}{(m-1)! a_1 a_2 \cdots a_m x^{m-1}} \left( \sum_{k=1}^{a_1-1} \frac{A_{a_1}(k)}{1 - \zeta_{a_1}^{-k} x^{a_1}} + \cdots + \sum_{k=1}^{a_m-1} \frac{A_{a_m}(k)}{1 - \zeta_{a_m}^{-k} x^{a_m}} \right),
\]

where \( \zeta_{a_l} = e^{2\pi i / a_l} \) (\( l = 1, 2, \ldots, m \)). We have two decompositions. The first decomposition into ordinary partial fractions is called the first type; the second one including the periodic sequences is called the second type or Herschelian type [4, p.109].

Multiplying both sides of (2) by \( 1 - \zeta_{a_1}^{-k} x \) and taking limits as \( x \to \zeta_{a_1}^k \) entails that

\[
A_{a_1}(k) = \lim_{x \to \zeta_{a_1}^k} \frac{1 - \zeta_{a_1}^{-k} x}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_m})} = \lim_{x \to \zeta_{a_1}^k} \frac{-\zeta_{a_1}^{-k}}{a_1 x^{a_1-1}(1 - x^{a_2}) \cdots (1 - x^{a_m})} = \frac{1}{a_1} \left( \frac{1}{1 - \zeta_{a_1}^{a_2 k}} \cdots (1 - \zeta_{a_1}^{a_m k}) \right).$$
In a similar manner we can obtain for \( P \)

\[
A_{ni}(k) = \frac{1}{a_1(1 - \zeta_{a_1}^k)} \cdots \frac{1}{(1 - \zeta_{a_l}^k)(1 - \zeta_{a_l + 1}^k)} \cdots \frac{1}{(1 - \zeta_{a_m}^k)}.
\]

Multiplying both sides of (2) by \((1 - x)^m\) and letting \( x \to 1 \) entails that \( c_m = 1/(a_1 \cdots a_m) \). To calculate \( c_l \) \((l = m - 1, m - 2, \ldots, 1)\), we multiply both sides of (2) by \((1 - x)^m\), differentiate \( m - l \) times and take limits as \( x \to 1 \). Namely, we have

\[
(-1)^{m-l}(m-l)!c_l = \frac{\partial^{m-l}}{\partial x^{m-l}} \left( \frac{(1-x)^m}{(1-x^{a_1}) \cdots (1-x^{a_m})} \right)_{x=1}
\]

Then one can obtain \( c_{m-1} = (a_1 + \cdots + a_m - m)/(2a_1 \cdots a_m) \). We should be able to obtain \( c_{m-2}, c_{m-3}, \ldots \) in a similar manner, but it seems that it becomes extremely difficult to calculate them practically. The details are given in the next section.

Notice that for \( l = 1, 2, \ldots, m \)

\[
\frac{1}{(1-x)^l} = \sum_{n=0}^{\infty} \binom{n+l-1}{n} x^n.
\]

Hence,

\[
N(a_1, a_2, \ldots, a_m; b) = \sum_{l=1}^{m} \left( c_l \binom{b + l - 1}{b} + \sum_{k=1}^{a_l-1} A_{ni}(k) \zeta_{a_l}^{-bk} \right)
\]

\[
= \sum_{j=0}^{m-1} d_j b^j + \sum_{l=1}^{m} \sum_{k=1}^{a_l-1} A_{ni}(k) \zeta_{a_l}^{-bk}.
\]

This form has been already known (see e.g. [4], [14], [18]).

3. The calculation of \( d_j \)

We consider the terms derived from the first type of two decompositions. First of all, notice that

\[
\binom{b + l - 1}{b} = \frac{1}{(l-1)!} \left( b^{l-1} + b^{l-2} \sum_{j=1}^{l-1} \binom{b}{j} + b^{l-3} \sum_{1 \leq j_1 < j_2 < l} j_1 j_2 + \cdots \right.
\]

\[
\left. + b \sum_{1 \leq j_1 < \cdots < j_{l-2} < l} j_1 \cdots j_{l-2} + (l-1)! \right).
\]

Denote \( P = a_1 a_2 \cdots a_m \) and \( S_j = a_1^j + a_2^j + \cdots + a_m^j \) \((j = 1, 2, \ldots)\). By obtaining

\[
c_m = \frac{1}{P}, \quad c_{m-1} = \frac{S_1 - m}{2P},
\]
In a similar manner, one can find
\[ c_{m-2} = \frac{3S_1^2 - S_2 - 6(m - 1)(S_1 - m) - m(3m - 1)}{24P} \quad \text{and} \]
\[ c_{m-3} = \frac{1}{48P} \left( S_1^3 - S_1S_2 - (m - 2)(3S_1^2 - S_2) + (m - 1)(3m - 8)(S_1 - m) + 2m(m^2 - 3m + 1) \right), \]
one can find that (Cf. [4, p.113])
\[
\begin{align*}
d_{m-1} &= \frac{e_m}{(m-1)!} = \frac{1}{(m-1)!P} \\
d_{m-2} &= \frac{c_m}{(m-1)!} \sum_{j=1}^{m-1} j + \frac{c_m}{(m-2)!} \\
&= \frac{c_m}{(m-1)!} \frac{m(m-1)}{2} + \frac{c_m}{(m-2)!} = \frac{S_1}{2(m-2)!P}, \\
d_{m-3} &= \frac{c_m}{(m-1)!} \sum_{1 \leq j_1 < j_2 \leq m-1} j_1j_2 + \frac{c_{m-1}}{(m-2)!} \sum_{j=1}^{m-2} j + \frac{c_{m-2}}{(m-3)!} \\
&= \frac{c_m}{(m-1)!} \frac{(m-2)(m-1)m(3m-1)}{24} + \frac{c_{m-1}}{(m-2)!} \frac{(m-1)(m-2)}{2} + \frac{c_{m-2}}{(m-3)!} \\
&= \frac{3S_1^2 - S_2}{24(m-3)!P}, \\
d_{m-4} &= \frac{e_m}{(m-1)!} \sum_{1 \leq j_1 < j_2 < j_3 \leq m-1} j_1j_2j_3 + \frac{c_{m-1}}{(m-2)!} \sum_{1 \leq j_1 < j_2 \leq m-2} j_1j_2 \\
&+ \frac{c_{m-2}}{(m-3)!} \sum_{j=1}^{m-3} j + \frac{c_{m-3}}{(m-4)!} \\
&= \frac{c_m}{(m-1)!} \frac{m^2(m-1)^2(m-2)(m-3)}{48} + \frac{c_{m-1}}{(m-2)!} \frac{(m-3)(m-2)(m-1)(3m-4)}{24} \\
&+ \frac{c_{m-2}}{(m-3)!} \frac{(m-2)(m-3)}{2} + \frac{c_{m-3}}{(m-4)!} \\
&= \frac{S_1(S_1^2 - S_2)}{48(m-4)!P}. 
\end{align*}
\]

In a similar manner, one can find
\[ d_{m-5} = \frac{2S_4 + 5S_1^2 - 30S_1^2S_2 + 15S_1^4}{240 \cdot 4!(m-5)!P}, \]
Finally, we have the following identity.

\[ d_{m-6} = \frac{S_1(2S_4 + 5S_2^2 - 10S_1^2S_2 + 3S_1^4)}{96 \cdot 5!(m - 6)!P}, \]
\[ d_{m-7} = \frac{-16S_6 - 42S_2S_4 + 126S_2^2S_4 - 35S_3^2 + 315S_4^2S_2^2 - 315S_4S_2^3 + 63S_6^0}{4032 \cdot 6!(m - 7)!P}, \]
\[ d_{m-8} = \frac{S_1(-16S_6 - 42S_2S_4 + 42S_4^2S_2^2 - 35S_3^2 + 105S_4^2S_2^2 - 63S_4S_2^3 + 9S_6^0)}{1152 \cdot 7!(m - 8)!P}, \]

\[ \ldots \]

After obtaining \( c_m, c_{m-1}, \ldots, c_{m-t+1} \), one can find \( d_{m-l} \) as

\[ d_{m-l} = \frac{c_m}{(m-1)!} \sum_{1 \leq j_1 < \cdots < j_{l-1} \leq m-1} j_1 \cdots j_{l-1} + \frac{c_{m-1}}{(m-2)!} \sum_{1 \leq j_1 < \cdots < j_{l-2} \leq m-2} j_1 \cdots j_{l-2} + \cdots + \frac{c_{m-l+1}}{(m-l+1)!} \sum_{j=1}^{m-l+1} j + \frac{c_{m-l+1}}{(m-l)!}. \]

Finally, \( d_0 = c_m + c_{m-1} + \cdots + c_1 \).

But it was very hard to find an explicit form of the general \( d_j \). One nice-looking form can be derived from the main result in [1]. Define Bell polynomials \( Y_n(y_1, y_2, \ldots, y_n) \) by

\[ \exp \left( \sum_{k=1}^{\infty} \frac{y_k x^k}{k!} \right) = \sum_{n=0}^{\infty} Y_n(y_1, y_2, \ldots, y_n) \frac{x^n}{n!}, \]

where \( Y_0 = 1 \) and

\[ Y_n(y_1, y_2, \ldots, y_n) = \sum_{k_1 + 2k_2 + \cdots + nk_n = n} \prod_{i=1}^{n} \frac{n! y_i^{k_i}}{k_1! (i)!^{k_1}}. \]

We have the following identity.

**Proposition 1.** For \( l = 0, 1, 2, \ldots \) we have

\[ d_{m-l-1} = \frac{(-1)^l}{(m-l-1)!!P} Y_l(B_1S_1, -B_2S_2, \ldots, -B_lS_l), \]

where \( P = \prod_{j=1}^{m} a_j \), \( S_n = \sum_{j=1}^{m} a_j^n \) and \( B_n \) is the \( n \)-th Bernoulli number (\( n = 1, 2, \ldots \)).

**Proposition 2.** For \( l = 1, 2, \ldots \) we have

\[ d_{m-l} = \frac{2}{m-l} \frac{\partial}{\partial S_1} d_{m-l-1}. \]
4. The calculation of $\sum A_{a_l}(k)\zeta_{a_l}^{-bk}$

We consider the terms derived from the Herschellian type of two decompositions. We assume that $\gcd(a_h, a_l) = 1$ ($h \neq l$). Put $A_{a_l} = \sum_{k=1}^{a_l-1} A_{a_l}(k)\zeta_{a_l}^{-bk}$ ($l = 1, 2, \ldots, m$) for convenience. Without loss of generality, set $a = a_1$. When $a = 1$, this term does not exist. When $a = 2$, by the assumption all of $a_2, a_3, \ldots, a_m$ are odd. From $\zeta_2 = -1$ we have

$$A_2 = \frac{1}{a} \sum_{k=1}^{a-1} A_2(k)\zeta_2^{-bk} = \frac{1}{2} \left( 1 - \zeta_2^{a_2} \right) \left( 1 - \zeta_2^{a_1} \right) \cdots \left( 1 - \zeta_2^{a_m} \right) = \frac{(-1)^b}{2^m}.$$ 

Let $a_1$ be odd with $a_1 \geq 3$. Denote $s_l$ ($l = 1, 2, \ldots, a - 1$) by

$$s_l := \# \{ a_j | 2 \leq j \leq m, \quad a_j \equiv l \pmod{a} \},$$

satisfying $\sum_{l=1}^{a-1} s_l = m - 1$. By the assumption, $a_j \not\equiv 0 \pmod{a}$ for any $j$ with $2 \leq j \leq m$. Put $\zeta = \zeta_{a_1}$ for simplicity.

With these notations we can write

$$A_a = \frac{1}{a} \sum_{k=1}^{a-1} \frac{\zeta^{-bk}}{(1 - \zeta^k)^{s_1} (1 - \zeta^{2k})^{s_2} \cdots (1 - \zeta^{(a-1)k})^{s_{a-1}}}.$$ 

**Lemma 1.** For any integer $k$ we have

$$1 - \zeta_k = 2 \sin \frac{k}{a} \pi \cdot e^{-\frac{\pi}{a} 2k^i \pi}.$$ 

**Proof.** Put $1 - \zeta_k = re^{i\theta}$. Then

$$r = \sqrt{ \left( 1 - \cos \frac{2k}{a} \pi \right)^2 + \left( \sin \frac{2k}{a} \pi \right)^2 } = 2 \sin \frac{k}{a} \pi.$$ 

By

$$\sin(-\theta) = \frac{1}{r} \sin \frac{2k}{a} \pi = \cos \frac{k}{a} \pi,$$

we have

$$-\theta = \frac{\pi}{2} - \frac{k}{a} \pi = \frac{a - 2k}{2a} \pi.$$ 

By this lemma together with the facts

$$\sin \frac{l(a-k)}{a} \pi = (-1)^l \sin \frac{lk}{a} \pi$$

and

$$e^{(a-2l(a-k))(s_l-s_{a-l}) \pi/(2a)} = (-1)^{l-1} e^{-(a-2lk)(s_l-s_{a-l}) \pi/(2a)},$$

we obtain

$$(1 - \zeta_k)^{(s_l-s_{a-l})} e^{-(a-2lk)(s_l-s_{a-l}) \pi/(2a)}.$$
From $\zeta^{-b(a-k)} = \zeta^{bk}$, if $a$ is odd, then

$$A_a = \frac{1}{a} \sum_{k=1}^{a-1} \zeta^{-bk} \prod_{i=1}^{(a-1)/2} \left( e^{(a-2ik)(s_i-s_{a-i})\pi/(2a)} \right)^{(a-1)/2} \prod_{i=1}^{(a-1)/2} \left( 2 \sin \frac{lk}{a} \pi \right)^{s_i+s_{a-i}}$$

$$= \frac{2}{a} \sum_{k=1}^{(a-1)/2} \cos \left( \frac{\sum_{i=1}^{(a-1)/2} (a-2ik)(s_i-s_{a-i})-4bk}{2a} \right) \prod_{i=1}^{(a-1)/2} \left( 2 \sin \frac{lk}{a} \pi \right)^{s_i+s_{a-i}}.$$  

We can interchange $a_1$ and any $a_h$ ($2 \leq h \leq m$) without loss of generality. Therefore, we obtain the following.

**Theorem 1.** If $a = a_h$ is odd with $a \geq 3$ and $\gcd(a_j, a) = 1$ ($1 \leq j \leq m$, $j \neq h$), then

$$A_a = \frac{2}{a} \sum_{k=1}^{(a-1)/2} \cos \left( \frac{\sum_{i=1}^{(a-1)/2} (a-2ik)(s_i-s_{a-i})-4bk}{2a} \right) \prod_{i=1}^{(a-1)/2} \left( 2 \sin \frac{lk}{a} \pi \right)^{s_i+s_{a-i}}.$$  

This form seems still very complicated, but we can calculate $A_a$ very easily when $a$ is small even if the number $m$ is very big.

**Corollary 1.** When $a = 3$, we have

$$A_3 = \sum_{k=1}^{2} A_3^{(1)} \zeta_{s_3}^{bk} = \frac{1}{3} \sum_{k=1}^{2} \frac{\zeta_{s_3}^{bk}}{(1-\zeta_{s_3}^{bk})(1-\zeta_{s_3}^{2bk})\cdots(1-\zeta_{s_3}^{ak})}$$

$$= \frac{2}{3} \cos \left( \frac{2b}{3} \frac{s_1-s_2}{6} \right) \pi.$$  

**Proof.** When $a = 3$, we have $l = k = 1$, and $2 \sin (lk/a)\pi = \sqrt{3}$. \qed

**Corollary 2.** When $a = 5$, we have

$$A_5 = \frac{2}{5} \sum_{k=1}^{2} \cos \left( \frac{4bk+(2k-5)(s_1-s_2)+(4k-5)(s_2-s_3)}{2(2\sin \frac{k}{5}\pi)^{s_1+s_2}(2\sin \frac{4k}{5}\pi)^{s_2+s_3}} \right).$$  

**Remark 1.** Notice that

$$\left( 2 \sin \frac{\pi}{5} \right) \left( 2 \sin \frac{2\pi}{5} \right) = \sqrt{\frac{5-\sqrt{5}}{2}} \sqrt{\frac{5+\sqrt{5}}{2}} = \sqrt{5}$$

for further calculations.

**Corollary 3.** When $a = 7$, we have

$$A_7 = \frac{2}{7} \sum_{k=1}^{3} \cos \left( \frac{4bk+(2k-7)(s_1-s_2)+(4k-7)(s_2-s_3)+(6k-7)(s_3-s_4)}{2(2\sin \frac{k}{7}\pi)^{s_1+s_2}(2\sin \frac{4k}{7}\pi)^{s_2+s_3}(2\sin \frac{6k}{7}\pi)^{s_3+s_4}} \right).$$
Remark 2. It is convenient to use relations

\[
2\sin \frac{\pi}{7} \cdot 2\sin \frac{2\pi}{7} \cdot 2\sin \frac{3\pi}{7} = \sqrt{7} \quad \text{and} \quad \sin \frac{2\pi}{7} + \sin \frac{3\pi}{7} - \sin \frac{\pi}{7} = \frac{\sqrt{7}}{2}.
\]

for further calculations.

Let \(a\) be even with \(a \geq 4\). By the assumption, \(s_l = 0\) if \(l\) is even or \(l = a/2\). In a similar manner we obtain

\[
A_a = \frac{1}{a} \sum_{k=1}^{a-1} \frac{\zeta^{-bk}}{(1 - \zeta^k)(1 - \zeta^{3k})(1 - \zeta^{(a-1)k})^{a-1}}
= \frac{1}{a} \sum_{k=1}^{a-1} \frac{\zeta^{-bk}}{\prod_{l=1}^{2\lfloor a/4 \rfloor - 1} c(a-2(2l-1)k)(s_{2l-1} - s_{a-2l+1})\pi/(2a)}
\]

\[
= \frac{2^{a/2-1}}{a} \cos \left( \sum_{l=1}^{2\lfloor a/4 \rfloor - 1} (a-2(2l-1)k)(s_{2l-1} - s_{a-2l+1} - 4bk) \right) + \frac{(-1)^b}{a \cdot 2^{m-1}}.
\]

Notice that the last term arises for \(k = a/2\).

Theorem 2. If \(a = a_h\) is even with \(a \geq 4\) and \(\gcd(a_j, a) = 1\) (\(1 \leq j \leq m, j \neq h\)), then

\[
A_a = \frac{2}{a} \sum_{k=1}^{2^{m-1}} \frac{\cos \left( \frac{4bk + \sum_{l=1}^{2\lfloor a/4 \rfloor - 1} (2(2l-1)k - a)(s_{2l-1} - s_{a-2l+1})\pi}{2a} \right)}{\prod_{l=1}^{2\lfloor a/4 \rfloor - 1} \left( 2\sin \frac{(2l-1)k}{a} \pi \right)^{s_{2l-1} + s_{a-2l+1}} + \frac{(-1)^b}{a \cdot 2^{m-1}}.}
\]

5. Examples

Suppose that \(m = 3\). Then

\[
N(a_1, a_2, a_3; b) = \frac{a_1^2 + a_2^2 + a_3^2 + 3(a_1a_2 + a_2a_3 + a_3a_1)}{12a_1a_2a_3} + \frac{a_1 + a_2 + a_3}{2a_1a_2a_3}b
\]

\[
+ \frac{1}{2a_1a_2a_3}b^2 + \frac{1}{a_1} \sum_{k=1}^{a_1-1} \frac{\zeta^{-bk}}{(1 - \zeta^{a_2k})(1 - \zeta^{a_3k})}
\]

\[
+ \frac{1}{a_2} \sum_{k=1}^{a_2-1} \frac{\zeta^{-bk}}{(1 - \zeta^{a_1k})(1 - \zeta^{a_3k})}
+ \frac{1}{a_3} \sum_{k=1}^{a_3-1} \frac{\zeta^{-bk}}{(1 - \zeta^{a_1k})(1 - \zeta^{a_2k})}.
\]
Let \( a_1 = 3, a_2 = 5 \) and \( a_3 = 7 \). For \( a_1 = 3 \), by Corollary 1 with \( s_1 = s_2 = 1 \) we have
\[
A_3 = \frac{2}{9} \cos \frac{2}{3} b \pi.
\]

For \( a_2 = 5 \), by Corollary 2 with \( s_1 = s_4 = 0 \) and \( s_2 = s_3 = 1 \) we have
\[
A_5 = \frac{2}{5} \sum_{k=1}^{2} \frac{\cos \frac{4k}{5} \pi}{(2 \sin \frac{4k}{5} \pi)^2} = \frac{2}{25} \left( (2 \sin \frac{\pi}{5})^2 \cos \frac{2b}{5} \pi + (2 \sin \frac{2\pi}{5})^2 \cos \frac{4b}{5} \pi \right).
\]

For \( a_3 = 7 \), by Corollary 3 with \( s_1 = s_2 = s_4 = s_6 = 0 \) and \( s_3 = s_5 = 1 \) we have
\[
A_7 = \frac{2}{7} \sum_{k=1}^{3} \frac{\cos \frac{2k+1}{7} \pi}{(2 \sin \frac{2k}{7} \pi)(2 \sin \frac{4k}{7} \pi)}
\begin{align*}
&= \frac{2}{7 \sqrt{7}} \left( 2 \sin \frac{\pi}{7} \cos \frac{2b+1}{7} \pi + 2 \sin \frac{2\pi}{7} \cos \frac{2(2b+1)}{7} \pi \\
&\quad - 2 \sin \frac{3\pi}{7} \cos \frac{3(2b+1)}{7} \pi \right).
\end{align*}
\]

Therefore, we obtain
\[
N(3,5,7; b) = \frac{1}{210} b^2 + \frac{1}{14} b + \frac{74}{315} + \frac{2}{9} \cos \frac{2}{3} b \pi
+ \frac{2}{25} \left( (2 \sin \frac{\pi}{5})^2 \cos \frac{2b}{5} \pi + (2 \sin \frac{2\pi}{5})^2 \cos \frac{4b}{5} \pi \right)
+ \frac{2}{7 \sqrt{7}} \left( 2 \sin \frac{\pi}{7} \cos \frac{2b+1}{7} \pi + 2 \sin \frac{2\pi}{7} \cos \frac{2(2b+1)}{7} \pi \\
&\quad - 2 \sin \frac{3\pi}{7} \cos \frac{3(2b+1)}{7} \pi \right).
\]

With the notation due to Cayley (Cf. [9]), \((x_0, x_1, \ldots, x_{k-1})_{\text{pcr} k} = x_i \) if \( b \equiv i \) (mod \( k \)), this result matches the Comtet’s one [4, pp.114–115],
\[
N(3,5,7; b) = \frac{1}{210} b^2 + \frac{1}{14} b + \frac{74}{315} + \frac{1}{9}(2, -1, -1)_{\text{pcr} 3k}
\begin{align*}
&+ \frac{1}{5}(2, -1, 0, 0, -1)_{\text{pcr} 5k} + \frac{1}{7}(1, 0, -2, 2, -2, 0, 1)_{\text{pcr} 7k}.
\end{align*}
\]

It is quite easy to find
\[
N(1,2,3; b) = \frac{1}{12} b^2 + \frac{1}{2} b + \frac{47}{72} + \frac{(-1)^b}{8} + \frac{2}{5} \cos \frac{2}{3} b \pi
\]
for \( a_1 = 1, a_2 = 2 \) and \( a_3 = 3 \) (Cf. [4, p.110]).

If each \( a_j \) is small, it is not difficult to obtain the exact form of \( N(a_1, \ldots, a_m; b) \), even though the number \( m \) becomes large. For example, let \( a_1 = 2, a_2 = 3, a_3 = 5, \)
\( a_4 = 7, a_5 = 11, a_6 = 13, a_7 = 17 \) and \( a_8 = 19 \). Then one can get

\[
N(2, 3, 5, 7, 11, 13, 17, 19; b) = \frac{1}{48886437600} b^7 + \frac{1}{1813968000} b^6 + \frac{419}{698377680} b^5 \\
+ \frac{43}{1272960} b^4 + \frac{21901069}{20951330400} b^3 + \frac{174869}{10077600} b^2 + \frac{134507}{978120} b + \frac{310672961}{2176761600} \\
+ \frac{(-1)^b}{256} + \frac{2}{81\sqrt{3}} \cos \left( \frac{2}{3} b + \frac{1}{6} \right) \pi \\
+ \frac{2}{125\sqrt{5}} \left( (2 \sin \frac{\pi}{5})^3 \cos \frac{4b - 1}{10} \pi + (2 \sin \frac{2\pi}{5})^3 \cos \frac{8b + 3}{10} \pi \right) \\
+ \frac{2}{49\sqrt{7}} \left( (2 \sin \frac{\pi}{7})^2 \cos \frac{4b + 7}{14} \pi + (2 \sin \frac{2\pi}{7})^2 \cos \frac{8b + 7}{14} \pi \\
- (2 \sin \frac{3\pi}{7})^2 \cos \frac{12b + 7}{14} \pi \right) \\
+ \frac{2}{121} \sum_{k=1}^{5} (2 \sin \frac{k\pi}{11})^2 (2 \sin \frac{4k\pi}{11}) \cos \frac{4bk - 11}{22} \pi \\
+ \frac{2}{169} \sum_{k=1}^{6} (2 \sin \frac{k\pi}{13})^2 (2 \sin \frac{3k\pi}{13})(2 \sin \frac{4k\pi}{13}) (2 \sin \frac{5k\pi}{13}) \cos \frac{(4b + 24)k - 39}{26} \pi \\
+ \frac{2}{289} \sum_{k=1}^{8} (2 \sin \frac{k\pi}{17})^2 (2 \sin \frac{3k\pi}{17})(2 \sin \frac{4k\pi}{17}) (2 \sin \frac{5k\pi}{17})(2 \sin \frac{6k\pi}{17}) \\
\cdot (2 \sin \frac{7k\pi}{17}) (2 \sin \frac{8k\pi}{17})^2 \cos \frac{(4b + 18)k - 51}{34} \pi \\
+ \frac{2}{361} \sum_{k=1}^{9} (2 \sin \frac{k\pi}{19})^2 (2 \sin \frac{3k\pi}{19})(2 \sin \frac{4k\pi}{19})^2 (2 \sin \frac{5k\pi}{19})(2 \sin \frac{6k\pi}{19}) \\
\cdot (2 \sin \frac{7k\pi}{19}) (2 \sin \frac{8k\pi}{19})(2 \sin \frac{9k\pi}{19})^2 \cos \frac{(4b + 2)k - 19}{38} \pi \\
= \frac{1}{48886437600} b^7 + \frac{1}{1813968000} b^6 + \frac{419}{698377680} b^5 + \frac{43}{1272960} b^4 \\
+ \frac{43}{2176761600} b^3 + \frac{174869}{10077600} b^2 + \frac{134507}{978120} b + \frac{310672961}{2176761600} \\
+ \frac{(-1)^b}{256} + \frac{1}{81}(1, -1, 0)pcr3b + \frac{1}{25}(1, -1, 1, -1, 0)pcr5b \\
+ \frac{1}{49}(0, -1, -2, 4, -4, 2, 1)pcr7b + \frac{1}{11}(0, -1, 2, -1, 0, 0, 0, 1, -2, 1)pcr11b \\
+ \frac{1}{13}(3, -3, 1, -1, 1, 0, 1, 0, -1, -1, 1)pcr13b \\
+ \frac{1}{17}(4, -4, 2, -2, 0, 2, -2, 4, -4, 3, -1, 3, -3, 3, -3, 1, -3)pcr17b \\
+ \frac{1}{19}(2, 2, -2, 5, -5, 3, -1, 2, 0, 0, 0, -2, 1, -3, 5, -5, 2, -2, -2)pcr19b.
We omit the detail calculations above. For example, use the relation
\[
\prod_{k=1}^{(a-1)/2} \left(2 \sin \frac{k \pi}{a}\right) = \sqrt{a}.
\]

Let \(a_1 = 137\), \(a_2 = 251\) and \(a_3 = 256\), which triple is an example much used in the literature (see e.g. [13]). By Theorems 1 and 2 one gets
\[
N(137, 251, 256; b) = \frac{1}{17606144}b^2 + \frac{161}{4401536}b + \frac{182817}{35212288} + 2 \sum_{k=1}^{68} \cos \left(\frac{2b-41}{137}k - 1\right) \pi \left(\frac{2 \sin \frac{k \pi}{137}}{2 \sin \frac{194 \pi}{137}}\right) + 2 \sum_{k=1}^{125} \cos \left(\frac{2b-109}{251}k \pi\right) \left(\frac{2 \sin \frac{k \pi}{251}}{2 \sin \frac{144 \pi}{251}}\right)
\]
\[
+ \frac{1}{128} \sum_{k=1}^{127} \cos \left(\frac{k-62}{128}b - 1\right) \pi \left(\frac{2 \sin \frac{k \pi}{128}}{2 \sin \frac{198 \pi}{128}}\right) + (-1)^b \frac{1}{1024}.
\]

It seems nearly impossible to continue this calculation by hand only. For example, Mathematica or Maple calculations can show immediately
\[
N(137, 251, 256; 4948) = 0, \quad N(137, 251, 256; 4949) = 2
\]
and so on. In fact, \(G(137, 251, 256) = 4948\).

References


