On the number of solutions of the Diophantine equation of Frobenius – General case*

TAKAO KOMATSU†

Abstract. *We determine the number of solutions of the equation $a_1x_1+a_2x_2+\cdots+a_mx_m=b$ in non-negative integers x_1, x_2, ..., x_n. If $m=2$, then the largest b for which no solution exists is $a_1a_2-a_1-a_2$, and an explicit formula for the number of solutions is known. In this paper we give the method for computing the desired number. The method is illustrated with several examples.*

Key words: *Diophantine problem of Frobenius, number of solutions*

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1. Introduction

Let a_1, a_2, ..., a_m be positive integers with $\gcd(a_1,a_2,\ldots,a_m)=1$. Furthermore, let $N(a_1,a_2,\ldots,a_m;b)$ denote the number of solutions of the equation

$$a_1x_1+a_2x_2+\ldots+a_mx_m=b \tag{1}$$

in non-negative integers x_1, x_2, ..., x_m. It is well known that $N(\underbrace{1,\ldots,1}_{m};b)=\binom{b+m-1}{m-1}$ for any non-negative integer b (see e.g. Theorem 13.1 in [11]). It is also well-known that equation (1) has a solution in non-negative integers if b is sufficiently large. Then, what is the generating function $\sum_{b=0}^{\infty}N(a_1,a_2,\ldots,a_m;b)\,x^b$? How can one determine the constant c as a function of a_1, a_2, ..., a_m such that $N(a_1,a_2,\ldots,a_m;b)\sim cb^{m-1}$ (Problem 15C, [11])? If $m=2$, the generating function can be expressed and $N(a_1,a_2;b)$ can be given in an explicit formula (see e.g. [14], [16], [18]). But, the problem seems to be fairly difficulty if $m\ge 3$.

Several authors determined the greatest integer, say, $G(a_1,a_2,\ldots,a_m)$, such that equation (1) has no such solution in non-negative integers. For $m=2$ a bound $G(a_1,a_2)=(a_1-1)(a_2-1)-1$ was given by Sylvester and this is the best

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†Department of Mathematical System Science, Faculty of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan, e-mail: `komatsu@cc.hirosaki-u.ac.jp`

possible. For $m > 2$ the problem has not been solved. Several bounds are given by many authors (see e.g. [3], [8], [15], [17]) and the good algorithm to calculate it is known if $m = 3$ ([6], [13]). There is, however, no good algorithm for its calculation if $m \geq 4$. – The general solution of an equation (1), where each x_j can take a negative integer too, was obtained by Bond [2]. An algorithm by Djawadi and Hofmeister [7] can calculate some bound under the condition $a_1 = 1$. In fact, if $m \geq 3$, $G(a_1, a_2, \ldots, a_m)$ cannot be given by closed formulas of a certain type ([5]) and the problem to determine G is NP-hard ([12]).

In this paper we are interested in determining the number of solutions in (1), when $\gcd(a_h, a_l) = 1$ $(h \neq l)$. Sertöz [14] and Tripathi [16] independently obtained an explicit formula in the case $m = 2$. Israilov [10] found one in the general m, but it was too long and complicated. We shall give a general form which is well computable practically to find the real values of $N(a_1, \ldots, a_m; b)$ even if $m \geq 3$.

2. Preliminaries

By the counting theorem one has

$$\begin{aligned}\mathcal{N}(x) := \sum_{b=0}^{\infty} N(a_1, a_2, \ldots, a_m; b) x^b &= \frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_m})} \\ &= \frac{c}{(1-x)^m} + O((1-x)^{-m+1}) \,.\end{aligned}$$

By Schur's theorem one has

$$N(a_1, a_2, \ldots, a_m; b) \sim \frac{b^{m-1}}{(m-1)! a_1 a_2 \cdots a_m} \quad (b \to \infty) \,.$$

In particular, there exists an integer N such that every $b \geq N$ is so representable in at least one way ([18, pp.93–99]).

Assume that $\gcd(a_h, a_l) = 1$ $(h \neq l)$. Then we can write

$$\begin{aligned}\mathcal{N}(x) &= \sum_{b=0}^{\infty} N(a_1, a_2, \ldots, a_m; b) x^b = \frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_m})} \qquad (2) \\ &= \frac{c_1}{1-x} + \cdots + \frac{c_m}{(1-x)^m} + \sum_{k=1}^{a_1 - 1} \frac{A_{a_1}(k)}{1 - \zeta_{a_1}^{-k} x} + \cdots + \sum_{k=1}^{a_m - 1} \frac{A_{a_m}(k)}{1 - \zeta_{a_m}^{-k} x} \,,\end{aligned}$$

where $\zeta_{a_l} = e^{2\pi i / a_l}$ $(l = 1, 2, \ldots, m)$. We have two decompositions. The first decomposition into ordinary partial fractions is called the *first type*; the second one including the periodic sequences is called the *second type* or *Herschellian type* [4, p.109].

Multiplying both sides of (2) by $1 - \zeta_{a_1}^{-k} x$ and taking limits as $x \to \zeta_{a_1}^k$ entails that

$$\begin{aligned}A_{a_1}(k) &= \lim_{x \to \zeta_{a_1}^k} \frac{1 - \zeta_{a_1}^{-k} x}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_m})} \\ &= \lim_{x \to \zeta_{a_1}^k} \frac{-\zeta_{a_1}^{-k}}{-a_1 x^{a_1 - 1}(1 - x^{a_2}) \cdots (1 - x^{a_m})} = \frac{1}{a_1} \frac{1}{(1 - \zeta_{a_1}^{a_2 k}) \cdots (1 - \zeta_{a_1}^{a_m k})} \,.\end{aligned}$$

In a similar manner we can obtain for $l = 1, 2, \dots, m$

$$A_{a_l}(k) = \frac{1}{a_l} \frac{1}{(1-\zeta_{a_l}^{a_1 k}) \cdots (1-\zeta_{a_l}^{a_{l-1} k})(1-\zeta_{a_l}^{a_{l+1} k}) \cdots (1-\zeta_{a_l}^{a_m k})} .$$

Multiplying both sides of (2) by $(1-x)^m$ and letting $x \to 1$ entails that $c_m = 1/(a_1 \cdots a_m)$. To calculate c_l $(l = m-1, m-2, \dots, 1)$, we multiply both sides of (2) by $(1-x)^m$, differentiate $m-l$ times and take limits as $x \to 1$. Namely, we have

$$\begin{aligned}(-1)^{m-l}(m-l)!c_l &= \frac{\partial^{m-l}}{\partial x^{m-l}} \left(\frac{(1-x)^m}{(1-x^{a_1}) \cdots (1-x^{a_m})} \right) \Bigg|_{x=1} \\ &= \frac{\partial^{m-l}}{\partial x^{m-l}} \left(\frac{1}{(1+x+\cdots+x^{a_1-1}) \cdots (1+x+\cdots+x^{a_m-1})} \right) \Bigg|_{x=1} .\end{aligned}$$

Then one can obtain $c_{m-1} = (a_1 + \cdots + a_m - m)/(2a_1 \cdots a_m)$. We should be able to obtain c_{m-2}, c_{m-3}, ... in a similar manner, but it seems that it becomes extremely difficult to calculate them practically. The details are given in the next section.

Notice that for $l = 1, 2, \dots, m$

$$\frac{1}{(1-x)^l} = \sum_{n=0}^{\infty} \binom{n+l-1}{n} x^n .$$

Hence,

$$\begin{aligned}N(a_1, a_2, \dots, a_m; b) &= \sum_{l=1}^{m} \left(c_l \binom{b+l-1}{b} + \sum_{k=1}^{a_l - 1} A_{a_l}(k) \zeta_{a_l}^{-bk} \right) \\ &= \sum_{j=0}^{m-1} d_j b^j + \sum_{l=1}^{m} \sum_{k=1}^{a_l - 1} A_{a_l}(k) \zeta_{a_l}^{-bk} .\end{aligned}$$

This form has been already known (see e.g. [4], [14], [18]).

3. The calculation of d_j

We consider the terms derived from the first type of two decompositions. First of all, notice that

$$\begin{aligned}\binom{b+l-1}{b} = \frac{1}{(l-1)!} \Bigg(& b^{l-1} + b^{l-2} \sum_{j=1}^{l-1} j + b^{l-3} \sum_{1 \le j_1 < j_2 < l} j_1 j_2 + \cdots \\ & + b \sum_{1 \le j_1 < \cdots < j_{l-2} < l} j_1 \cdots j_{l-2} + (l-1)! \Bigg) .\end{aligned}$$

Denote $P = a_1 a_2 \cdots a_m$ and $S_j = a_1^j + a_2^j + \cdots + a_m^j$ $(j = 1, 2, \dots)$. By obtaining

$$c_m = \frac{1}{P}, \quad c_{m-1} = \frac{S_1 - m}{2P},$$

$$c_{m-2} = \frac{3S_1^2 - S_2 - 6(m-1)(S_1 - m) - m(3m-1)}{24P} \quad \text{and}$$
$$c_{m-3} = \frac{1}{48P}\big(S_1^3 - S_1S_2 - (m-2)(3S_1^2 - S_2) \\ +(m-1)(3m-8)(S_1 - m) + 2m(m^2 - 3m + 1)\big)\,,$$

one can find that (*Cf.* [4, p.113])

$$d_{m-1} = \frac{c_m}{(m-1)!} = \frac{1}{(m-1)!P},$$
$$d_{m-2} = \frac{c_m}{(m-1)!}\sum_{j=1}^{m-1} j + \frac{c_{m-1}}{(m-2)!}$$
$$= \frac{c_m}{(m-1)!}\frac{m(m-1)}{2} + \frac{c_{m-1}}{(m-2)!} = \frac{S_1}{2(m-2)!P},$$

$$d_{m-3} = \frac{c_m}{(m-1)!}\sum_{1\le j_1<j_2\le m-1} j_1j_2 + \frac{c_{m-1}}{(m-2)!}\sum_{j=1}^{m-2} j + \frac{c_{m-2}}{(m-3)!}$$
$$= \frac{c_m}{(m-1)!}\frac{(m-2)(m-1)m(3m-1)}{24}$$
$$+\frac{c_{m-1}}{(m-2)!}\frac{(m-1)(m-2)}{2} + \frac{c_{m-2}}{(m-3)!}$$
$$= \frac{3S_1^2 - S_2}{24(m-3)!P},$$

$$d_{m-4} = \frac{c_m}{(m-1)!}\sum_{1\le j_1<j_2<j_3\le m-1} j_1j_2j_3 + \frac{c_{m-1}}{(m-2)!}\sum_{1\le j_1<j_2\le m-2} j_1j_2$$
$$+\frac{c_{m-2}}{(m-3)!}\sum_{j=1}^{m-3} j + \frac{c_{m-3}}{(m-4)!}$$
$$= \frac{c_m}{(m-1)!}\frac{m^2(m-1)^2(m-2)(m-3)}{48}$$
$$+\frac{c_{m-1}}{(m-2)!}\frac{(m-3)(m-2)(m-1)(3m-4)}{24}$$
$$+\frac{c_{m-2}}{(m-3)!}\frac{(m-2)(m-3)}{2} + \frac{c_{m-3}}{(m-4)!}$$
$$= \frac{S_1(S_1^2 - S_2)}{48(m-4)!P}\,.$$

In a similar manner, one can find

$$d_{m-5} = \frac{2S_4 + 5S_2^2 - 30S_1^2S_2 + 15S_1^4}{240\cdot 4!(m-5)!P},$$

$$
\begin{aligned}
d_{m-6} &= \frac{S_1(2S_4 + 5S_2^2 - 10S_1^2 S_2 + 3S_1^4)}{96 \cdot 5!(m-6)!P},\\
d_{m-7} &= \frac{-16S_6 - 42S_2S_4 + 126S_1^2S_4 - 35S_2^3 + 315S_1^2S_2^2 - 315S_1^4S_2 + 63S_1^6}{4032 \cdot 6!(m-7)!P},\\
d_{m-8} &= \frac{S_1(-16S_6 - 42S_2S_4 + 42S_1^2S_4 - 35S_2^3 + 105S_1^2S_2^2 - 63S_1^4S_2 + 9S_1^6)}{1152 \cdot 7!(m-8)!P},\\
&\dots .
\end{aligned}
$$

After obtaining c_m, c_{m-1}, $\dots$, c_{m-l+1}, one can find d_{m-l} as

$$
\begin{aligned}
d_{m-l} = \frac{c_m}{(m-1)!} &\sum_{1 \le j_1 < \cdots < j_{l-1} \le m-1} j_1 \cdots j_{l-1}\\
&+ \frac{c_{m-1}}{(m-2)!} \sum_{1 \le j_1 < \cdots < j_{l-2} \le m-2} j_1 \cdots j_{l-2} + \cdots\\
&+ \frac{c_{m-l+2}}{(m-l+1)!} \sum_{j=1}^{m-l+1} j + \frac{c_{m-l+1}}{(m-l)!}.
\end{aligned}
$$

Finally, $d_0 = c_m + c_{m-1} + \cdots + c_1$.

But it was very hard to find an explicit form of the general d_j. One nice-looking form can be derived from the main result in [1]. Define Bell polynomials $\mathbf{Y}_n(y_1, y_2, \dots, y_n)$ by

$$
\exp\left(\sum_{k=1}^{\infty} y_k \frac{x^k}{k!}\right) = \sum_{n=0}^{\infty} \mathbf{Y}_n(y_1, y_2, \dots, y_n)\frac{x^n}{n!}
$$

where $\mathbf{Y}_0 = 1$ and

$$
\mathbf{Y}_n(y_1, y_2, \dots, y_n) = \sum_{\substack{k_1 + 2k_2 + \cdots + nk_n = n\\ k_1, k_2, \dots, k_n \ge 0}} \prod_{i=1}^{n} \frac{n! y_i^{k_i}}{k_i!(i!)^{k_i}}.
$$

We have the following identity.

Proposition 1. *For $l = 0, 1, 2, \dots$ we have*

$$
d_{m-l-1} = \frac{(-1)^l}{(m-l-1)!l!P}\mathbf{Y}_l(B_1S_1, -\frac{B_2S_2}{2}, \dots, -\frac{B_lS_l}{l}),
$$

where $P = \prod_{j=1}^m a_j$, $S_n = \sum_{j=1}^m a_j^n$ and B_n is the n-th Bernoulli number ($n = 1, 2, \dots$).

Proposition 2. *For $l = 1, 2, \dots$ we have*

$$
d_{m-l} = \frac{2}{m-l}\frac{\partial}{\partial S_1} d_{m-l-1}.
$$

4. The calculation of $\sum A_{a_l}(k)\zeta_{a_l}^{-bk}$

We consider the terms derived from the Herschellian type of two decompositions. We assume that $\gcd(a_h, a_l) = 1$ $(h \neq l)$. Put $A_{a_l} = \sum_{k=1}^{a_l-1} A_{a_l}(k)\zeta_{a_l}^{-bk}$ $(l = 1, 2, \dots, m)$ for convenience. Without loss of generality, set $a = a_1$. When $a = 1$, this term does not exist. When $a = 2$, by the assumption all of a_2, a_3, …, a_m are odd. From $\zeta_2 = -1$ we have

$$A_2 = \sum_{k=1}^{1} A_2(k)\zeta_2^{-bk} = \frac{1}{2}\frac{\zeta_2^{-b}}{(1-\zeta_2^{a_2})(1-\zeta_2^{a_3})\cdots(1-\zeta_2^{a_m})} = \frac{(-1)^b}{2^m}\,.$$

Let a_1 be odd with $a_1 \geq 3$. Denote s_l $(l = 1, 2, \dots, a-1)$ by

$$s_l := \#\{a_j | 2 \leq j \leq m,\ a_j \equiv l \pmod{a}\}\,,$$

satisfying $\sum_{l=1}^{a-1} s_l = m - 1$. By the assumption, $a_j \not\equiv 0 \pmod{a}$ for any j with $2 \leq j \leq m$. Put $\zeta = \zeta_a$ for simplicity.

With these notations we can write

$$A_a = \frac{1}{a}\sum_{k=1}^{a-1}\frac{\zeta^{-bk}}{(1-\zeta^k)^{s_1}(1-\zeta^{2k})^{s_2}\cdots(1-\zeta^{(a-1)k})^{s_{a-1}}}\,.$$

Lemma 1. *For any integer k we have*

$$1 - \zeta_a^k = 2\sin\frac{k}{a}\pi \cdot e^{-\frac{a-2k}{2a}i\pi}\,.$$

Proof. Put $1 - \zeta_a^k = re^{i\theta}$. Then

$$r = \sqrt{\left(1 - \cos\frac{2k}{a}\pi\right)^2 + \left(\sin\frac{2k}{a}\pi\right)^2} = 2\sin\frac{k}{a}\pi\,.$$

By

$$\sin(-\theta) = \frac{1}{r}\sin\frac{2k}{a}\pi = \cos\frac{k}{a}\pi\,,$$

we have

$$-\theta = \frac{\pi}{2} - \frac{k}{a}\pi = \frac{a-2k}{2a}\pi\,.$$

□

By this lemma together with the facts

$$\sin\frac{l(a-k)}{a}\pi = (-1)^l \sin\frac{lk}{a}\pi$$

and

$$e^{(a-2l(a-k))(s_l - s_{a-l})i\pi/(2a)} = (-1)^{l-1}e^{-(a-2lk)(s_l-s_{a-l})i\pi/(2a)}\,,$$

we obtain

$$(1-\zeta^{lk})^{s_l}(1-\zeta^{(a-l)k})^{s_{a-l}} = \left(2\sin\frac{lk}{a}\pi\right)^{s_a+s_{a-l}} e^{-(a-2lk)(s_l-s_{a-l})i\pi/(2a)}\,.$$

From $\zeta^{-b(a-k)}=\zeta^{bk}$, if a is odd, then

$$A_a=\frac{1}{a}\sum_{k=1}^{a-1}\frac{\zeta^{-bk}\prod_{l=1}^{(a-1)/2}e^{(a-2lk)(s_l-s_{a-l})i\pi/(2a)}}{\prod_{l=1}^{(a-1)/2}\left(2\sin\frac{lk}{a}\pi\right)^{s_l+s_{a-l}}}$$
$$=\frac{2}{a}\sum_{k=1}^{(a-1)/2}\frac{\cos\left(\frac{\sum_{l=1}^{(a-1)/2}(a-2lk)(s_l-s_{a-l})-4bk}{2a}\pi\right)}{\prod_{l=1}^{(a-1)/2}\left(2\sin\frac{lk}{a}\pi\right)^{s_l+s_{a-l}}}\,.$$

We can interchange a_1 and any a_h $(2\le h\le m)$ without loss of generality. Therefore, we obtain the following.

Theorem 1. *If $a=a_h$ is odd with $a\ge 3$ and $\gcd(a_j,a)=1$ $(1\le j\le m$, $j\ne h)$, then*

$$A_a=\frac{2}{a}\sum_{k=1}^{(a-1)/2}\frac{\cos\left(\frac{4bk+\sum_{l=1}^{(a-1)/2}(2lk-a)(s_l-s_{a-l})}{2a}\pi\right)}{\displaystyle\prod_{l=1}^{(a-1)/2}\left(2\sin\frac{lk}{a}\pi\right)^{s_l+s_{a-l}}}\,.$$

This form seems still very complicated, but we can calculate A_a very easily when a is small even if the number m is very big.

Corollary 1. *When $a=3$, we have*

$$A_3=\sum_{k=1}^{2}A_k^{(1)}\zeta_3^{-bk}=\frac{1}{3}\sum_{k=1}^{2}\frac{\zeta_3^{-bk}}{(1-\zeta_3^{a_2k})(1-\zeta_3^{a_3k})\cdots(1-\zeta_3^{a_mk})}$$
$$=\frac{2}{3^{(m+1)/2}}\cos\left(\frac{2}{3}b-\frac{s_1-s_2}{6}\right)\pi\,.$$

Proof. When $a=3$, we have $l=k=1$, and $2\sin(lk/a)\pi=\sqrt{3}$. □

Corollary 2. *When $a=5$, we have*

$$A_5=\frac{2}{5}\sum_{k=1}^{2}\frac{\cos\frac{4bk+(2k-5)(s_1-s_4)+(4k-5)(s_2-s_3)}{10}\pi}{\left(2\sin\frac{k}{5}\pi\right)^{s_1+s_4}\left(2\sin\frac{2k}{5}\pi\right)^{s_2+s_3}}\,.$$

Remark 1. *Notice that*

$$\left(2\sin\frac{\pi}{5}\right)\left(2\sin\frac{2\pi}{5}\right)=\sqrt{\frac{5-\sqrt{5}}{2}}\sqrt{\frac{5+\sqrt{5}}{2}}=\sqrt{5}$$

for further calculations.

Corollary 3. *When $a=7$, we have*

$$A_7=\frac{2}{7}\sum_{k=1}^{3}\frac{\cos\frac{4bk+(2k-7)(s_1-s_6)+(4k-7)(s_2-s_5)+(6k-7)(s_3-s_4)}{14}\pi}{\left(2\sin\frac{k}{7}\pi\right)^{s_1+s_6}\left(2\sin\frac{2k}{7}\pi\right)^{s_2+s_5}\left(2\sin\frac{3k}{7}\pi\right)^{s_3+s_4}}\,.$$

Remark 2. *It is convenient to use relations*

$$2\sin\frac{\pi}{7}\cdot 2\sin\frac{2\pi}{7}\cdot 2\sin\frac{3\pi}{7}=\sqrt{7}\quad\textit{and}\quad \sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}=\frac{\sqrt{7}}{2}\,.$$

for further calculations.

Let a be even with $a\ge 4$. By the assumption, $s_l=0$ if l is even or $l=a/2$. In a similar manner we obtain

$$\begin{aligned}
A_a &= \frac{1}{a}\sum_{k=1}^{a-1}\frac{\zeta^{-bk}}{(1-\zeta^k)^{s_1}(1-\zeta^{3k})^{s_3}\cdots(1-\zeta^{(a-1)k})^{s_{a-1}}}\\
&= \frac{1}{a}\sum_{k=1}^{a-1}\frac{\zeta^{-bk}\prod_{l=1}^{2\lfloor a/4\rfloor-1}e^{(a-2(2l-1)k)(s_{2l-1}-s_{a-2l+1})i\pi/(2a)}}{\prod_{l=1}^{2\lfloor a/4\rfloor-1}\left(2\sin\frac{(2l-1)k}{a}\pi\right)^{s_{2l-1}+s_{a-2l+1}}}\\
&= \frac{2}{a}\sum_{k=1}^{a/2-1}\frac{\cos\left(\frac{\sum_{l=1}^{2\lfloor a/4\rfloor-1}(a-2(2l-1)k)(s_{2l-1}-s_{a-2l+1})-4bk}{2a}\pi\right)}{\prod_{l=1}^{2\lfloor a/4\rfloor-1}\left(2\sin\frac{(2l-1)k}{a}\pi\right)^{s_{2l-1}+s_{a-2l+1}}}+\frac{(-1)^b}{a\cdot 2^{m-1}}\,.
\end{aligned}$$

Notice that the last term arises for $k=a/2$.

Theorem 2. *If $a=a_h$ is even with $a\ge 4$ and $\gcd(a_j,a)=1$ $(1\le j\le m$, $j\ne h)$, then*

$$\begin{aligned}
A_a = \frac{2}{a}\sum_{k=1}^{\frac{a}{2}-1}&\frac{\cos\left(\frac{4bk+\sum_{l=1}^{2\lfloor a/4\rfloor-1}(2(2l-1)k-a)(s_{2l-1}-s_{a-2l+1})}{2a}\pi\right)}{\displaystyle\prod_{l=1}^{2\lfloor a/4\rfloor-1}\left(2\sin\frac{(2l-1)k}{a}\pi\right)^{s_{2l-1}+s_{a-2l+1}}}\\
&+\frac{(-1)^b}{a\cdot 2^{m-1}}\,.
\end{aligned}$$

5. Examples

Suppose that $m=3$. Then

$$\begin{aligned}
N(a_1,a_2,a_3;b) &= \frac{a_1^2+a_2^2+a_3^2+3(a_1a_2+a_2a_3+a_3a_1)}{12a_1a_2a_3}+\frac{a_1+a_2+a_3}{2a_1a_2a_3}b\\
&\quad+\frac{1}{2a_1a_2a_3}b^2+\frac{1}{a_1}\sum_{k=1}^{a_1-1}\frac{\zeta_{a_1}^{-bk}}{(1-\zeta_{a_1}^{a_2k})(1-\zeta_{a_1}^{a_3k})}\\
&\quad+\frac{1}{a_2}\sum_{k=1}^{a_2-1}\frac{\zeta_{a_2}^{-bk}}{(1-\zeta_{a_2}^{a_3k})(1-\zeta_{a_2}^{a_1k})}+\frac{1}{a_3}\sum_{k=1}^{a_3-1}\frac{\zeta_{a_3}^{-bk}}{(1-\zeta_{a_3}^{a_1k})(1-\zeta_{a_3}^{a_2k})}\,.
\end{aligned}$$

Let $a_1=3$, $a_2=5$ and $a_3=7$. For $a_1=3$, by *Corollary 1* with $s_1=s_2=1$ we have

$$A_3=\frac{2}{9}\cos\frac{2}{3}b\pi\,.$$

For $a_2=5$, by *Corollary 2* with $s_1=s_4=0$ and $s_2=s_3=1$ we have

$$A_5=\frac{2}{5}\sum_{k=1}^{2}\frac{\cos\frac{4bk}{10}\pi}{(2\sin\frac{2k}{5}\pi)^2}=\frac{2}{25}\left((2\sin\frac{\pi}{5})^2\cos\frac{2b}{5}\pi+(2\sin\frac{2\pi}{5})^2\cos\frac{4b}{5}\pi\right)\,.$$

For $a_3=7$, by *Corollary 3* with $s_1=s_2=s_4=s_6=0$ and $s_3=s_5=1$ we have

$$\begin{aligned}A_7&=\frac{2}{7}\sum_{k=1}^{3}\frac{\cos\frac{2b+1}{7}\pi}{(2\sin\frac{2k}{7}\pi)(2\sin\frac{3k}{7}\pi)}\\&=\frac{2}{7\sqrt{7}}\left(2\sin\frac{\pi}{7}\cos\frac{2b+1}{7}\pi+2\sin\frac{2\pi}{7}\cos\frac{2(2b+1)}{7}\pi\right.\\&\qquad\left.-2\sin\frac{3\pi}{7}\cos\frac{3(2b+1)}{7}\pi\right)\,.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}N(3,5,7;b)=&\frac{1}{210}b^2+\frac{1}{14}b+\frac{74}{315}+\frac{2}{9}\cos\frac{2}{3}b\pi\\&+\frac{2}{25}\left((2\sin\frac{\pi}{5})^2\cos\frac{2b}{5}\pi+(2\sin\frac{2\pi}{5})^2\cos\frac{4b}{5}\pi\right)\\&+\frac{2}{7\sqrt{7}}\left(2\sin\frac{\pi}{7}\cos\frac{2b+1}{7}\pi+2\sin\frac{2\pi}{7}\cos\frac{2(2b+1)}{7}\pi\right.\\&\qquad\left.-2\sin\frac{3\pi}{7}\cos\frac{3(2b+1)}{7}\pi\right)\,.\end{aligned}$$

With the notation due to Cayley (*Cf.* [9]), $(x_0,x_1,\dots,x_{k-1})\mathrm{pcr}k_b=x_i$ if $b\equiv i \pmod k$, this result matches the Comtet's one [4, pp.114–115],

$$\begin{aligned}N(3,5,7;b)=&\frac{1}{210}b^2+\frac{1}{14}b+\frac{74}{315}+\frac{1}{9}(2,-1,-1)\mathrm{pcr}3_b\\&+\frac{1}{5}(2,-1,0,0,-1)\mathrm{pcr}5_b+\frac{1}{7}(1,0,-2,2,-2,0,1)\mathrm{pcr}7_b\,.\end{aligned}$$

It is quite easy to find

$$N(1,2,3;b)=\frac{1}{12}b^2+\frac{1}{2}b+\frac{47}{72}+\frac{(-1)^b}{8}+\frac{2}{9}\cos\frac{2}{3}b\pi$$

for $a_1=1$, $a_2=2$ and $a_3=3$ (*Cf.* [4, p.110]).

If each a_j is small, it is not difficult to obtain the exact form of $N(a_1,\dots,a_m;b)$, even though the number m becomes large. For example, let $a_1=2$, $a_2=3$, $a_3=5$,

$a_4 = 7$, $a_5 = 11$, $a_6 = 13$, $a_7 = 17$ and $a_8 = 19$. Then one can get

$$
\begin{aligned}
N(2,3,5,7,11,13,17,19;b) &= \frac{1}{48886437600}b^7 + \frac{1}{181396800}b^6 + \frac{419}{698377680}b^5 \\
&\quad + \frac{43}{1272960}b^4 + \frac{21901069}{20951330400}b^3 + \frac{174869}{10077600}b^2 + \frac{134507}{978120}b + \frac{810672961}{2176761600} \\
&\quad + \frac{(-1)^b}{256} + \frac{2}{81\sqrt{3}}\cos\left(\frac{2}{3}b + \frac{1}{6}\right)\pi \\
&\quad + \frac{2}{125\sqrt{5}}\left(\bigl(2\sin\frac{\pi}{5}\bigr)^3\cos\frac{4b-1}{10}\pi + \bigl(2\sin\frac{2\pi}{5}\bigr)^3\cos\frac{8b+3}{10}\pi\right) \\
&\quad + \frac{2}{49\sqrt{7}}\left(\bigl(2\sin\frac{\pi}{7}\bigr)^2\cos\frac{4b+7}{14}\pi + \bigl(2\sin\frac{2\pi}{7}\bigr)^2\cos\frac{8b+7}{14}\pi\right. \\
&\qquad\qquad \left. - \bigl(2\sin\frac{3\pi}{7}\bigr)^2\cos\frac{12b+7}{14}\pi\right) \\
&\quad + \frac{2}{121}\sum_{k=1}^{5}\bigl(2\sin\frac{k\pi}{11}\bigr)^2\bigl(2\sin\frac{4k\pi}{11}\bigr)\cos\frac{4bk-11}{22}\pi \\
&\quad + \frac{2}{169}\sum_{k=1}^{6}\bigl(2\sin\frac{k\pi}{13}\bigr)^2\bigl(2\sin\frac{3k\pi}{13}\bigr)\bigl(2\sin\frac{4k\pi}{13}\bigr)\bigl(2\sin\frac{5k\pi}{13}\bigr)\cos\frac{(4b+24)k-39}{26}\pi \\
&\quad + \frac{2}{289}\sum_{k=1}^{8}\bigl(2\sin\frac{k\pi}{17}\bigr)^2\bigl(2\sin\frac{3k\pi}{17}\bigr)\bigl(2\sin\frac{4k\pi}{17}\bigr)\bigl(2\sin\frac{5k\pi}{17}\bigr)\bigl(2\sin\frac{6k\pi}{17}\bigr) \\
&\qquad\qquad \cdot\bigl(2\sin\frac{7k\pi}{17}\bigr)\bigl(2\sin\frac{8k\pi}{17}\bigr)^2\cos\frac{(4b+18)k-51}{34}\pi \\
&\quad + \frac{2}{361}\sum_{k=1}^{9}\bigl(2\sin\frac{k\pi}{19}\bigr)^2\bigl(2\sin\frac{3k\pi}{19}\bigr)\bigl(2\sin\frac{4k\pi}{19}\bigr)^2\bigl(2\sin\frac{5k\pi}{19}\bigr)\bigl(2\sin\frac{6k\pi}{19}\bigr) \\
&\qquad\qquad \cdot\bigl(2\sin\frac{7k\pi}{19}\bigr)\bigl(2\sin\frac{8k\pi}{19}\bigr)\bigl(2\sin\frac{9k\pi}{19}\bigr)^2\cos\frac{(4b+2)k-19}{38}\pi \\
&= \frac{1}{48886437600}b^7 + \frac{1}{181396800}b^6 + \frac{419}{698377680}b^5 + \frac{43}{1272960}b^4 \\
&\quad + \frac{21901069}{20951330400}b^3 + \frac{174869}{10077600}b^2 + \frac{134507}{978120}b + \frac{810672961}{2176761600} \\
&\quad + \frac{(-1)^b}{256} + \frac{1}{81}(1,-1,0)\mathrm{pcr}3_b + \frac{1}{25}(1,-1,1,-1,0)\mathrm{pcr}5_b \\
&\quad + \frac{1}{49}(0,-1,-2,4,-4,2,1)\mathrm{pcr}7_b + \frac{1}{11}(0,-1,2,-1,0,0,0,0,1,-2,1)\mathrm{pcr}11_b \\
&\quad + \frac{1}{13}(3,-3,1,-1,1,0,1,0,-1,0,-1,1,-1)\mathrm{pcr}13_b \\
&\quad + \frac{1}{17}(4,-4,2,-2,0,2,-2,4,-4,3,-1,3,-3,3,-3,1,-3)\mathrm{pcr}17_b \\
&\quad + \frac{1}{19}(2,2,-2,5,-5,3,-1,2,0,0,0,-2,1,-3,5,-5,2,-2,-2)\mathrm{pcr}19_b\,.
\end{aligned}
$$

We omit the detail calculations above. For example, use the relation

$$\prod_{k=1}^{(a-1)/2}\left(2\sin\frac{k\pi}{a}\right)=\sqrt{a}\,.$$

Let $a_1=137$, $a_2=251$ and $a_3=256$, which triple is an example much used in the literature (see e.g. [13]). By *Theorems 1* and *2* one gets

$$\begin{aligned}N(137,251,256;b)&=\frac{1}{17606144}b^2+\frac{161}{4401536}b+\frac{182817}{35212288}\\&+\frac{2}{137}\sum_{k=1}^{68}\frac{\cos\left(\frac{2b-41}{137}k-1\right)\pi}{\left(2\sin\frac{18k}{137}\pi\right)\left(2\sin\frac{23k}{137}\pi\right)}+\frac{2}{251}\sum_{k=1}^{125}\frac{\cos\frac{2b-109}{251}k\pi}{\left(2\sin\frac{5k}{251}\pi\right)\left(2\sin\frac{114k}{251}\pi\right)}\\&+\frac{1}{128}\sum_{k=1}^{127}\frac{\cos\left(\frac{b-62}{128}k-1\right)\pi}{\left(2\sin\frac{5k}{256}\pi\right)\left(2\sin\frac{119k}{256}\pi\right)}+\frac{(-1)^b}{1024}\,.\end{aligned}$$

It seems nearly impossible to continue this calculation by hand only. For example, Mathematica or Maple calculations can show immediately

$$N(137,251,256;4948)=0,\quad N(137,251,256;4949)=2$$

and so on. In fact, $G(137,251,256)=4948$.

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