DGS–trapezoids in GS–quasigroups

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Abstract. The concept of the DGS–trapezoid is defined and investigated in any GS–quasigroup and geometrical interpretation in the GS–quasigroup $C(\frac{1}{2}(1 + \sqrt{5}))$ is also given. The connection of this concept with GS–trapezoids in the general GS–quasigroup is obtained.

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GS–quasigroups are defined in [1]; in [2] different properties of GS–trapezoids in the GS–quasigroup are explored. In this paper some “geometric” concepts in the general GS–quasigroup will be defined.

A quasigroup $(Q, \cdot)$ is said to be a GS–quasigroup if it is idempotent and if it satisfies the (mutually equivalent) identities

\[ a(ab \cdot c) \cdot c = b, \quad a \cdot (a \cdot bc)c = b. \quad (1) \]

In a GS–quasigroup we also have the mediality and elasticity

\[ ab \cdot cd = ac \cdot bd, \quad (2) \]

\[ a \cdot ba = ab \cdot a, \quad (3) \]

as well as identities

\[ a(ab \cdot c) = b \cdot bc, \quad (4) \]

\[ (c \cdot ba)a = cb \cdot b, \quad (4)’ \]

and equivalencies

\[ ab = c \iff a = c \cdot cb, \quad (5) \]

\[ ab = c \iff b = ac \cdot c. \quad (5)’ \]

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If $C$ is the set of all points in Euclidean plane and if groupoid $(C, \cdot)$ is defined so that $aa = a$ for any $a \in C$ and for any two different points $a, b \in C$ we define $ab = c$ if the point $b$ divides the pair $a, c$ in the ratio of golden section. In [1] it is proved that $(C, \cdot)$ is a GS–quasigroup. We shall denote that quasigroup by $C(\frac{1}{2}(1 + \sqrt{5}))$ because we have $c = \frac{1}{2}(1 + \sqrt{5})$ if $a = 0$ and $b = 1$. Figures in this quasigroup $C(\frac{1}{2}(1 + \sqrt{5}))$ can be used for illustration of “geometrical” relations in any GS–quasigroup.

From now on let $(Q, \cdot)$ be any GS–quasigroup. Elements of the set $Q$ are said to be points.

Points $a, b, c, d$ successively are said to be the vertices of the golden section trapezoid which is denoted by $GST(a, b, c, d)$ if the identity $a \cdot ab = d \cdot dc$ holds (Figure 1). Because of (5), this identity is equivalent to the identity $d = (a \cdot ab)c$.

![Figure 1](image)

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Points $a, b, c, d$ are said to be the vertices of a trapezoid of double golden section or shorter a DGS–trapezoid and we write $DGST(a, b, c, d)$ if the equality $ab = dc$ holds (Figure 2). Namely, because of (5), the equality $d = ab \cdot (ab \cdot c)$.

Obviously the following theorems hold.

**Theorem 1.** From $DGST(a, b, c, d)$ there follows $DGST(d, c, b, a)$.

**Theorem 2.** A DGS–trapezoid is uniquely determined with any three of its vertices.

Based on Theorem 16. from [2] it follows immediately:

**Theorem 3.** Any two of the three statements $GST(a, e, f, d)$, $GST(e, b, c, f)$, $DGST(a, b, c, d)$ imply the remaining statement (Figure 2).
Corollary 1. The statement \( DGST(a, b, c, d) \) is valid if and only if there are points \( e, f \) such that the statements \( GST(a, e, f, d) \), \( GST(e, b, c, f) \) are valid (Figure 2).

This corollary justifies the name of the trapezoid of double golden section.

Theorem 4. Any two of the three statements \( DGST(a, b, c, d) \), \( DGST(e, f, g, h) \), \( DGST(ae, bf, cg, dh) \) imply the remaining statement (Figure 3).

Proof. We must prove that any two of the three equalities \( ab = dc \), \( ef = hg \) and \( ae \cdot bf = dh \cdot cg \) imply the remaining equality. This is obvious, because of (2) the third equality is equivalent to \( ab \cdot ef = dc \cdot hg \). \( \square \)
For any point \( p \) we have obviously \( \text{DGST}(p, p, p, p) \) and from Theorem 4 it follows further:

**Corollary 2.** For any point \( p \) the statements \( \text{DGST}(a, b, c, d) \), \( \text{DGST}(pa, pb, pc, pd) \) and \( \text{DGST}(ap, bp, cp, dp) \) are mutually equivalent.

![Figure 4](image)

**Theorem 5.** Any two of the three statements \( \text{DGST}(a, b, c, d) \), \( \text{DGST}(b, c, d, e) \), \( \text{GST}(a, b, d, e) \) imply the remaining statement (Figure 4).

**Proof.** Because of symmetry \( a \leftrightarrow e \), \( b \leftrightarrow d \), it is sufficient under assumption \( \text{DGST}(a, b, c, d) \) i.e. \( d = ab \cdot (ab \cdot c) \) to prove the equivalency of the statements \( \text{DGST}(b, c, d, e) \) and \( \text{GST}(a, b, d, e) \) i.e. \( e = bc \cdot (bc \cdot d) \) and \( e = (a \cdot ab)d \).
However, we have successively

\[
bc \cdot (bc \cdot d) = bc \cdot (bc)[ab \cdot (ab \cdot c)] \overset{(2)}{=} bc \cdot (bc)[(a \cdot ab) \cdot bc] \\
\overset{(3)}{=} bc \cdot [bc \cdot (a \cdot ab)][bc] \overset{(4)}{=} (a \cdot ab) \cdot (a \cdot ab)(bc) \\
\overset{(2)}{=} (a \cdot ab) \cdot (ab)(ab \cdot c) = (a \cdot ab)d.
\]

\( \square \)

**References**
