On the Ishikawa iterative approximation with mixed errors for solutions to variational inclusions with accretive type mappings in Banach spaces

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Abstract. Using the new analysis techniques, the existence and iterative approximation problem of a solution for a class of nonlinear variational inclusions with accretive type mappings are discussed in arbitrary Banach spaces. The results extend and improve some recent results.

Key words: variational inclusion; accretive mapping; Ishikawa iterative sequence with mixed errors; Mann iterative sequence with mixed errors.

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1. Introduction

Throughout this paper we suppose that \( X \) is a real Banach space, \( X^* \) is its dual space, \( \langle \cdot, \cdot \rangle \) is the pairing of \( X \) and \( X^* \). \( D(T) \) and \( R(T) \) denote the domain and the range of \( T \), respectively.

Let \( T, A : X \to X, g : X \to X^* \) be three mappings and \( \varphi : X^* \to R \cup \{+\infty\} \) be a proper convex lower semicontinuous function.

In 1999, Chang [1] introduced and studied the existence and approximation problem of solutions for a class of nonlinear variational inclusions with accretive mappings in uniformly smooth Banach space as follows:

For any given \( f \in X \), to find an \( u \in X \) such that

\[
\begin{aligned}
g(u) & \in D(\partial \varphi), \\
\langle Tu - Au - f, v - g(u) \rangle & \geq \varphi(g(u)) - \varphi(v), \quad \forall v \in X^*,
\end{aligned}
\]

(1)

where \( \partial \varphi \) denotes the subdifferential of \( \varphi \).

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The purpose of this paper is to study further the existence and uniqueness of solutions and the convergence problem of Ishikawa and Mann iterative processes with mixed errors for a class of accretive type variational inclusion in arbitrary Banach spaces. The results presented in this paper not only extend and improve the main results in Chang [1], but also extend and improve the corresponding results in Chang [2,3], Chang, Cho, Lee et al [4], Ding [5,6], Hassouni and Moudafi [7], Huang [8-10], Kazmi [12], Noor [15,16], Siddiqi and Ansari [17], Siddiqi, Ansari and Kazmi [18] and Zeng [19].

2. Preliminaries

A mapping $J : X \to 2^{X^*}$ is said to be a normalized duality mapping, if it is defined by

$$J(x) = \{ f \in X^* : \langle x, f \rangle = \|f\|^2 = \|x\|^2 \}, \quad \forall x \in X.$$ 

Definition 1. A mapping $T : D(T) \subset X \to X$ is said to be accretive, if for any $x,y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0.$$ 

If $T$ is accretive and $R(I + rT) = X$ for all $r > 0$, then $T$ is called $m$-accretive.

In the sequel we shall use the following Proposition and Lemmas.

Proposition 1 [14]. Let $X$ be a real Banach space, $T : D(T) \subset X \to X$ is accretive and continuous, and $D(T) = X$. Then $T$ is $m$-accretive.

Lemma 1 [13]. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the inequality:

$$a_{n+1} \leq (1-t_n)a_n + b_nt_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\} \subset [0,1]$, $\sum_{n=0}^\infty t_n = \infty$, $\lim_{n \to \infty} b_n = 0$ and $\sum_{n=0}^\infty c_n < \infty$. Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2 [20]. Let $X$ be a real Banach space, $T : D(T) \subset X \to X$ an $m$-accretive mapping. Then the equation $x + Tx = f$ has a unique solution in $D(T)$ for any $f \in X$.

Lemma 3 [2]. Let $X$ be an arbitrary real Banach space, $\partial \varphi \circ g : X \to 2^X$ a mapping, then the following conclusions are equivalent to each other:

(i) $x^* \in X$ is a solution of variational inclusion problem (1);

(ii) $x^* \in X$ is a fixed point of the mapping $S : X \to 2^X$:

$$S(x) = f - (Tx - Ax + \partial \varphi(g(x))) + x;$$

(iii) $x^* \in X$ is a solution of the equation $f \in Tx - Ax + \partial \varphi(g(x))$. 
3. Main results

**Theorem 1.** Let $X$ be an arbitrary real Banach space, $T, A : X \to X$, $g : X \to X^*$ three mappings, and $\varphi : X^* \to R \cup \{+\infty\}$ a function with a continuous Gâteaux differential $\partial \varphi$. For any given $f \in X$, define a mapping $S : X \to X$ by

$$Sx = f - (Tx - Ax + \partial \varphi(g(x))) + x.$$ 

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Ishikawa iterative sequence with mixed errors defined by

$$\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nSy_n + u_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nSx_n + v_n, \quad \forall n \geq 0, \\
\end{align*}$$

(2)

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0, 1]$, and $\{u_n\}$, $\{v_n\}$ are two sequences in $X$ such that $u_n = u_n^* + u_n''$ for any sequences $\{u_n^*\}$ and $\{u_n''\}$ in $X$ satisfying the following conditions

(i) $T - A + \partial \varphi \circ g - I : X \to X$ is accretive,

(ii) $T - A + \partial \varphi \circ g : X \to X$ is a Lipschitz operator with constant $L$,

(iii) $K_n = (1 + L_1)(1 + L_2^2)\alpha_n + L_1 (1 + L_2)\beta_n \leq 1 - r$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \|u_n^*\| < \infty$, $\|u_n''\| = \gamma_n \alpha_n$ and $\|v_n\| \to 0$ $(n \to \infty)$,

where $L_1 = 1 + L$, $r \in (0, 1)$ is a constant and $\gamma_n \to 0$ $(n \to \infty)$. Then the following conclusions hold:

(1) The nonlinear variational inclusion problem (1) has a unique solution $x^* \in X$.

(2) The Ishikawa iterative sequence $\{x_n\}$ with mixed errors converges strongly to the unique solution $x^* \in X$ of the variational inclusion problem (1).

**Proof.** (1) First we prove that the variational inclusion problem (1) has a unique solution $x^* \in X$.

From conditions (i) and (ii), the mapping $T - A + \partial \varphi \circ g - I : X \to X$ is continuous and accretive. By Proposition 1 we know that $T - A + \partial \varphi \circ g - I$ is $m$-accretive. Therefore, by Lemma 2, for any given $f \in X$, the equation

$$f = x + (T - A + \partial \varphi \circ g - I)(x)$$

has a unique solution $x^* \in X$. Hence, by Lemma 3, we know that $x^*$ is a unique solution of the variational inclusion problem (1), and it is also a fixed point of $S$, i.e., $Sx^* = x^*$.

(2) Next we prove that the Ishikawa iterative sequence $\{x_n\}$ with mixed errors converges strongly to $x^*$.

By condition (i), for any $x, y \in X$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle = - \langle (T - A + \partial \varphi \circ g - I)x - (T - A + \partial \varphi \circ g - I)y, j(x - y) \rangle \leq 0.$$
It follows from Lemma 1.1 of Kato [11] that
\[ \|x - y\| \leq \|x - y - t(Sx - Sy)\| \] (3)
for all \( x, y \in X \) and \( t > 0 \). Using (2), we easily conclude that for all \( n \geq 0 \),
\[
x_n = x_{n+1} + \alpha_n x_n - \alpha_n S y_n - u_n
\]
\[
= (1 + \alpha_n)x_{n+1} - \alpha_n S x_{n+1} + \alpha_n^2 (x_n - S y_n)
+ \alpha_n(S x_{n+1} - S y_n) - (1 + \alpha_n)u_n.
\] (4)
Note that
\[
x^* = (1 + \alpha_n)x^* - \alpha_n S x^*
\]
for all \( n \geq 0 \). It follows from (3), (4) and (5) that
\[
\|x_n - x^*\| \geq (1 + \alpha_n)\left\|x_{n+1} - x^* - \frac{\alpha_n}{1 + \alpha_n} (S x_{n+1} - S x^*)\right\|
\]
\[
- \alpha_n^2 \|x_n - S y_n\| - \alpha_n \|S x_{n+1} - S y_n\| - (1 + \alpha_n)\|u_n\|
\]
\[
\geq (1 + \alpha_n)\|x_{n+1} - x^*\| - \alpha_n^2 \|x_n - S y_n\|
- \alpha_n \|S x_{n+1} - S y_n\| - (1 + \alpha_n)\|u_n\|
\]
which implies that
\[
\|x_{n+1} - x^*\| \leq \frac{1}{1 + \alpha_n} \|x_n - x^*\| + \alpha_n^2 \|x_n - S y_n\| + \alpha_n \|S x_{n+1} - S y_n\| + \|u_n\|. \] (6)

Since \( T - A + \partial \varphi \circ g \) is a Lipschitz mapping with the constant \( L \), it is easy to verify that \( S \) is also Lipschitz with the constant \( L_* = 1 + L \). Furthermore, we have the following estimates:
\[
\|x_n - S y_n\| \leq \|x_n - x^*\| + \|S y_n - S x^*\|
\]
\[
\leq \|x_n - x^*\| + L_* \|y_n - x^*\|
\]
\[
\leq \|x_n - x^*\| + L_* (1 - \beta_n) \|x_n - x^*\| + \beta_n L_* \|x_n - x^*\| + \|v_n\|
\]
\[
\leq (1 + L_*^2) \|x_n - x^*\| + L_* \|v_n\| \] (7)
and
\[
\|S x_{n+1} - S y_n\| \leq L_* \|x_{n+1} - y_n\| = L_* \|\alpha_n (S y_n - x_n) + \beta_n (x_n - S x_n) + u_n - v_n\|
\]
\[
\leq L_* \alpha_n \|x_n - S y_n\| + L_* \beta_n (\|x_n - x^*\| + \|S x_n - x^*\|) + L_* \|u_n\| + L_* \|v_n\|
\]
\[
\leq [L_* (1 + L_*^2) \alpha_n + L_*(1 + L_* \beta_n)] \|x_n - x^*\| + L_* (1 + L_* \alpha_n) \|v_n\| + L_* \|u_n\|. \] (8)

Substituting (7) and (8) into (6), and by conditions (iii) and (iv), we infer that
\[
\|x_{n+1} - x^*\| \leq \frac{1}{1 + \alpha_n} \left\{ 1 + (1 + L_*) (1 + L_*^2) \alpha_n + L_* (1 + L_* \beta_n) \alpha_n \right\} \|x_n - x^*\|
\]
\[
+ \frac{1}{1 + \alpha_n} L_* (1 + (1 + L_*) \alpha_n) \|v_n\| + \frac{1}{1 + \alpha_n} [1 + (1 + L_*) \alpha_n] \|u_n\|
\]
\[
\leq \left( 1 - \frac{K_n}{1 + \alpha_n} \right) \|x_n - x^*\| + (2 + L_*) \alpha_n \|v_n\| + (2 + L_*) \|u_n\| + (2 + L_*) \|u_n\|
\]
\[
\leq \left( 1 - \frac{r}{2 \alpha_n} \right) \|x_n - x^*\| + (2 + L_*) \alpha_n \|v_n\| + (2 + L_*) \|u_n\|. \] (9)
Set
\[ a_n = \|x_n - x^*\|, \quad t_n = \frac{r}{2} \alpha_n, \quad b_n = \frac{2}{r} (L \|v_n\| + \gamma_n)(2 + L), \quad \text{and} \quad c_n = (2 + L)\|u'_n\| \]
Then (9) is equivalent to the following inequality:
\[ a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq 0. \]
In view of Lemma 1, conditions (iii) and (iv), we know that \( a_n \to 0 \) \((n \to \infty)\), that is, \( x_n \to x^* \) \((n \to \infty)\). This completes the proof.

**Remark 1.** Theorem 1 improves and extends the corresponding results of [1] in its four aspects:

1. It abolishes the condition that \( X \) is uniformly smooth,
2. The Ishikawa iterative process is replaced by the more general Ishikawa iterative process with mixed errors,
3. It abolishes the condition that the range \( R(S) \) of \( S \) is bounded,
4. Sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) need not converge to zero.

**Remark 2.** Theorem 1 extends and improves the main results of [2] in the following ways:

1. Sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) need not converge to zero,
2. It abolishes the condition that the \( \{Sx_n\} \) and \( \{Sy_n\} \) are bounded,
3. The Ishikawa and Mann iterative process with errors is replaced by the more general Ishikawa iterative process with mixed errors.

**Remark 3.** Theorem 1 also extends and improves the corresponding results of Chang [3], Chang, Cho and Lee et al [4], Ding [5,6], Hassouni and Moudafi [7], Huang [8-10], Kazmi [12], Noor [15,16], Siddiqi and Ansari [17], Siddiqi, Ansari and Kazmi [18] and Zeng [19].

In Theorem 1, if \( \beta_n \equiv 0 \), \( v_n \equiv 0 \), \( \forall n \geq 0 \), then \( y_n = x_n \), hence we have the following result.

**Theorem 2.** Let \( X \) be an arbitrary real Banach space, \( T, A : X \to X, \) \( g : X \to X^* \) be three mappings, and \( \varphi : X^* \to \mathbb{R} \cup \{+\infty\} \) a function with a continuous Gâteaux differential \( \partial \varphi \). For any given \( f \in X \), define a mapping \( S : X \to X \) by
\[ Sx = f - (Tx - Ax + \partial \varphi(g(x))) + x. \]
Let \( x_0 \in X \) be any given point and \( \{x_n\} \) the Mann iterative sequence with mixed errors defined by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n + u_n, \quad \forall n \geq 0, \] (10)
where \( \{\alpha_n\} \) is a real sequence in [0, 1], and \( \{u_n\} \) is a sequence in \( X \) such that \( u_n = u'_n + u''_n \) for any sequences \( \{u'_n\} \) and \( \{u''_n\} \) in \( X \) satisfying the following conditions:
(i) \( T - A + \partial \varphi \circ g - I : X \to X \) is accretive,

(ii) \( T - A + \partial \varphi \circ g : X \to X \) is a Lipschitz operator with the constant \( L \),

(iii) \( \alpha_n \leq \frac{1 - r}{(1 + L_s)(1 + L_s^2)} \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \),

(iv) \( \sum_{n=0}^{\infty} \| u_n \| < \infty \) and \( \| u_n^\prime \| = \gamma_n \alpha_n \), where \( L_s = 1 + L, r \in (0, 1) \) is a constant and \( \gamma_n \to 0 \) (\( n \to \infty \)).

Then the following conclusions hold:

(1) The nonlinear variational inclusion problem (1) has a unique solution \( x^* \in X \),

(2) The Mann iterative sequence \( \{ x_n \} \) with mixed errors converges strongly to the unique solution \( x^* \in X \) of the variational inclusion problem (1).

If \( \varphi \equiv 0 \) in Theorem 1, we have the following result.

**Theorem 3.** Let \( X \) be an arbitrary real Banach space and let \( T, A : X \to X \), \( g : X \to X^* \) be three mappings. For any given \( f \in X \), define a mapping \( S : X \to X \) by

\[
Sx = f - (Tx - Ax) + x.
\]

Let \( x_0 \in X \) be any given point and \( \{ x_n \} \) the Ishikawa iterative sequence with mixed errors defined by

\[
\begin{cases}
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nSy_n + u_n, \\
y_n = (1 - \beta_n)x_n + \beta_nSx_n + v_n,
\end{cases}
\]

where \( \{ \alpha_n \} \), \( \{ \beta_n \} \) are two real sequences in \([0, 1]\), and \( \{ u_n \} \), \( \{ v_n \} \) are two sequences in \( X \) such that \( u_n = u_n^\prime + u_n^\prime\prime \) for any sequences \( \{ u_n^\prime \} \) and \( \{ u_n^\prime\prime \} \) in \( X \) satisfying the following conditions:

(i) \( T - A - I : X \to X \) is accretive,

(ii) \( T - A : X \to X \) is a Lipschitz operator with the constant \( L \),

(iii) \( K_n = (1 + L_s)(1 + L_s^2)\alpha_n + L_s(1 + L_s)\beta_n \leq 1 - r \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \),

(iv) \( \sum_{n=0}^{\infty} \| u_n^\prime \| < \infty \), \( \| u_n^\prime\prime \| = \gamma_n \alpha_n \) and \( \| v_n \| \to 0 \) (\( n \to \infty \)), where \( L_s = 1 + L, r \in (0, 1) \) is a constant and \( \gamma_n \to 0 \) (\( n \to \infty \)).

Then the following conclusions hold:

(1) The variational inequality

\[
\langle Tx - Ax - f, v - g(x) \rangle \geq 0, \forall v \in X^*
\]

has a unique solution \( x^* \in X \),

(2) The Ishikawa iterative sequence \( \{ x_n \} \) with mixed errors converges strongly to the unique solution \( x^* \in X \) of the variational inequality (12).
In Theorem 3, if $\beta_n \equiv 0$, $v_n \equiv 0$, $\forall n \geq 0$, then $y_n = x_n$, hence we have the following result.

**Theorem 4.** Let $X$ be an arbitrary real Banach space, and let $T, A : X \to X$, $g : X \to X^*$ be three mappings. For any given $f \in X$, define a mapping $S : X \to X$ by

$$Sx = f - (Tx - Ax) + x.$$  

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Mann iterative sequence with mixed errors defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nSx_n + u_n, \ \forall n \geq 0, \quad (13)$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$, and $\{u_n\}$ is a sequence in $X$ such that $u_n = u'_n + u''_n$ for any sequences $\{u'_n\}$ and $\{u''_n\}$ in $X$ satisfying the following conditions:

(i) $T - A - I : X \to X$ is accretive,

(ii) $T - A : X \to X$ is a Lipschitz operator with the constant $L$,

(iii) $\alpha_n \leq \frac{1 - r}{(1 + L^*)(1 + L^2^*)}$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \|u'_n\| < \infty$ and $\|u''_n\| = \gamma_n \alpha_n$, where $L^* = 1 + L$, $r \in (0, 1)$ is a constant and $\gamma_n \to 0 (n \to \infty)$.

Then the following conclusions hold:

(1) The variational inequality (12) has a unique solution $x^* \in X$,

(2) The Mann iterative sequence $\{x_n\}$ with mixed errors converges strongly to the unique solution $x^* \in X$ of the variational inequality (12).

**Remark 4.** The following example reveals that Theorem 1 extends properly Theorem 3.1 of Chang [1] and Theorem 2.1 of Chang [2].

**Example 1.** Let $X, T, A, g, f, S, \phi$ be as in Theorem 1 and

$$\alpha_n = \frac{1 - r}{2(1 + L^*)(1 + L^2^*)}, \quad \beta_n = \frac{1 - r}{2L^*(1 + L^*)},$$

$$\|u'_n\| = \frac{1}{(n + 1)^2}, \quad \|u''_n\| = \frac{1 - r}{(n + 1)2(1 + L^*)(1 + L^2^*)}, \quad \|v_n\| = \frac{1}{n + 1}$$

for all $n \geq 0$. Then the conditions of Theorem 1 are satisfied. But Theorem 3.1 in [1] and Theorem 2.1 in [2] are not applicable since $\{\alpha_n\}$ and $\{\beta_n\}$ do not converge to 0.

**References**


