Lines with the butterfly property

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Abstract. In this paper it is explored which lines have the butterfly property with respect to quadrangles (inscribed into a given conic curve).

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Let \(ABCD\) be a plane quadrangle, \(w\) a line intersecting all sides and diagonals of \(ABCD\) (considered as lines), and \(S\) a point on \(w\). Let \(H, K, U, V, X,\) and \(Y\) denote intersections of \(w\) with lines \(AB, CD, AC, BD, AD,\) and \(BC,\) respectively. We consider the statements

\[ \mathcal{B}(w, ABCD): \] If the midpoints of any two of the following segments \(HK, UV,\) and \(XY\) coincide, then they all coincide.

\[ \mathcal{B}(w, S, ABCD): \] If \(S\) is the midpoint of any of the following segments \(HK, UV,\) and \(XY,\) then it is the midpoint of them all.

Figure 1. Quadrangle \(ABCD\) and six points of intersection of its sides with line \(w\)

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The first statement $B(w, ABCD)$ is not very interesting because we have the following result (see Figure 1).

**Theorem 1.** The statement $B(w, ABCD)$ is true for every line $w$ and every quadrangle $ABCD$.

**Proof.** Without loss of generality, we can assume that $A, B, C, D$ are points in the Gauss complex plane with affixes $0$ (zero), $1$ (one), $c$, and $d$ and that the line $w$ has the equation $z + t \bar{z} = s$, where $t$ is a unimodular complex number and $S = (s)$ is the point symmetric to the origin with respect to the line $w$.

This unusual equation for a line in the complex plane is explained on page 76 of the reference [5] and could be seen as follows. Without loss of generality, one can assume that no vertex of $ABCD$ belongs to $w$. Then one can consider $w$ as the perpendicular bisector of segment $AS$ which leads to the equation in the given form.

The points of intersection have the following affixes $h = \frac{s}{1+t}$, $k = \frac{(t(d-s)) \bar{c} + (s-ct) d}{d-c+t(d-c)}$, $u = \frac{s}{1+t}$, $v = \frac{(s-t) d + d - s}{d - 1 + t(d - 1)}$, $x = \frac{sd}{d - t}$, and $y = \frac{c(1-t) s + (s-ct)}{d-c+t(1-c)}$. Now $h_2 = \frac{1}{2} (h + k)$, $u_2 = \frac{1}{2} (u + v)$, and $x_2 = \frac{1}{2} (x + y)$ are the affixes of the midpoints $H_2, U_2$, and $X_2$ of $HK, UV$, and $XY$ respectively. Using as denominators for $h_2 - u_2$ and $h_2 - x_2$ just the products of the denominators in the given descriptions of $h$, $k$, $u$, $v$, $x$, and $y$, one finds that fractions describing $h_2 - u_2$ and $h_2 - x_2$ have the same numerator (possibly up to the sign). From this the conclusion of the theorem follows immediately. Indeed, if $H_2$ and $U_2$ coincide, then the numerator of $h_2 - u_2$ vanishes and so does the numerator of $h_2 - x_2$ implying finally $H_2 = X_2$. □

**Remark 1.** The hypothesis that the line $w$ intersects all sides and diagonals is essential in Theorem 1. In the case of an isosceles trapezium $ABCD$ and $w||AB||CD$ the midpoints of $UV$ and $XY$ coincide while the points $H$ and $K$ do not exist.

Our goal now is to prove the following three theorems.

**Theorem 2.** For every parabola $k$ and every point $S$ there is a unique line $w$ such that $B(w, S, ABCD)$ is true for every quadrangle $ABCD$ inscribed into $k$.

![Figure 2. Parabola $k$ and point $S$ with line $w(k, S)$ and two inscribed quadrangles having the butterfly property with respect to this line and the point](image)
Theorem 3. Let $O$ be the centre of either an ellipse or hyperbola $k$. For every line $w$ through $O$ the statement $B(w, O, ABCD)$ is true for every quadrangle $ABCD$ inscribed into $k$.

Figure 3. Hyperbola $k$ and line $w$ through the centre $O$ with an inscribed quadrangle such that $B(w, O, ABCD)$ holds

Theorem 4. If $k$ is either an ellipse or a hyperbola with the centre $O$, then for every point $S$ different from $O$ there is a unique line $w$ such that $B(w, S, ABCD)$ is true for every quadrangle $ABCD$ inscribed into $k$.

Figure 4. Ellipse $k$, point $S$ and line $w = w(k, S)$ through $S$ with inscribed quadrangles such that $B(w, S, ABCD)$ and $B(w, S, A'B'C'D')$ hold
Proof. Before proving these theorems we shall recall some facts from the analytic geometry of conics. It is well-known that if we take a focus of conic $k$ as the pole (the origin) and the main axis (the line of symmetry through the focus) $\mu$ as the polar axis of a polar coordinate system, then $k$ has the equation $\varrho = p/(1 + \varepsilon \cos \vartheta)$, where $\varrho$ is the polar radius, $\vartheta$ is the polar angle, and $p$ and $\varepsilon$ are nonnegative real numbers. Hence, in the associated rectangular coordinate system points $A$, $B$, $C$, and $D$ have coordinates $(p \cos \vartheta/(1 + \varepsilon \cos \vartheta), p \sin \vartheta/(1 + \varepsilon \cos \vartheta))$, where $\vartheta$ is $\alpha$, $\beta$, $\gamma$, and $\delta$. We could continue using trigonometric functions but it is easier at this point to employ the universal trigonometric substitution to write

$$
\cos \alpha = \frac{1 - a^2}{1 + a^2}, \quad \sin \alpha = \frac{2a}{1 + a^2},
$$

and similarly for the remaining three points (and their corresponding letters). We conclude that points $A$, $B$, $C$, and $D$ have coordinates

$$
\left( \frac{p(1 - t^2)}{\varepsilon(1 - t^2) + t^2 + 1}, \frac{2pt}{\varepsilon(1 - t^2) + t^2 + 1} \right)
$$

for $t$ equal to $a$, $b$, $c$, and $d$.

Let us assume that line $w$ has the equation $fx + gy + h = 0$ and that point $S$ has coordinates $(m, n)$. Since point $S$ belongs to line $w$ it follows that $h = -fm - gn$. Line $AB$ has the equation

$$(ab(\varepsilon - 1) + \varepsilon + 1)x + (a + b)y - p(ab + 1) = 0.$$ 

The other lines $CD$, $AC$, $BD$, $AD$, and $BC$ have analogous equations. The point of intersection $H$ of lines $w$ and $AB$ has the coordinates

$$
\left[ \frac{g p(a b + 1) + h(a + b)}{g(\varepsilon - 1) a b - f(a + b) + g(\varepsilon + 1)}, \frac{-h((\varepsilon - 1) a b + \varepsilon + 1) - f p(a b + 1)}{g(\varepsilon - 1) a b - f(a + b) + g(\varepsilon + 1)} \right].
$$

Notice that the denominators of the above fractions do not vanish since the considered point of intersection exists by hypothesis. The other points of intersection $K$, $U$, $V$, $X$, and $Y$ have similar coordinates.

Let $H_2(h_2, k_2)$, $U_2(u_2, v_2)$, and $X_2(x_2, y_2)$ be the midpoints of the segments $HK$, $UV$, and $XY$. Then $h_2 - m = \frac{a(b + 1)}{2\lambda u}M_H$ and $k_2 - n = \frac{b(d + 1)}{2\lambda u}M_H$, where

$$
P_H = (c + d)f + cd(1 - \varepsilon)g - (1 + \varepsilon)g, \quad N_H = (a + b)f + ab(1 - \varepsilon)g - (1 + \varepsilon)g,
$$

$$
M_H = mQ_H + nR_H + pS_H, \quad Q_H = -Z \varepsilon^2 + (Pf + 2Ug)e + Dg - R,
$$

$$
R_H = Sf + (R - Pf)e, \quad S_H = Z - Pf - Ug,
$$

with $Z = 2(ab + 1)(cd + 1)$, $D = 2(ab - 1)(cd - 1)$, $S = 2(ab + 1)(c + d)$, $U = 2(abcd - 1)$, $P = abc + abd + acd + bcd + a + b + c + d$, and $R = abc + abd + acd + bcd - a - b - c - d$.

Notice that

$$
M_H - M_U = 2(d - a)(b - c)(nf + [m(\varepsilon^2 - 1) - p\varepsilon]g)
$$
and
\[ M_H - M_X = 2 (d - b) (a - c) \left( x + m (\varepsilon^2 - 1) - p \varepsilon \right). \]

Without loss of generality, we now assume \( H_2 = S \), i.e., that \( M_H = 0 \). Then we have to look for conditions on line \( w \) implying \( U_2 = X_2 = S \), i.e., \( M_U = M_X = 0 \).

When \( k \) is a parabola, then \( \varepsilon = 1 \) so that we distinguish two possibilities: (a) \( n = 0 \) and (b) \( n \neq 0 \).

In the first case, point \( S \) belongs to axis \( \mu \) of \( k \) and it follows
\[ M_U = M_X = 0 \implies g = 0. \]

But \( g = 0 \) means that \( w \) is the line perpendicular to the axis of the parabola passing through point \( S \).

In the second case, point \( S \) is not on axis \( \mu \) of \( k \) and points \( U_2 \), and \( X_2 \) coincide with point \( S \) if and only if \( f = \frac{p \varepsilon}{n} \) (i.e., if and only if \( w \) has the equation \( px + ny = m p + n^2 \)). This proves Theorem 2.

When \( k \) is either an ellipse or a hyperbola, then \( \varepsilon \neq 1 \) and its centre is at point \( O(\frac{p \varepsilon}{n}, 0) \). Now we distinguish four cases: (i) \( (m, n) = (\frac{p \varepsilon}{n}, 0) \) (i.e., \( S = O \)), (ii) \( n = 0 \) and \( m \neq \frac{p \varepsilon}{n} \), (iii) \( n \neq 0 \) and \( m = \frac{p \varepsilon}{n} \) and (iv) \( n \neq 0 \) and \( m \neq \frac{p \varepsilon}{n} \).

In case (i), we have \( M_U = M_X = 0 \) so that \( B(w, O, ABCD) \) is true for every line \( w \) which goes through the center \( O \) of either an ellipse or a hyperbola \( k \) and for every quadrangle \( ABCD \) inscribed into it. This proves Theorem 3.

In case (ii), point \( S \) is on the principal axis \( \mu \) of \( k \) and points \( H_2, U_2, \) and \( X_2 \) coincide with point \( S \) if and only if \( g = 0 \) (i.e., if and only if \( w \) is perpendicular to \( \mu \) at point \( S \)).

In case (iii), point \( S \) is on the secondary axis \( \nu \) of \( k \) and points \( H_2, U_2, \) and \( X_2 \) coincide with point \( S \) if and only if \( f = 0 \) (i.e., if and only if \( w \) is the perpendicular to \( \nu \) at point \( S \)).

Finally, in case (iv), point \( S \) is not on either axis of \( k \) and points \( H_2, U_2, \) and \( X_2 \) coincide with point \( S \) if and only if
\[ f = \frac{(p \varepsilon - m (\varepsilon^2 - 1)) g}{n} \]
(with \( g \neq 0 \)), i.e., if and only if \( w \) has the equation
\[ (p \varepsilon - m (\varepsilon^2 - 1)) x + ny = m (p \varepsilon - m (\varepsilon^2 - 1)) + n^2. \]

This proves Theorem 4.

Line \( w \) from Theorems 2 and 4 is denoted also as \( w(k, S) \). The above proof establishes also the following corollary which is the main result in [3] and [2].

Corollary 1. Let \( k \) be a conic and let \( S \) be a point different from the centre of \( k \) (if the centre exists). Line \( w(k, S) \) is perpendicular to axis \( \nu \) of \( k \) if and only if \( S \) lies on \( \nu \).

Our second corollary shows that the main result in [10] is also covered by the above theorems.

Corollary 2. Let \( k \) be a conic and let \( \ell \) be a line in the same plane. If \( S \) is the point of intersection of \( \ell \) with the diameter of \( k \) conjugate to \( \ell \) and \( S \) is different from the centre of \( k \) (when the centre exists), then \( w(k, S) = \ell \).
Proof. We know that line \( w(k, S) \) has the equation
\[
(p \varepsilon - m (\varepsilon^2 - 1)) x + n y - m (p \varepsilon - m (\varepsilon^2 - 1)) - n^2 = 0
\]
where \((m, n)\) are coordinates of \( S \). In order to find these coordinates, let us assume that line \( \ell \) has the equation \( f x + g y + h = 0 \). In the rectangular coordinate system \( k \) has the equation \((\varepsilon^2 - 1) x^2 - y^2 - 2 \varepsilon p x + p^2 = 0 \). When we compute the midpoint of the points of intersections of \( k \) and \( \ell \) and eliminate parameter \( h \) we obtain the equation \((\varepsilon^2 - 1) g x + f y - \varepsilon p g = 0 \) of the diameter of \( k \) conjugate to the given line \( \ell \). It intersects line \( \ell \) at the point
\[
S \left( -\frac{\varepsilon p g^2 + f h}{f^2 + g^2 (1 - \varepsilon^2)}, \frac{g (\varepsilon p f + h (\varepsilon^2 - 1))}{f^2 + g^2 (1 - \varepsilon^2)} \right).
\]
By substituting the coordinates of \( S \) for \( m \) and \( n \) on the left-hand side of the above equation of \( w(k, S) \) we shall get
\[
\frac{(\varepsilon p f + h (\varepsilon^2 - 1)) (f x + g y + h)}{f^2 + g^2 (1 - \varepsilon^2)} = 0.
\]
This clearly concludes the proof. \(\square\)

The next result shows the connection of our theorems with the version of the original Butterfly Theorem from [7] and the Three-Winged Butterfly Problem from [8] for conics.

Theorem 5. If \( S \) is the midpoint of chord \( PQ \) of conic \( k \), then \( w(k, S) \) is line \( PQ \).

Proof. From the proof of Theorems 2–4 we know that line \( w(k, S) \) has the equation \( f x + g y = f m + g n \) where \((m, n)\) are coordinates of \( S \) and
\[
f n + g (m (\varepsilon^2 - 1) - p \varepsilon) = 0. \tag{1}
\]
We assume that \( P \) and \( Q \) have coordinates
\[
\left( \frac{p (1 - t^2)}{\varepsilon (1 - t^2) + t^2 + 1}, \frac{2 p t}{\varepsilon (1 - t^2) + t^2 + 1} \right)
\]
for \( t \) equal to \( u \) and \( v \). It follows that by substituting for \( m \) and \( n \) the coordinates of the midpoint of the segment \( PQ \) into (1) we obtain
\[
p \left( (\varepsilon - 1) u v - \varepsilon - 1 \right) \left( (- (u + v) f + (u v (\varepsilon - 1) + \varepsilon + 1) g) \right) \overline{(\varepsilon (1 - t^2) + t^2 + 1)} = 0.
\]
Since the equation of line \( PQ \) is \((u v (\varepsilon - 1) + \varepsilon + 1) x + (u + v) y = p (u v + 1) \) it is obvious that \( w(k, S) = PQ \). \(\square\)

Remark 2. Line \( w(k, S) \) has the following simple construction. When \( k \) is a parabola with directrix \( d \), then the perpendicular through \( S \) to \( d \) intersects \( k \) at point \( P \) and \( w(k, S) \) is the parallel through \( S \) to the tangent at \( P \) to \( k \). When \( k \) is an ellipse or a hyperbola and \( S \) is different from the centre \( O \) of \( k \), then line \( OS \) intersects \( k \) at point \( P \) (which could be imaginary) and \( w(k, S) \) is the parallel through \( S \) to the tangent at \( P \) to \( k \).
Remark 3. This paper (without Corollary 2) was written in August 2001. In the meantime, [10] has appeared which is similar in that for a given line \(w\) it searches for a point \(S\) on it such that \(B(w, S, ABCD)\) is true while our approach is to find a line \(w\) through a given point \(S\) such that \(B(w, S, ABCD)\) holds.

References


