Approximation to minimum-norm common fixed point of a semigroup of nonexpansive operators

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Abstract. The purpose of this paper is to introduce a new iterative algorithm for a semigroup of nonexpansive operators in Hilbert space. We prove that the proposed iterative algorithm converges strongly to the minimum-norm common fixed point of the semigroup of nonexpansive operators. The results of this paper extend and improve some known results in the literature.

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1. Introduction

Many problems in various branches of mathematical and physical sciences can be reduced to finding a common fixed point in a given family of mappings. It is usually called the common fixed point problem (hereinafter referred to as: CFPP), that is

Find $x \in F := \bigcap_{i \in I} \text{Fix}(T_i) \neq \emptyset$, \hspace{1cm} (1)

where $\text{Fix}(T_i)$ denotes the fixed point set of $T_i$ and $I$ denotes the index of mappings $T_i$. For example, if we take $T_i = P_{C_i}$, for each $i \in I$, then the common fixed point becomes a well-known convex feasibility problem (CFP) of finding $x \in \bigcap_{i \in I} C_i \neq \emptyset$, where each $C_i$ is a nonempty closed convex subset of Hilbert space $H$ and $P_{\Omega}(x)$ is an orthogonal projection of a point $x \in H$ onto a closed convex set $\Omega \subseteq H$ which is defined by

$$P_{\Omega}(x) := \text{arg min}\{\|x - z\| \mid z \in \Omega\}, \hspace{1cm} (2)$$

where $\|\cdot\|$ denotes the norm in $H$. A complete and exhaustive study on algorithms and applications for solving the convex feasibility problem can be found in [3].

Throughout the paper, we always assume that $F \neq \emptyset$. Many iterative algorithms have appeared to solve the CFPP (1). For a finite family of firmly nonexpansive
mappings \( \{T_i\}_{i \in I} \), where \( I = \{1, 2, \cdots, N\}, \) \( N \geq 1 \) is an integer. Combettes [7] introduced a simultaneous iterative algorithm as follows:

\[
x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) \left( \lambda \sum_{i \in I} \omega_i T_i(x_n) + (1 - \lambda)x_n \right), \quad n \geq 0, x_0 \in C,
\]

where \( \{\alpha_n\} \subset (0, 1) \) satisfies

\[
(i) \lim_{n \to \infty} \alpha_n = 0, \quad (ii) \sum_{n=0}^{\infty} \alpha_n = +\infty, \quad (iii) \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty.
\]

\( \omega_i \in (0, 1) \) for all \( i \in I, \) \( \sum_{i \in I} \omega_i = 1 \) and \( 0 < \lambda \leq 2. \) Meanwhile, he defined a sequential algorithm by

\[
x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(T_1 \cdots T_N)(x_n), \quad n \geq 0, x_0 \in C,
\]

where \( \{\alpha_n\} \) is as in (4). He showed that any sequence \( \{x_n\}_{n \geq 0} \) generated by both algorithms (3) and (5) converges strongly to \( P_F x_0. \) Since every firmly nonexpansive mapping is nonexpansive, Bauschke [2] proposed a sequential method to find the common fixed point of a finite family of nonexpansive mappings. This iterative algorithm has the following form.

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)T_{[n]}x_n, \quad n \geq 0, u, x_0 \in C,
\]

where \( [n] = n(n \text{ mod } N) + 1, \) the mod \( N \) function takes values in \( \{1, 2, \cdots, N\}. \) He proved the sequence generated by (6) converges in norm to \( P_F u \) under assumptions on the mappings that

\[
F = \text{Fix}(T_N \cdots T_1) = \text{Fix}(T_1 T_N \cdots T_3 T_2) = \cdots = \text{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N),
\]

and \( \{\alpha_n\} \) is a sequence of parameters in \( (0, 1) \) which satisfies the following:

\[
(i) \lim_{n \to \infty} \alpha_n = 0, \quad (ii) \sum_{n=0}^{\infty} \alpha_n = +\infty, \quad (iii) \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+N}| < +\infty.
\]

**Remark 1.** If \( \{T_i\}_{i \in I} \) is a family of firmly nonexpansive mappings, then condition (7) is naturally met (see Proposition 2.2 of [6]). Even to nonexpansive mappings, by the results of [11] and [9], assumption (7) can be simplified by

\[
F = \text{Fix}(T_N \cdots T_1).
\]

On the other hand, if \( I \) is a countable infinite set, Shimoji and Takahashi [11] investigated the following iterative algorithm.

\[
x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)W_n(\lambda_n x_0 + (1 - \lambda_n)x_n), \quad n \geq 0, x_0 \in C,
\]

where \( W_n \) is a \( W \)-mapping defined by (16) below, \( \{\alpha_n\} \subset (0, 1) \) and \( \{\lambda_n\} \subset (0, 1) \) satisfy \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \lambda_n = 0, \prod_{n=0}^{\infty}(1 - \alpha_n)(1 - \lambda_n) = 0 \) and \( \sum_{n=0}^{\infty} |\alpha_n -
\[ \alpha_{n+1} + |\lambda_n - \lambda_{n+1}| < +\infty. \] They proved the sequence \( \{x_n\}_{n \geq 0} \) converges strongly to \( P_F x_0 \). When \( I \) is an unbounded subset of \( \mathbb{R}_+ \), where \( \mathbb{R}_+ \) denotes the set of non-negative real numbers. Aleyner and Censor [1] introduced the following algorithm for a family of nonexpansive semigroups \( \{T_t \mid t \in I\} \).

\[ x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{r_n} x_n, \quad n \geq 0, u, x_0 \in C, \quad (11) \]

where \( \{\alpha_n\} \subset (0, 1) \) satisfies the condition as in (4) and \( \{r_n\}_{n \geq 0} \subset I \) is some given sequence. If \( \{T_t \mid t \in I\} \) is a uniformly asymptotically regular semigroup of a nonexpansive operator, they proved the sequence \( \{x_n\}_{n \geq 0} \) converges strongly to \( P_F u \). Suzuki [14] proved the sequence \( \{x_n\}_{n \geq 0} \) generated by (11) converges strongly to \( P_F u \) with an assumption that \( \{T_t \mid t \in I\} \) is a one-parameter nonexpansive semigroup and the sequences \( \{\alpha_n\} \) and \( \{r_n\} \) satisfying

(i) \( 0 < \alpha_n < 1, 0 \leq r_n \) and \( s_n := \lim \inf_{m \to \infty} |t_m - t_n| > 0 \), for any \( n \geq 0 \);

(ii) \( \{r_n\} \) is bounded;

(iii) \( \lim_{n \to \infty} \alpha_n/s_n = 0 \),

since these iterative algorithms not only have strong convergence, but also converge to the projection of the starting point \( x_0 \) or any point \( u \) onto \( F \). In contrast to the common fixed point problem, it is in addition called the best approximation problem with respect to \( F \). Consider the projection operator \( P_F x \)

\[ P_F x = \arg \min \{\|x - z\| \mid z \in F\}, \]

where \( F \) is as in (1). Define \( x^* := P_F 0 = \arg \min \{\|z\| \mid z \in F\} \), i.e., \( x^* \) is the minimum-norm common fixed point of \( F \). If \( 0 \in C \), then the iterative algorithms (3), (5), (6), (10) and (11) do the job to find the minimum-norm common fixed point of \( \bigcap_{i \in I} F x(T_i) \). In fact, one can let \( x_0 = 0 = u = 0 \). However, if \( 0 \notin C \), then none of these algorithms work to find the minimum-norm element of \( F \). In order to overcome this difficulty caused by possible exclusion of the origin from \( C \), some authors have applied the metrical projection \( P_C \) on the right-hand side of the iterative algorithm (see for example [6, 10 – 12]). The role of the metrical projection \( P_C \) is to pull the substituted sequence back to \( C \), then the iterative sequences are well-defined. In these works, Liu and Cui [9] proposed two iterative algorithms, one was sequential; the other is simultaneous.

(i) The sequential method.

\[ x_{n+1} = P_C \left( (1 - t_n) T_{[n+1]} x_n \right), \quad n \geq 0, x_0 \in C, \quad (12) \]

where \( \{t_n\} \subset (0, 1) \) satisfies the following properties: (i) \( \lim_{n \to \infty} t_n = 0 \); (ii) \( \sum_{n=0}^\infty t_n = +\infty \); (iii) either \( \sum_{n=0}^\infty |t_n - t_{n+N}| < +\infty \) or \( \lim_{n \to \infty} t_n/t_{n+N} = 1 \). \( T_{[n]} := T_n \mod N \) with the mod \( N \) function taking values in the set \( \{1, 2, \cdots, N\} \).

(ii) The simultaneous method.

\[ x_{n+1} = P_C \left( (1 - t_n) \sum_{i=1}^N \lambda_i^{(n)} T_i x_n \right), \quad n \geq 0, x_0 \in C, \quad (13) \]
where $\lambda_i^{(n)}>0$ for all $n \geq 0$, $i = 1, 2, \cdots, N$, and $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all $n$ and satisfy (i) $\sum_{n=0}^{\infty} \sum_{i=1}^{N} |\lambda_i^{(n+1)} - \lambda_i^{(n)}| < +\infty$, $\inf_{n \geq 0} \lambda_i^{(n)} > 0$ for all $i$; (ii) $\lim_{n \to \infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = +\infty$; (iii) either $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$ or $\lim_{n \to \infty} (t_n/t_{n+1}) = 1$. Assume that $\{T_i\}_{i=1}^{N}$ satisfy condition (9), they proved that the sequence $\{x_n\}_{n \geq 0}$ generated by the sequential method and the simultaneous method converge strongly to the minimum-norm common fixed point of the mappings $\{T_i\}_{i=1}^{N}$.

Motivated and inspired by the above works, we introduce a new iterative algorithm for finding the minimum-norm common fixed point of a nonexpansive semigroup $\{T_t \mid t \in I\}$. The proposed algorithm combines the iterative algorithm given by Aleyner and Censor [1] and Liu and Cui [9]. The sequence $\{x_n\}$ is generated by the following recursive:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C((1 - t_n)T_{r_n}x_n), \quad n \geq 0, x_0 \in C, \quad (14)$$

where the parameters $\{\alpha_n\}$ and $\{t_n\}$ are sequences in $(0, 1)$, $\{r_n\}_{n \geq 0} \subset I$ is some given sequence. Furthermore, we present a new way to prove the strong convergence of the iterative algorithm (14) under a mild assumption on the parameters and its limit is also the minimum-norm common fixed point of a nonexpansive semigroup $\{T_t \mid t \in I\}$.

2. Preliminaries

In this section we present definitions and some tools that will be used later on in the proof of our main theorem. Throughout this paper, by $\mathbb{R}$ we denote the set of real numbers and by $\mathbb{R}_+$ the set of nonnegative real numbers. Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. In a Hilbert space, it is known that for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \quad (15)$$

Recall that the orthogonal projection $P_C x$ of $x$ onto $C$ is defined by the following

$$P_C x = \arg \min_{y \in C} \|x - y\|.$$

The orthogonal projection has the following well-known properties. For a given $x \in H$,

(i) $\langle x - P_C x, z - P_C x \rangle \leq 0$, for all $z \in C$;

(ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$, for all $x, y \in H$.

In what follows, we give some definitions and lemmas.

**Definition 1.** Let $C$ be a nonempty closed convex subset of $H$. $T : C \to C$ is called

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$,
(ii) firmly nonexpansive if \( \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle \), for all \( x, y \in C \).

**Remark 2.** It is easy to see that the projection operator is firmly nonexpansive, and the firmly nonexpansive mapping is a nonexpansive mapping. The relations between them can be expressed as the visual picture.

\[ \text{Projection operator} \implies \text{Firmly nonexpansive} \implies \text{Nonexpansive} \]

**Definition 2** (See [11]). Let \( C \) be a nonempty closed convex subset of Banach space \( E \). Let \( \{T_i\}_{i=1}^{\infty} \) be infinite mappings of \( C \) into themselves and let \( \alpha_1, \alpha_2, \cdots \) be real numbers such that \( 0 \leq \alpha_i \leq 1 \) for every \( i \). For any \( n \geq 1 \), define a mapping \( W_n \) of \( C \) into itself as follows:

\[
U_{n,n+1} = I,
\]
\[
U_{n,n} = \alpha_n U_{n,n+1} + (1 - \alpha_n) I,
\]
\[
U_{n,n-1} = \alpha_{n-1} U_{n,n} + (1 - \alpha_{n-1}) I,
\]
\[ \vdots \]
\[
U_{n,k} = \alpha_k U_{n,k+1} + (1 - \alpha_k) I,
\]
\[
U_{n,k-1} = \alpha_{k-1} U_{n,k} + (1 - \alpha_{k-1}) I,
\]
\[ \vdots \]
\[
U_{n,2} = \alpha_2 U_{n,3} + (1 - \alpha_2) I,
\]
\[
W_n = U_{n,1} = \alpha_1 U_{n,2} + (1 - \alpha_1) I,
\]

where \( I \) is the identity mapping. Such a mapping \( W_n \) is called a \( W \)-mapping generated by \( T_n, T_{n-1}, \cdots, T_1 \) and \( \alpha_n, \alpha_{n-1}, \cdots, \alpha_1 \).

A semigroup of nonexpansive operators could be recognized as special families of nonexpansive operators, see [10] and others.

**Definition 3.** Let \( I \) be an unbounded subset of \( \mathbb{R}^+ \) such that

(i) \( t + s \in I \), for all \( t, s \in I \),

(ii) \( t - s \in I \), for all \( t, s \in I \)

with \( t \geq s \), and let \( \Gamma = \{T_t \mid t \in I\} \) be a family of self-operators on a nonempty closed convex subset \( C \) of \( E \). The family \( \Gamma \) is called a semigroup of nonexpansive operators on \( C \) if the following conditions hold:

(i) \( T_t \) is a nonexpansive self-operator on \( C \), for all \( t \in I \),

(ii) \( T_{t+s}x = T_t T_s x \), for all \( t, s \in I \) and all \( x \in C \).

In addition,

(iii) for each \( x \in C \), the mapping \( t \mapsto T_t x \) from \( [0, +\infty) \) into \( C \) is strongly continuous.

Then the family of mappings \( \{T_t \mid t \in I\} \) is called a one-parameter strongly continuous semigroup of nonexpansive mappings (a one-parameter nonexpansive semigroup, for short).
The concept of a uniformly asymptotically regular semigroup of nonexpansive operators can be found in [4, 5].

**Definition 4.** Let \( \Gamma = \{ T_t \mid t \in I \} \) be a semigroup of nonexpansive operators on a nonempty closed convex subset \( C \) of \( H \). The family \( \Gamma \) is called a uniformly asymptotically regular semigroup of nonexpansive operators on \( C \) if

\[
\lim_{r \to \infty} \left( \sup_{x \in C} \| T_s T_r x - T_r x \| \right) = 0,
\]

uniformly for all \( s \in I \).

As a matter of fact, condition (17) implies that there exists a monotone sequence \( \{ r_n \}_{n \geq 0} \subseteq I \) such that

\[
0 \leq r_0 \leq r_1 \leq \cdots \leq r_n \leq \cdots, \quad \text{and} \quad \lim_{n \to \infty} r_n = \infty,
\]

and

\[
\sum_{n=0}^{\infty} \sup_{x \in C} \| T_s T_{r_n} x - T_{r_n} x \| < +\infty,
\]

uniformly for all \( s \in I \).

The following demiclosedness principle of a nonexpansive mapping played an important role in our work. We denote strong or weak convergence by "\( \to \)" or "\( \rightharpoonup \)\), respectively.

**Lemma 1.** Let \( T : C \to C \) a nonexpansive mapping with \( \text{Fix}(T) \neq \emptyset \). If \( x_n \to x \) and \((I - T)x_n \to 0\), then \( x = Tx \).

In order to prove the main results in this paper, we shall make use of the following lemmas.

**Lemma 2** (See [12]). Let \( \{ x_n \} \) and \( \{ y_n \} \) be bounded sequences in a Banach space \( E \) and let \( \{ \beta_n \} \) be a sequence in \( [0, 1] \) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose \( x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n \) for all \( n \geq 0 \) and

\[
\limsup_{n \to \infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0.
\]

Then \( \lim_{n \to \infty} \| y_n - x_n \| = 0 \).

**Lemma 3** (See [15]). Let \( \{ a_n \} \) be a sequence of nonnegative real sequences satisfying the following inequality:

\[
a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad n \geq 0,
\]

where \( \{ \gamma_n \} \) is a sequence in \((0, 1)\) and \( \{ \delta_n \} \) is a sequence such that

\begin{enumerate}
  \item \( \sum_{n=0}^{\infty} \gamma_n = +\infty \);
  \item \( \limsup_{n \to \infty} \delta_n \leq 0 \) or \( \sum_{n=0}^{\infty} \gamma_n |\delta_n| < +\infty \).
\end{enumerate}

Then \( \lim_{n \to \infty} a_n = 0 \).
3. Main results

The main result of our work is the next convergence theorem for the iterative algorithm (14). Now, we are in the position to prove the following theorem.

**Theorem 1.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\Gamma = \{T_t \mid t \in I\}$ be a uniformly asymptotically regular semigroup of nonexpansive operators on $C$ such that $F := \bigcap_{t \in I} \text{Fix}(T_t) \neq \emptyset$. Let the sequence $\{x_n\}_{n \geq 0}$ be generated by the iterative algorithm (14), where $\{\alpha_n\}$ and $\{t_n\} \subset (0,1)$ satisfy the conditions:

(i) $\lim_{n \to \infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = +\infty$;

(ii) $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$.

Then any sequence $\{x_n\}_{n \geq 0}$ generated by (14) converges strongly to the minimum-norm common fixed point of $F$.

**Proof.** We divide the proof into five steps.

Step 1. We prove that the sequence $\{x_n\}_{n \geq 0}$ is bounded. In fact, take $p \in F$, by (14), we have

$$
\|x_{n+1} - p\| = \|(1 - \alpha_n)(x_n - p) + \alpha_n(T_{t_n}(1 - t_n)x_n) - p\| \\
\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|(1 - t_n)T_{t_n}x_n - p\| \\
\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n(1 - t_n)\|x_n - p\| + \alpha_n t_n \|p\| \\
= (1 - \alpha_n t_n)\|x_n - p\| + \alpha_n t_n \|p\| \\
\leq \max\{\|x_n - p\|, \|p\|\}.
$$

By induction, we get

$$
\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\}, \text{ for all } n \geq 0.
$$

Hence, $\{x_n\}$ is bounded. So is the sequence $\{T_{t_n}x_n\}$. Let $M > 0$, such that $M \geq \sup_{n \geq 0}\{\|x_n\|, \|T_{t_n}x_n\|\}$.

Set $z_n := PC((1 - t_n)T_{t_n}x_n)$, we obtain

$$
\|z_n - p\| = \|PC((1 - t_n)T_{t_n}x_n) - p\| \\
\leq \|(1 - t_n)T_{t_n}x_n - p\| \\
\leq (1 - t_n)\|x_n - p\| + t_n \|p\| \\
\leq \max\{\|x_n - p\|, \|p\|\}.
$$

Since $\{x_n\}$ is bounded, we get that $\{z_n\}$ is also bounded.

**Step 2.** We show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Let $\overline{C}$ be any bounded subset of $C$ which contains the sequence $\{x_n\}_{n \geq 0}$. Since $z_n = PC((1 - t_n)T_{t_n}x_n)$,
we get
\[
\|z_{n+1} - z_n\| = \|P_C((1 - t_{n+1})T_{r_{n+1}}x_{n+1}) - P_C((1 - t_n)T_{r_n}x_n)\| \\
\leq \|(1 - t_{n+1})T_{r_{n+1}}x_{n+1} - (1 - t_n)T_{r_n}x_n\| \\
\leq \|(1 - t_{n+1})T_{r_{n+1}}x_{n+1} - (1 - t_{n+1})T_{r_{n+1}}x_n\| \\
+ \|(1 - t_{n+1})T_{r_{n+1}}x_n - (1 - t_n)T_{r_n}x_n\| \\
\leq (1 - t_{n+1})\|x_{n+1} - x_n\| + \|(1 - t_{n+1})T_{r_{n+1}}x_n - (1 - t_n)T_{r_{n+1}}x_n\| \\
+ \|(1 - t_{n+1})T_{r_{n+1}}x_n - (1 - t_n)T_{r_n}x_n\| \\
\leq (1 - t_{n+1})\|x_{n+1} - x_n\| + t_n - t_{n+1}M + (1 - t_n)\|T_{r_{n+1}}x_n - T_{r_n}x_n\|.
\]

Since \( \Gamma \) is a semigroup, and by using (18), we are able to rewrite the last term as follows
\[
\|T_{r_{n+1}}x_n - T_{r_n}x_n\| = \|T_{r_{n+1} - r_n}T_{r_n}x_n - T_{r_n}x_n\|.
\]

It follows that
\[
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq |t_n - t_{n+1}|M + (1 - t_n)\sup_{x \in C} \|T_{r_{n+1} - r_n}T_{r_n}x_n - T_{r_n}x_n\|.
\]

By using (19) and condition (i), we deduce that
\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

With the help of Lemma 2, we get
\[
\lim_{n \to \infty} \|x_n - z_n\| = 0.
\]

Hence, from (14), we have
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \alpha_n \|x_n - z_n\| = 0.
\]

**Step 3.** We show that for each fixed \( s \in I \), \( T_s x_n - x_n \to 0 \) as \( n \to \infty \). In fact,
\[
\|x_n - T_{r_n}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n}x_n\| \\
\leq \|x_n - x_{n+1}\| + (1 - \alpha_n)\|x_n - T_{r_n}x_n\| \\
+ \alpha_n\|P_C((1 - t_n)T_{r_n}x_n) - T_{r_n}x_n\| \\
\leq \|x_n - x_{n+1}\| + (1 - \alpha_n)\|x_n - T_{r_n}x_n\| + \alpha_n t_n M,
\]

which implies that
\[
\|x_n - T_{r_n}x_n\| \leq \frac{\|x_n - x_{n+1}\|}{\alpha_n} + t_n M \to 0 \quad \text{as} \quad n \to \infty. \quad (21)
\]

On the other hand, by using (19) and (21), we have
\[
\|T_s x_n - x_n\| \leq \|T_s x_n - T_s T_{r_n}x_n\| + \|T_s T_{r_n}x_n - T_{r_n}x_n\| + \|T_{r_n}x_n - x_n\| \\
\leq 2\|x_n - T_{r_n}x_n\| + \sup_{x \in C} \|T_s T_{r_n}x_n - T_{r_n}x_n\| \to 0 \quad \text{as} \quad n \to \infty. \quad (22)
\]
Step 4. We prove that lim sup$_{n \to \infty} \langle x^* - x_n, x^* \rangle \leq 0$, where $x^* = P_F 0$. Indeed, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that
\[
\limsup_{n \to \infty} \langle x^* - x_n, x^* \rangle = \lim_{j \to \infty} \langle x^* - x_{n_j}, x^* \rangle.
\]
Since $\{x_{n_j}\}$ is bounded, there exists a subsequence of $\{x_{n_j}\}$ which converges weakly to a point $\tilde{x}$. Without loss of generality, we may assume that $\{x_{n_j}\}$ converges weakly to $\tilde{x}$. Therefore, from (22) and Lemma 1, we have $x_{n_j} \to \tilde{x} \in F$. Since $x^* = P_F 0$, it follows from the properties of the projection operator that
\[
\limsup_{n \to \infty} \langle x^* - x_n, x^* \rangle = \langle x^* - \tilde{x}, x^* \rangle \leq 0. \tag{23}
\]

Step 5. Finally, we prove that $x_n \to x^*$. We observe that
\[
\langle x^* - T_{r_n} x_n, x^* \rangle = \langle x^* - x_n, x^* \rangle + \langle x_n - T_{r_n} x_n, x^* \rangle
\leq \langle x^* - x_n, x^* \rangle + \|x_n - T_{r_n} x_n\| \|x^*\|.
\]
Taking the limsup on both sides of the above inequality, and together with (21), (23), we get
\[
\limsup_{n \to \infty} \langle x^* - T_{r_n} x_n, x^* \rangle \leq 0.
\]

From (15) and (14), we have
\[
\|x_{n+1} - x^*\|^2 = \|(1 - \alpha_n)(x_n - x^*) + \alpha_n (P_C((1 - t_n)T_{r_n} x_n) - x^*)\|^2
\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n \|P_C((1 - t_n)T_{r_n} x_n) - x^*\|^2
\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n \|(1 - t_n)(T_{r_n} x_n - x^*) - t_n x^*\|^2
= (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n (1 - t_n)^2 \|T_{r_n} x_n - x^*\|^2
+ 2\alpha_n (1 - t_n) t_n \langle x^* - T_{r_n} x_n, x^* \rangle + \alpha_n t_n^2 \|x^*\|^2
\leq (1 - \alpha_n t_n)\|x_n - x^*\|^2 + 2\alpha_n (1 - t_n) t_n \langle x^* - T_{r_n} x_n, x^* \rangle + \alpha_n t_n^2 \|x^*\|^2.
\]
It is clear that all conditions of Lemma 3 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$ as $n \to \infty$. This completes the proof.

Remark 3. Theorem 1 improves the results of Aleyner and Censor [1] by discarding the assumption that "\[\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty\]." The proposed iterative algorithm (14) is a sequential algorithm which combines the Krasnoselskii-Mann algorithm with iterative algorithm (12) for solving the minimum-norm common fixed point problem with respect to the common fixed point set of infinitely countable or non-countable families of nonexpansive mappings in a real Hilbert space. Therefore, Theorem 1 also generalizes the corresponding results of Liu and Cui [9] and removes the conditions on \{t_n\} that "\[\sum_{n=0}^{\infty} |t_n - t_{n+1}| < +\infty\ or \lim_{n \to \infty} t_n/t_{n+1} = 1\]."

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