

On mapping properties and the property of Kelley

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Abstract. *Mapping conditions are studied under which a continuum having the property of Kelley has this property hereditarily. The obtained results, related mainly to confluent mappings, extend some known assertions of the subject.*

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1. Introduction

All considered spaces are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected space.

A surjective mapping $f : X \rightarrow Y$ between topological spaces is said to be:

- *open*, provided that the images of open sets under f are open;
- *monotone*, provided that for each point $y \in Y$ the set $f^{-1}(y)$ is connected;
- *confluent*, provided that for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto Q under f ;
- *semi-confluent*, provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;
- *weakly confluent*, provided that for each subcontinuum Q of Y some component of $f^{-1}(Q)$ is mapped onto Q under f .

The reader is referred to [11] for properties of and relations between the above and some other classes of mappings.

A continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing

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x_n and converging to the continuum K (see e.g. [6, p. 167] or [12, Definition 16.10, p. 538]). A continuum is defined to have *the property of Kelley hereditarily* provided that each of its subcontinua has the property of Kelley, see [1].

The property, introduced by J. L. Kelley as property 3.2 in [8, p. 26], has been used there to study hyperspaces, in particular their contractibility (see e.g. [12, Chapter 16], where references for further results in this area are given). Now the property, which has been recognized as an important tool in investigation of various properties of continua, is interesting by its own right, and has numerous applications to continuum theory. Many of them are not related to hyperspaces.

2. Mappings that preserve the property of Kelley

A class \mathfrak{M} of mappings between topological spaces is said to have the *(weak) component restriction property* provided that for each mapping $f : X \rightarrow Y$ belonging to \mathfrak{M} and for each subset B of Y , if $A \subset X$ is the union of some components (is a component) of $f^{-1}(B)$, then the restriction $f|_A : A \rightarrow f(A)$ belongs to \mathfrak{M} . Obviously, the component restriction property implies the weak component restriction property. The component restriction property for various classes of mappings is studied in [4].

A class \mathfrak{M} of mappings between topological spaces is said to *preserve a topological property \mathcal{P}* provided that for each surjection $f \in \mathfrak{M}$ if the domain of f has the property \mathcal{P} , then also the range of f has \mathcal{P} . In particular, we say that a class \mathfrak{M} of mappings between continua *preserves a property \mathcal{P} hereditarily* provided that for each mapping $f : X \rightarrow Y$ belonging to \mathfrak{M} and for each subcontinuum K of X the restriction $f|_K : K \rightarrow f(K) \subset Y$ is in \mathfrak{M} .

Theorem 1. *Let \mathfrak{M} denote a subclass of the class of weakly confluent mappings between continua and let a topological property \mathcal{P} be given. If*

(1.1) \mathfrak{M} has the weak component restriction property, and

(1.2) \mathfrak{M} preserves \mathcal{P} ,

then \mathfrak{M} preserves \mathcal{P} hereditarily.

Proof. Let a mapping $f : X \rightarrow Y$ between continua X and Y be weakly confluent. Assume that X has the property \mathcal{P} hereditarily. Let Q be a subcontinuum of Y . Fix a component K of $f^{-1}(Q)$ such that $f(K) = Q$. Then K has the property \mathcal{P} according to the assumption on X . Further, the restriction $f|_K : K \rightarrow f(K) \subset Y$ is in \mathfrak{M} by (1.1). Thus, $f(K)$ has \mathcal{P} by (1.2). Finally, since $f(K) = Q$, it follows that Q has \mathcal{P} , and the proof is complete. \square

If we take in *Theorem 1* the class of confluent mappings as \mathfrak{M} and the property of Kelley as \mathcal{P} , then (1.1) is satisfied since \mathfrak{M} has the component restriction property (so it has the weak component restriction property), see [2, I, p. 213], and (1.2) holds since confluent mappings preserve the property of Kelley, see [14, Theorem 4.3, p. 296]. Thus we get the following result as a corollary, see [1, Theorem 7.1, p. 159].

Corollary 1. *Confluent mappings preserve the property of Kelley hereditarily.*

Remark 1. *In connection with a question of Nadler who asks in [12, (16.38), p. 559] what classes of mappings between continua preserve the property of Kelley, it is interesting to know what class can be substituted for the class of confluent ones in Corollary 1. Easy examples show that neither semi-confluent nor weakly confluent mappings preserve the property of Kelley. Furthermore, even if the classes of hereditarily semi-confluent or hereditarily weakly confluent mappings are considered, then the preservation of the property of Kelley does not hold. Indeed, if X is the closure of the set $\{(x, \sin(\frac{1}{x})) : 0 < |x| \leq 1\}$ in the plane, and $Y = (\{0\} \times [-1, 2]) \cup \{(x, \sin(\frac{1}{x})) : 0 < x \leq 1\}$, then X has the property of Kelley while Y does not, and each mapping from X onto Y is hereditarily weakly confluent, see [13, Theorem 4, p. 236] and compare [11, Theorem 6.16, p. 56].*

3. Generalized homogeneity

Recall that a continuum X is called a *triod* provided that X contains a subcontinuum C such that $X \setminus C$ has at least three components. A continuum is said to be *atriodic* provided that it contains no triod. For example each solenoid and each arc-like continuum is atriodic.

The following assertion (see [1, Corollary 5.2, p. 158]) is a consequence of a more general result (viz. [1, Theorem 5.1, p. 157]).

Proposition 1. *Each atriodic continuum with the property of Kelley has the property of Kelley hereditarily.*

Let \mathfrak{M} be a class of mappings between topological spaces. A space X is said to be *\mathfrak{M} homogeneous* provided that for every two points p and q in X there is a mapping $f \in \mathfrak{M}$ such that $f(p) = q$. The next result follows from Proposition 1.

Theorem 2. *Let \mathfrak{M} be a class of mappings between continua. If*

(2.1) *each atriodic \mathfrak{M} homogeneous continuum has the property of Kelley,*

then

(2.2) *each atriodic \mathfrak{M} homogeneous continuum has the property of Kelley hereditarily.*

It is known that each homogeneous (see [14, Theorem 2.5, p. 293]) and, moreover, each open homogeneous (see [3, Statement, p. 380]) continuum has the property of Kelley. Consequently, condition (2.1) holds if \mathfrak{M} means the class of open mappings between continua. Thus we have a corollary that extends [1, Theorem 10.1, p. 161] from homogeneous to open homogeneous continua.

Corollary 2. *Each atriodic open homogeneous continuum has the property of Kelley hereditarily.*

It is interesting to know what other classes \mathfrak{M} of mappings between continua satisfy condition (2.1). Recall that the above mentioned implication [3, Statement, p. 380] cannot be extended to confluent homogeneous continua because examples are known of confluent homogeneous continua that do not have the property of Kelley, see [7, Section 1, p. 52]. Namely, two such examples are presented there. One of them contains a 2-cell, see [7, Figure 1, p. 53], and the other contains the Menger universal curve, see [7, Figure 2, p. 57]. Thus none of them is atriodic,

and we do not know if condition (2.1) holds if \mathfrak{M} stands for the class of confluent mappings. So, the following question can be asked.

Question 1. *Does each atriodic confluent homogeneous continuum have the property of Kelley?*

A similar situation is with the class of monotone mappings. An example of a monotone homogeneous continuum without the property of Kelley is constructed in [5], but again the continuum is very far from being atriodic. So, we have the next question.

Question 2. *Does each atriodic monotone homogeneous continuum have the property of Kelley?*

4. A remark on hyperspaces

Let \mathbb{N} denote the set of all positive integers. Given a continuum X , we denote by 2^X the hyperspace of all nonempty closed subsets of X metrized by the Hausdorff metric (see [6] or [12]), and for an $n \in \mathbb{N}$ we put

$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$$

and

$$C_\infty(X) = \{A \in 2^X : A \text{ has finitely many components}\}.$$

Thus

$$C(X) = C_1(X) \subset \cdots \subset C_n(X) \subset C_{n+1}(X) \subset \cdots \subset C_\infty(X). \quad (1)$$

The reader is referred to [9] and [10] for properties of these hyperspaces.

It is shown in [1, Theorem 8.1, p. 159] that for any continuum X the hyperspace $C(X)$ does not have the property of Kelley hereditarily. Therefore the inclusions (1) imply the following extension of this result.

Proposition 2. *For each continuum X and for each $n \in \mathbb{N}$ the hyperspaces $C_n(X)$ and $C_\infty(X)$ do not have the property of Kelley hereditarily.*

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