On the theorem of N. Singh and K. M. Sharma

Živorad Tomovski*

Abstract. A new short proof of the Theorem of N. Singh and K. M. Sharma (see [7]) is given.

Key words: quasi-convex sequence, Moore’s class, $L^1$-convergence of Fourier series

AMS subject classifications: 26D15, 42A20

Received November 2, 2000 Accepted October 15, 2002

1. Introduction and preliminaries

The problem of $L^1$-convergence, via Fourier coefficients, consists of finding the properties of Fourier coefficients such that the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

is a Fourier series of some $f \in L^1(0, \pi)$ and

$$\|S_n - f\| = o(1), \quad n \to \infty \quad \text{if and only if} \quad a_n \log n = o(1), \quad n \to \infty.$$ (2)

Here, $S_n$ denotes the $n$-th partial sum of the series (1) and $\|\|$ is the $L^1$-norm.

Several authors have studied the question of $L^1$-convergence of the series (1).

The sequence $\{a_n\}$ that satisfies the condition $\sum_{n=1}^{\infty} (n+1)|\Delta^2a_n| < \infty$, where

$$\Delta^2a_n = \Delta(\Delta a_n) = \Delta a_n - \Delta a_{n+1} = a_n - 2a_{n+1} + a_{n+2}, \quad \text{for all } n,$$

is called quasi-convex.

A classical result concerning the integrability and $L^1$-convergence of a series (1) is the following well-known theorem of Kolmogorov (see [5]).

**Theorem 1 [see [4]].** If $\{a_n\}$ is a quasi-convex null-sequence, then the series (1) is the Fourier series of some $f \in L^1(0, \pi)$ and (2) holds.

The following class $S$ of $L^1$-convergence, was defined by Telyakovskii [9]. A null-sequence $\{a_n\}$ belongs to the class $S$ if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all $n$.

*Faculty of Mathematical and Natural Sciences, Department of Mathematics, P.O. BOX 162, 1 000 Skopje, Macedonia, e-mail: tomovski@iunona.pmf.ukim.edu.mk
Theorem 2 [see [8]]. Let \( \{a_n\} \in S \). Then the series (1) is the Fourier series of some \( f \in L^1(0, \pi) \) and (2) holds.

The difference of noninteger order \( k \geq 0 \) of the sequence \( \{a_n\}_{n=0}^{\infty} \) is defined as follows:

\[
\Delta^k a_n = \sum_{m=0}^{\infty} \binom{m - k - 1}{m} a_{n+m} \quad (n = 0, 1, 2, \ldots) \tag{3}
\]

where

\[
\binom{m + \alpha}{m} = \frac{(1 + \alpha) \cdots (m + \alpha)}{m!}.
\]

It is obvious that if \( a_n \to 0 \) as \( n \to \infty \), then series (3) is convergent and \( \lim_{n \to \infty} \Delta^k a_n = 0 \).

C. N. Moore in [6] generalized quasi-convexity of null-sequences \( \{a_n\} \) in the following way

\[
\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty, \quad \text{for some } k > 0. \tag{M}
\]

It is well-known [3] that if \( \{a_n\} \) is a null-sequence satisfying the condition (M), then

\[
\sum_{n=1}^{\infty} n^r |\Delta^{r+1} a_n| < \infty, \quad \text{for } 0 \leq r < k. \tag{4}
\]


Theorem 3 [see [7]]. Let \( k \) be a real number such that \( k > 0 \). If

(i) \( \lim_{n \to \infty} a_n = 0 \),

(ii) \( \sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty \),

then the series (1) is the Fourier series of some \( f \in L^1(0, \pi) \) and (2) holds.

2. Proof of Theorem 3

Applying Theorem 2, it suffices to show that the conditions (i) and (ii) of Theorem 3 imply condition S. Firstly, we suppose that for some \( k, 0 < k \leq 1 \), the series in (M) converges.

For \( 0 < k \leq 1 \), we construct the sequence

\[
A_n = \sum_{i=n}^{\infty} \binom{i - n + k - 1}{i - n} |\Delta^{k+1} a_i|.
\]

Then, we need the following properties for binomial coefficients \( \binom{\alpha + n}{\alpha} \) (see [2], page 885 and [5], page 68):
On the theorem of N. Singh and K. M. Sharma

a) $\alpha > -1 \Rightarrow \left( \frac{\alpha + n}{\alpha} \right) = \frac{(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)}{n!} > 0$,

b) $\left( \frac{\alpha + n}{\alpha} \right) = \frac{n^{\alpha}}{\Gamma(\alpha + 1)} + O(1), 0 < \alpha \leq 1$,

c) $\sum_{i=0}^{n} \left( \frac{i + \alpha}{\alpha} \right) = \left( \frac{n + \alpha + 1}{n} \right), n \in \mathbb{N}, \alpha \in \mathbb{R}$.

We have

$$\sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \sum_{i=n}^{\infty} \left( \frac{i - n + k - 1}{i - n} \right) |\Delta^{k+1} a_i|$$

$$= \sum_{i=0}^{\infty} |\Delta^{k+1} a_i| \sum_{n=0}^{i} \left( \frac{i - n + k - 1}{i - n} \right)$$

$$= \sum_{i=0}^{\infty} |\Delta^{k+1} a_i| \sum_{n=0}^{i} \left( \frac{n + k - 1}{n} \right) = \sum_{i=0}^{\infty} \left( \frac{i + k}{k} \right) |\Delta^{k+1} a_i|$$

$$= \frac{1}{\Gamma(k+1)} \sum_{i=0}^{\infty} i^k |\Delta^{k+1} a_i| + O \left( \sum_{i=0}^{\infty} |\Delta^{k+1} a_i| \right).$$

Since series (3) is convergent, by condition (M), we obtain

$$\sum_{i=0}^{\infty} |\Delta^{k+1} a_i| = |\Delta^{k+1} a_0| + \sum_{i=1}^{\infty} |\Delta^{k+1} a_i|$$

$$\leq \sum_{m=0}^{\infty} \left( \frac{m - k - 2}{m} \right) a_m + \sum_{i=1}^{\infty} i^k |\Delta^{k+1} a_i| < \infty.$$ 

Thus, $\sum_{n=0}^{\infty} A_n < \infty$ and $A_n \downarrow 0$.

Then (see [1], Lemma 1)

$$\Delta a_n = \sum_{i=n}^{\infty} \left( \frac{i - n + k - 1}{i - n} \right) \Delta^{k+1} a_i,$$

and hence

$$|\Delta a_n| \leq \sum_{i=n}^{\infty} \left( \frac{i - n + k - 1}{i - n} \right) |\Delta^{k+1} a_i| = A_n, \text{ for all } n.$$ 

If $k > 1$, by Bosanquet result (4), we obtain $\sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty$, i.e. \{a_n\} $\in S$.

Finally, \{a_n\} $\in S$, for all $k > 0$. 

References


