Factorizations of the complete graphs into factor of subdiameter two and factors of diameter three

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Abstract. *We search for the minimal number of vertices of the complete graph that can be decomposed into one factor of subdiameter* 2 *and* k *factors of diameter* 3. *We find as follows: exact values for* $k \leq 3$, *upper and lower bounds for small values of* k *and*

$$\lim_{k\to\infty} \frac{\phi(k)}{k} = 2.$$

Key words: *factorization, decomposition, graph*

AMS subject classifications: 05C70

Received June 21, 2002 Accepted October 24, 2002

1. Introduction and main results

Factorizations of graphs into factors with given diameters have been extensively studied. An excellent book [2] about this topic has been written and there exist numerous papers about this subject. The problem of factorization into the factors of equal diameters, where diameter of each factor is at least three has been solved in [5] . The problem of factorization into factors of diameter two is much harder and a lot of attention has been given to that problem. Denote by $f(k)$ the smallest natural number such that the complete graph with n vertices can be factorized into k factors of diameter 2. In [9] , it was proved that

$$f(k) \leq 7k.$$

Then, in [3] , this was improved to

$$f(k) \leq 6k.$$

In [11] , it was proved that this upper bound is quite close to the exact value of $f(k)$ since

$$f(k) \geq 6k - 7, \text{ for } k \geq 664$$

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and in [12] the correct value of $f(k)$ is given for large values of k, namely

$$f(k) = 6k, \text{ for } k \geq 10^{17}.$$

To conclude, factorization of the graph into factors with a small diameter is very interesting.

Factorizations into a small number of factors have been extensively studied as well. The case of factorization of the complete graph into two factors with given diameters was solved completely in [4] ; and the case of factorization of the complete graph into three factors with given diameters was partially solved in [8] . One of the hardest problems was determining the exact value of $f(3)$; this problem was attacked and settled by a computer in [6] and [7]. The development of the fast computer gave a new boost to this area of mathematics, because computers can be of great assistance in attacking some very hard problems where a graph is factorized into a small number of factors. Therefore, it is very interesting to observe factorizations into a small number of factors.

This paper is a kind of a sequel of the paper [10] . In that paper factorization of the complete graph into k factors of diameter 3 and one factor of diameter 2 was observed. In this paper, we observe factorization of the complete graph into k factors of diameter 3 and one factor of diameter 2 in such a way that deletion of any of its edges does not increase its diameter. Hence, the results given here are more complicated than the results given there.

This is the problem we want to model. We have a system of n devices and $k+1$ communicational networks. The first k communicational networks can have diameter 3. The remaining network is privileged and should work even if one of the links fails to work and even then it should have diameter 2.

We want to find out for what values of n and k this is possible.

Let $\phi(k)$ be the smallest number such that $K_{\phi(k)}$ can be factorized into k factors of diameter 3 and one factor of subdiameter 2.

It can be easily proved, that for each $l \geqslant \phi(k)$, K_l can be factorized into k factors of diameter 3 and one factor of subdiameter 2.

In this paper we prove that

$$\begin{aligned}
\phi(1) &= 7 \\
\phi(2) &= 10 \\
\phi(3) &= 13 \\
14 &\leq \phi(4) \leq 16 \\
2k+6 &\leq \phi(k) \leq 3k+3, \ k \geqslant 5 \\
\lim_{k\to\infty} \frac{\phi(k)}{k} &= 2.
\end{aligned}$$

2. Preliminaries and basic definitions

Let G be a graph. By $V(G)$ we denote a set of vertices of G and by $E(G)$ a set of edges of G. By $v(G)$ we denote the number of vertices of G and by $e(G)$ the number of edges in G. By $d_G(x)$ we denote a degree of vertex x (in G), by $\delta(G)$

the minimal degree of G and by $\Delta(G)$ the maximal degree of G. We say that G is k-uniform if $d_G(x) = k$, for each $x \in V(G)$. By $N_G(x)$ we denote the set of neighbors of x (in G). We say that two vertices $x, y \in V(G)$ are adjacent (in G) if $xy \in E(G)$. In this case we also say that x is a neighbor of y (in G).

By $d_G(x,y)$, we denote the distance (in G) of vertices x and y. The subdistance (in G) of vertices x and y, denoted by $\mathrm{subd}_G(x,y)$, is given by

$$\mathrm{subd}_G(x,y) = \max\left\{ \begin{array}{c} d_{G'}(x,y) : G' \text{ is obtained from } G \text{ by} \\ \text{deletion of the single edge} \end{array} \right\}.$$

Define diameter of G and subdiameter of G by

$$\begin{aligned} \mathrm{diam}\, G &= \max_{x,y \in V} \{d_G(x,y)\} \\ \mathrm{subdiam}\, G &= \max_{x,y \in V} \{\mathrm{subd}_G(x,y)\}. \end{aligned}$$

Let $A \subseteq V(G)$. By $G[A]$ we denote the subgraph of G spanned by the set of vertices A. Let $a_1, ..., a_k \in V(G)$. By $G[a_1, ..., a_k]$ we denote $G[\{a_1, ..., a_k\}]$.

Let $A, B \subseteq V(G)$. By $E_G(A,B)$ we denote the set of edges that have one incident vertex in A and the other in B. Let $e_G(A,B) = |E_G(A,B)|$ and $e_G(A) = e(G[A])$.

The factor of graph G is any spanning subgraph of G. We say that the set of factors $F_0, F_1, ..., F_k$ of G is factorization (or decomposition) of G if each edge of G is contained in exactly one of factors $F_0, F_1, ..., F_k$. Then we say that G is factorized (or decomposed) into factors $F_0, F_1, ..., F_k$.

By K_n we denote the complete graph with n vertices and by $K_{n,n}$ the complete bipartite graph with n vertices in each class.

We shall need a simple, but very useful Lemma given in [10] :

Lemma 1. *Let K_n have a factorization with a factor of diameter two. Then any factor (of that factorization) of diameter three has at least n edges.*

3. The value of $\phi(1)$

First, we prove that $\phi(1) \geqslant 7$. Suppose, to the contrary, that K_6 can be factorized into two factors F_0 and F_1 such that $\mathrm{subdiam}\, F_0 = 2$ and $\mathrm{diam}\, F_1 = 3$. It can be easily shown that $\delta(F_0) \geqslant 3$. Let us prove this.

Claim 1. *F_0 is not a 3-uniform graph.*

Proof. If none of the vertices has degree 3, the claim is trivial. Denote by x an arbitrary vertex such that $d_{F_0}(x) = 3$. There are no isolated vertices in $F_0[N_{F_0}(x)]$, hence $e(F_0[N_{F_0}(x)]) \geqslant 2$. Note that each of vertices in $V(F_0) \setminus (N_{F_0}(x) \cup \{x\})$ has at least two neighbors in $N_{F_0}(x)$. Therefore

$$\sum_{v \in N_{F_0}(x)} d_{F_0}(v) \geqslant 3 + 4 + 2 \cdot 2 = 11.$$

Hence, F_0 is not a 3-uniform graph. $\square$

It follows that $e(F_0) \geqslant 10$. From *Lemma 1*, it follows that $e(F_1) \geqslant 6$, but this is in contradiction with $e(K_6) = 15$. Hence, indeed $\phi(1) \geqslant 7$. The opposite inequality follows from the following *Figure*.

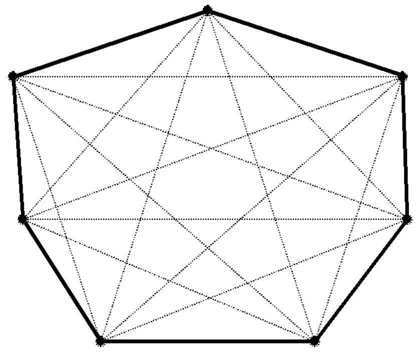

Figure 1. *The edges of F_0 are drawn with a dashed line and of F_1 by a bold line*

4. The value of $\phi(2)$

Let us prove that $\phi(2) \geqslant 10$. Suppose to the contrary that K_9 can be factorized into factors F_0, F_1 and F_2 such that $\operatorname{subdiam} F_0 = 2$ and $\operatorname{diam} F_1 = \operatorname{diam} F_2 = 3$. Note that $e(K_9) = 36$. It can be easily shown that $\delta(F_0) \geqslant 3$, $\delta(F_1) \geqslant 1$ and $\delta(F_2) \geqslant 1$, hence $\Delta(F_0) \leq 6$, $\Delta(F_1) \leq 4$ and $\Delta(F_2) \leq 4$.

Claim 2. $e(F_1) \geqslant 10$ *and* $e(F_2) \geqslant 10$.

Proof. Suppose to the contrary that $e(F_1) < 10$ or $e(F_2) < 10$. Without loss of generality, we may assume that $e(F_1) < 10$. From *Lemma 1*, it follows that $e(F_1) \geqslant 9$, hence $e(F_1) = 9$.

Denote the unique cycle in F_1 by C. Cycle C has less than 8 vertices, because $\operatorname{diam} F_1 \leq 3$. Hence, there are vertices in $V(F_1) \setminus C$. Now, however, it follows that C is of length at most 5. Distinguish three subcases:

3.1) C has three vertices.

We have

$$\sum_{v \in C} d_{F_0}(v) \geqslant 2 \cdot 3 + (9 - 3) = 12,$$

hence $d_{F_0}(v) = 4$, for each $v \in C$. Therefore, F_1 is isomorphic to the graph in *Figure 2*.

Since $\operatorname{subd}_{F_0}(x, y) \leq 2$, it follows that $xc, yc, xd, yd \in E(F_0)$ and since $\operatorname{subd}_{F_0}(x, z) \leq 2$, it follows that $xf, zf, xe, ze \in E(F_0)$. But then $d_{F_2}(x) = 0$, which is a contradiction.

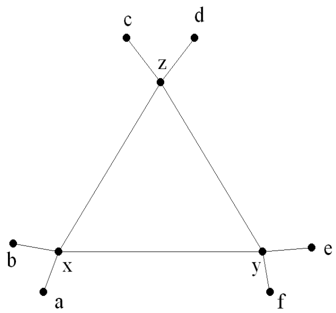

Figure 2.

3.2) C has four vertices.

There are two adjacent vertices in C such that each vertex in $V(F_1) \setminus C$ is adjacent to one of them. Denote these two vertices by x and y. Note that

$$\begin{aligned} 16 &= d_{K_9}(x) + d_{K_9}(y) \\ &= (d_{F_1}(x) + d_{F_1}(y)) + (d_{F_2}(x) + d_{F_2}(y)) + (d_{F_0}(x) + d_{F_0}(y)) \\ &\geqslant (2 + 2 + (9-4)) + (1+1) + (3+3) = 17, \end{aligned}$$

but this is a contradiction.

3.3) C has five vertices.

There are two adjacent vertices in C such that each vertex in $V(F_1) \setminus C$ is adjacent to one of them. Denote these two vertices by x and y. There is a single vertex that is not in $N_{F_1}(x) \cup N_{F_1}(y)$, but this is in contradiction with subdiam $F_0 = 2$.

We have exhausted all the cases and we have proved the claim. □

From this claim, it follows that $e(F_0) \leq 16$. Hence, $\delta(F_0) = 3$. Denote by x any vertex such that $d_{F_0}(x) = 3$. There are no isolated vertices in $F_0[N_{F_0}(x)]$, hence $e(F_0[N_{F_0}(x)]) = 2$. Hence, each vertex in $V(F_0) \setminus (N_{F_0}(x) \cup \{x\})$ has at least two neighbors (in F_0) in $N_{F_0}(x)$, therefore

$$\sum_{v \in V(F_0)} d(v) \geqslant \sum_{v \in N_{F_0}} d(v) + 6 \cdot 3 = (3 + 4 + 2 \cdot 6) + 3 \cdot 6 = 37,$$

but this is in contradiction with $e(F_0) \leq 16$. So, it is indeed, $\phi(2) \geqslant 10$. The opposite inequality follows from the following *Figure*.

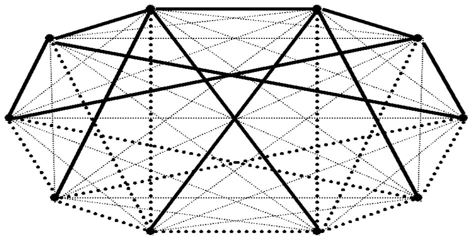

Figure 3. *The edges of factor of subdiameter 2 are drawn with a dashed line and factors of diameter 3 are drawn with a bold line and a bold dotted line, respectively*

5. The value of $\phi(3)$

First, we shall prove that $\phi(3) \geqslant 13$. Suppose, to the contrary that K_{12} can be factorized into factors F_0, F_1, F_2 and F_3, such that subdiam $F_0 = 2$ and diam $F_1 =$ diam $F_2 =$ diam $F_3 = 3$. Note that $\delta(F_1) \geqslant 1$, $\delta(F_2) \geqslant 1$ and $\delta(F_3) \geqslant 1$, hence $\Delta(F_0) \leq 8$.

Claim 3. $e(F_0) \geqslant 29$.

Proof. Distinguish four cases:

1) $\delta(F_0) \leq 2$

It can be easily shown that this case is impossible.

2) $\delta(F_0) = 3$.

Distinguish two subcases:

2.1) There are two vertices x_1 and x_2 such that $d(x_1) = d(x_2) = 3$ and $|N_{F_0}(x_1) \cap N_{F_0}(x_2)| = 2$.

Denote $N_{F_0}(x_1) \cap N_{F_0}(x_2) = \{y_1, y_2\}$, $N_{F_0}(x_1) \setminus N_{F_0}(x_2) = \{u\}$ and $N_{F_0}(x_2) \setminus N_{F_0}(x_1) = \{v\}$. Note that $d_{F_0[y_1,y_2,u,v]}(u) \geqslant 2$, because $\mathrm{subd}_{F_0}(x_2, u) \leq 2$. Also, note that $d_{F_0[y_1,y_2,u,v]}(v) \geqslant 2$, because $\mathrm{subd}_{F_0}(x_1, v) \leq 2$. A simple analysis shows that $d_{F_0[y_1,y_2,u,v]}(y_1)+d_{F_0[y_1,y_2,u,v]}(y_2) \geqslant 3$, $d_{F_0[y_1,y_2,u,v]}(y_1) \geqslant 1$ and $d_{F_0[y_1,y_2,u,v]}(y_2) \geqslant 1$. Therefore, $e(F_0[y_1, y_2, u, v]) \geqslant 4$. Denote

$$\begin{aligned}
S &= \{x_1, x_2, y_1, y_2, u, v\} \\
A &= \{x \in V(F_0) \setminus S : \{y_1, y_2, u, v\} = N_{F_0}(x) \cap \{y_1, y_2, u, v\}\} \\
B_1 &= \{x \in V(F_0) \setminus S : \{y_1, y_2, u\} = N_{F_0}(x) \cap \{y_1, y_2, u, v\}\} \\
B_2 &= \{x \in V(F_0) \setminus S : \{y_1, y_2, v\} = N_{F_0}(x) \cap \{y_1, y_2, u, v\}\} \\
C &= \{x \in V(F_0) \setminus S : \{y_1, y_2\} = N_{F_0}(x) \cap \{y_1, y_2, u, v\}\} \\
D_1 &= \{x \in V(F_0) \setminus S : \{y_1, u, v\} = N_{F_0}(x) \cap \{y_1, y_2, u, v\}\} \\
D_2 &= \{x \in V(F_0) \setminus S : \{y_2, u, v\} = N_{F_0}(x) \cap \{y_1, y_2, u, v\}\} \\
a &= |A|,\ b_1 = |B_1|,\ b_2 = |B_2|,\ c = |C|,\ d_1 = |D_1|,\ d_2 = |D_2|.
\end{aligned}$$

Note that $A \cup B_1 \cup B_2 \cup C \cup D_1 \cup D_2 = V(F_0) \setminus \{y_1, y_2, u, v\}$ and that A, B_1, B_2, C, D_1 and D_2 are pairwise disjoint sets, hence

$$a + b_1 + b_2 + c + d_1 + d_2 = 6. \tag{1}$$

Since, $\Delta(F_0) \leq 8$, it follows that

$$\begin{aligned} a + b_1 + b_2 + c + d_1 &\leq 5 & (2)\\ a + b_1 + b_2 + c + d_2 &\leq 5 & (3)\\ 2a + 2b_1 + 2b_2 + 2c + d_1 + d_2 &\leq 9. & (4) \end{aligned}$$

From (1) and (2), it follows that $d_2 \geqslant 1$, from (1) and (3), it follows that $d_1 \geqslant 1$ and from (1) and (4), it follows that $d_1 + d_2 \geqslant 3$. Without loss of generality, we may assume that $d_1 \geqslant d_2$, hence $d_1 \geqslant 2$.

We have

$$e_{F_0}(S, V(F_0) \setminus S) = 4a + 3b_1 + 3b_2 + 2c + 3d_1 + 3d_2.$$

Distinguish two subsubcases:

2.1.1) $c \neq 0$.

Note that $\operatorname{subdiam} F_0 = 2$ implies that all vertices in C are connected with all vertices in D_1 in the graph $F_0[V(F_0) \setminus S]$ and, also, that all vertices in C are connected with all vertices in D_2 in the graph $F_0[V(F_0) \setminus S]$, hence

$$e(F_0[V(F_0) \setminus S]) \geqslant c + d_1 + d_2 - 1.$$

Therefore,

$$\begin{aligned} e(F_0) &\geqslant 6 + e(F_0[y_1, y_2, u, v]) + (4a + 3b_1 + 3b_2 + 2c + 3d_1 + 3d_2)\\ &\quad + (c + d_1 + d_2 - 1)\\ &\geqslant 9 + 3(a + b_1 + b_2 + c + d_1 + d_2) + (d_1 + d_2) \geqslant 30. \end{aligned}$$

2.1.2) $c = 0$.

Suppose to the contrary that $e(F_0) \leq 28$. Note that

$$\begin{aligned} e(F_0) &\geqslant 6 + e(F_0[y_1, y_2, u, v]) + (4a + 3b_1 + 3b_2 + 3d_1 + 3d_2)\\ &\geqslant 10 + 3(a + b_1 + b_2 + d_1 + d_2) = 28, \end{aligned}$$

hence $e(F_0) = 28$, $a = 0$, $e(F_0[y_1, y_2, u, v]) = 4$ and $e(F_0[V(F_0) \setminus S]) = 0$. Note, that

$$d_{F_0}(y_1) + d_{F_0}(y_2) + d_{F_0}(u) + d_{F_0}(v) = 6 + 8 + 3 \cdot 6 = 32.$$

From $\Delta(F_0) = 8$, it follows that

$$d_{F_0}(y_1) = d_{F_0}(y_2) = d_{F_0}(u) = d_{F_0}(v) = 8,$$

hence

$$d_{F_i}(y_1) = d_{F_i}(y_2) = d_{F_i}(u) = d_{F_i}(v) = 1,\ i = 1, 2, 3.$$

Since $F_0[y_1, y_2, u, v]$ is not a complete graph, there are two adjacent vertices of degree 1 in one of the factors F_1, F_2 and F_3 and that is a contradiction.

2.2) $\delta(F_0) = 3$ and every two vertices of degree 3 are adjacent or have the same set of neighbors.

Denote by x_1 an arbitrary vertex such that $d_{F_0}(x_1) = 3$. Denote its neighbors by $Y = \{y_1, y_2, y_3\}$ and denote by $X = \{x_1, x_2, ..., x_p\}$ the set of vertices that have the same set of neighbors as x_1. Also, denote $Z = V(F_0) \setminus (X \cup Y) = \{z_1, ..., z_{9-p}\}$ Note that $e(F_0[Y]) \geqslant 2$ and $e_{F_0}(X, Y) = 3p$, hence

$$d_{F_0}(y_1) + d_{F_0}(y_2) + d_{F_0}(y_3) \geqslant 4 + 3p + 2 \cdot (9 - p). \tag{5}$$

We also have

$$\sum_{i=1}^{p} d_{F_0}(x_i) = 3p \tag{6}$$

$$\sum_{i=1}^{9-p} d_{F_0}(z_i) \geqslant 4 \cdot (9 - p). \tag{7}$$

Adding up (5), (6) and (7), we get

$$\sum_{v \in V(F_0)} d_{F_0}(v) \geqslant 58,$$

hence $e(F_0) \geqslant 29$.

3) $\delta(F_0) = 4$. Suppose to the contrary that $e(F_0) \leq 28$. Distinguish two cases:

3.1) There is a vertex x of degree 4 such that $d_{F_0}(y) \leq 7$, for each $y \in N_{F_0}(x)$.

Denote $S = V(F_0) \setminus (\{x\} \cup N_{F_0}(x))$. From $\mathrm{subd}_{F_0}(x, y) \leq 2$, for each $y \in V$, it follows that none of the vertices in $N_{F_0}(x)$ can be an isolated vertex in $F_0[N_{F_0}(x)]$ and that each vertex in S has at least two neighbors (in F_0) in $x \cup N_{F_0}(x)$. Therefore

$$e(F_0[x \cup N_{F_0}]) \geqslant 6 \tag{8}$$

$$e_{F_0}(N_{F_0}(x), S) \geqslant 14 \tag{9}$$

$$|N_{F_0}(v) \cap S| \leq 5, \text{ for each } v \in N_{F_0}(x). \tag{10}$$

From (8), it follows that

$$e_{F_0}(S) + e_{F_0}(N_{F_0}(x), S) \leq 22$$

or equivalently

$$\sum_{v \in N_{F_0}(x)} |N_{F_0}(v) \cap S| + \frac{1}{2} \cdot \sum_{v \in S} d_{F_0[S]}(v) \leq 22. \tag{11}$$

Suppose that $e_{F_0}(N_{F_0}(x), S) \geqslant 18$, then $\sum_{v \in N_{F_0}(x)} d(x) \geqslant 8 + 18 = 26$, hence $\sum_{v \in V(F_0)} d(x) \geqslant 58$, which is in contradiction with $e(F_0) \leq 28$. Therefore,

$$14 \leq e_{F_0}(N_{F_0}(x), S) \leq 17$$

or equivalently

$$14 \leq \sum_{v \in N_{F_0}(x)} |N_{F_0}(v) \cap S| \leq 17. \tag{12}$$

For each two vertices $s_1, s_2 \in S$, there are two disjoint paths of length at most 2 connecting s_1 and s_2, hence

$$\sum_{v \in N_{F_0}(x)} \binom{|N_{F_0}(v) \cap S|}{2} + e_{F_0}(S) + \sum_{v \in S} \binom{d_{F_0[S]}(v)}{2} \geqslant 2 \cdot \binom{7}{2}. \tag{13}$$

From $\delta(F_0) \geqslant 4$ and $|N_{F_0}(v) \cap N_{F_0}(x)| \geqslant 2$, for each $v \in S$, it follows that

$$\sum_{v \in S} \max\left\{0, 2 - d_{F_0[S]}(v)\right\} \leq e_{F_0}(N_{F_0}(x), S) - 14$$

or equivalently

$$\sum_{v \in S} \max\left\{0, 2 - d_{F_0[S]}(v)\right\} \leq \sum_{v \in N_{F_0}(x)} |N_{F_0}(v) \cap S| - 14. \tag{14}$$

Also, from $\delta(F_0) \geqslant 4$, follows that

$$\sum_{v \in N_{F_0}(x)} \max\{3 - |N_{F_0}(v) \cap S|\} \quad \leq \quad 2 \cdot e(F_0[N_{F_0}(x)]) \tag{15}$$

$$\max_{v \in N_{F_0}(x)} \{0, 3 - |N_{F_0}(v) \cap S|\} \quad \leq \quad e(F_0[N_{F_0}(x)]) \tag{16}$$

and that

$$\max_{v \in N_{F_0}(x)} \{0, 3 - |N_{F_0}(v) \cap S|\} \leq e(F_0[N_{F_0}(x)]) + 1 \text{ or } e(F_0[N_{F_0}(x)]) \geqslant 3. \tag{17}$$

Now let us observe multisets $Deg_1 = \{|N_{F_0}(v) \cap S| : v \in N_{F_0}(x)\}$ and $Deg_2 = \left\{d_{F_0[S]}(v) : V \in S\right\}$. A tedious check shows that relations $(10) - (17)$ are satisfied only if

$$Deg_1 \;=\; \{2, 2, 5, 5\} \tag{18}$$

$$Deg_2 \;=\; \{2, 2, 2, 2, 2, 2, 4\}. \tag{19}$$

In this case relation (13) is actually an equality, hence there are exactly two paths of length at most 2 connecting each two vertices in S. From (19), it follows that $F_0[S]$ is one of the following two graphs:

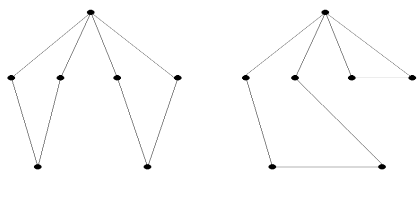

Figure 4.

Denote by s_1 a vertex in S such that $d_{F_0[S]}(s_1) = 4$. There are, in both cases, exactly 8 paths of length at most 2 (in $F_0[S]$) starting from the vertex s_1. Therefore, there are exactly four paths of the form s_1uv, where $u \in N_{F_0}(x)$ and $v \in S$, hence

$$\sum_{w \in N_{F_0}(x) \cap N_{F_0}(s_1)} |N_{F_0}(w) - 1| = 4. \tag{20}$$

Recall that

$$|N_{F_0}(x) \cap N_{F_0}(s_1)| \geqslant 2. \tag{21}$$

Relations (18), (20) and (21) are inconsistent, hence the claim is proved in this case.

3.2) For each vertex $x \in V(F_0)$ such that $d_{F_0}(x) = 4$, there is a vertex y adjacent to x such that $d_{F_0}(y) = 8$.

Distinguish three subcases:

3.2.1) There are vertices x, y and z such that $d_{F_0}(x) = 4$, $d_{F_0}(y) = 8$, $d_{F_0}(z) = 4$, $xy \in E(F_0)$, xz, $yz \notin E(F_0)$.

There are no isolated vertices in $F_0[N_{F_0}(z)]$, hence $e(F_0[N_{F_0}(z)]) \geqslant 2$. Each vertex in $V(F_0) \setminus (N_{F_0}(z) \cup \{z\})$ has at least two neighbors (in F_0) in $N_{F_0}(z)$, therefore

$$\begin{aligned} \sum_{v \in V(F_0)} d_{F_0}(v) &= \sum_{v \in N_{F_0}(z)} d_{F_0}(v) + d(y) + \sum_{v \in V(F_0) \setminus \left(N_{F_0}(z) \cup \{y\}\right)} d_{F_0}(v) \\ &\leq (4 + 2 \cdot 2 + 7 \cdot 2) + 8 + 7 \cdot 4 = 58, \end{aligned}$$

but this is in contradiction with $e(F_0) \leq 28$, so the claim is proved in this case.

3.2.2) There are vertices x, y and z such that $d_{F_0}(x) = 4$, $d_{F_0}(y) = 8$, $d_{F_0}(z) = 5$, $xy \in E(F_0)$, xz, $yz \notin E(F_0)$.

There are no isolated vertices in $F_0[N_{F_0}(z)]$, hence $e(F_0[N_{F_0}(z)]) \geqslant 3$. Each vertex in $V(F_0) \setminus (N_{F_0}(z) \cup \{z\})$ has at least two neighbors (in F_0) in $N_{F_0}(z)$ and vertex y has at least three neighbors (in F_0) in $N_{F_0}(z)$, hence

$$\begin{aligned} \sum_{v \in V(F_0)} d_{F_0}(v) &= \sum_{v \in N_{F_0}(z)} d_{F_0}(v) + d(y) + d(z) + \sum_{v \in V(F_0) \setminus \left(N_{F_0}(z) \cup \{y,z\}\right)} d_{F_0}(v) \\ &\leq (5 + 2 \cdot 3 + 3 + 2 \cdot 5) + 8 + 5 + 5 \cdot 4 = 57. \end{aligned}$$

This is in contradiction with $e(F_0) \leq 28$, so the claim is proved in this case.

3.2.3) For each two adjacent vertices x and y such that $d_{F_0}(x) = 4$ and $d_{F_0}(y) = 8$ and each vertex z such that $xz, yz \notin E(F_0)$, we have $d_{F_0}(z) \geqslant 6$.

Let x be an arbitrary vertex such that $d_{F_0}(x) = 4$. Denote

$$\begin{aligned} S &= V(F_0) \setminus (N_{F_0}(x) \cup \{x\}) \\ p &= e_{F_0}(N_{F_0}(x)) \\ q &= e_{F_0}(N_{F_0}(x), S) \\ r &= e_{F_0}(S). \end{aligned}$$

Note that

$$4+p+q+r \leq 28 \tag{22}$$
$$p \geqslant 2 \tag{23}$$
$$q \geqslant 14 \tag{24}$$
$$q+2r = \sum_{v\in S} d_{F_0}(v) \geqslant 6\cdot 4+6. \tag{25}$$

Solving $(22)-(25)$, we get $p=2$, $q=14$, $r=8$ and all inequalities $(22)-(25)$ are, in fact, equalities. Hence,

$$\begin{aligned} d_{F_0}(v) &= 4, \text{ for each } x\in S\setminus\{z\} \\ \sum_{v\in N_{F_0}(x)} d_{F_0}(x) &= 14+2\cdot 2+4=22. \end{aligned}$$

Since each vertex of degree 4 is adjacent to a vertex of degree 8, we may conclude that F_0 is the supergraph of the graph given in the following *Figure*:

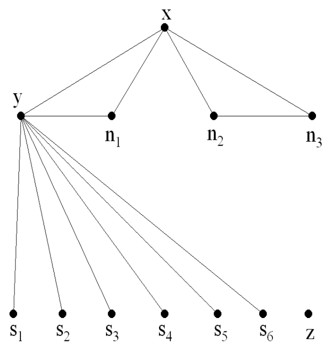

Figure 5.

where $d_{F_0}(n_1)=d_{F_0}(s_1)=d_{F_0}(s_2)=...=d_{F_0}(s_6)=4$ and $d_{F_0}(n_2)=d_{F_0}(n_3)=5$. Since x was an arbitrary vertex of degree four, it follows that each vertex of degree 4 has to be adjacent to two vertices of degree 5, but there are only two vertices of degree five and eight vertices of degree 4. This is a contradiction, so the claim is proved in this case.

4) $\delta(F_0)\geqslant 5$.

The claim is trivial in this case.

Hence, we have exhausted all the cases and we have proved our claim. $\square$

Denote by G_{12} the graph in the following *Figure.*

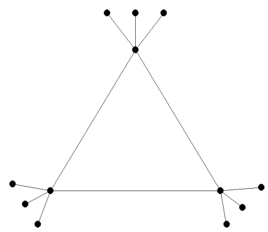

Figure 6.

Let us prove

Claim 4. *If* $e(F_i) \leq 12$, *then* $F_i \cong G_{12}$, *for each* $i = 1, 2, 3$.

Proof.

Suppose the contrary. Without loss of generality, we may assume that $e(F_1) \leq 12$ and $F_1 \ncong G_{12}$. Note that F_1 has at most one cycle. The length of that cycle (if it exists) is at most 5. Also, note that $\Delta(F_1) \leq 11 - 3 - 1 - 1 = 6$. Distinguish the following cases:

1) There are no cycles in F_1.

Then $e(F_1) < 12$, but this is in contradiction with *Lemma 1.*

2) There is a triangle in F_1.

Denote vertices of triangle by c_1, c_2 and c_3 and denote $C = \{c_1, c_2, c_3\}$. Distinguish two subcases:

2.1) Only one of the vertices in C has neighbors that are not in C.

From $\Delta(F_1) \leq 6$, it follows that F_1 is given by the following *Figure.*

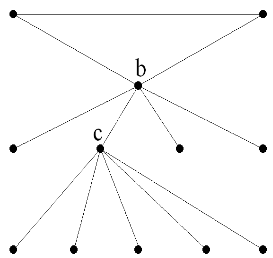

Figure 7.

But now, $d_{F_0}(b, c) > 2$ and this is a contradiction.

2.2) More than one vertex in C has neighbors that are not in C.

It follows that all vertices in $V(F_1) \setminus C$ are adjacent to exactly one vertex in C, because F_1 contains a single cycle and $\operatorname{diam} F_1 = 3$. If there is c_i, $i = 1, ..., 3$ such that $d(c_i) \leq 3$, then $\operatorname{subd}_{F_0}(c_j, c_k) > 2$, where $\{i, j, k\} = \{1, 2, 3\}$, which is a contradiction. Therefore, without loss of generality, we may assume that $4 \leq$

$d(c_1) \leq d(c_2) \leq d(c_3)$. Note that $d(c_1) + d(c_2) + d(c_3) = 15$. Distinguish three subsubcases:

2.2.1) $d(c_1) = 4$, $d(c_2) = 4$, $d(c_3) = 7$.

From $\mathrm{subd}_{F_0}(c_1, c_3) \leq 2$, it follows that

$$\begin{aligned} |N_{F_0}(c_1) \cap (N_{F_1}(c_2) \setminus \{c_1, c_3\})| &\geqslant 2 \quad (1) \\ |N_{F_0}(c_3) \cap (N_{F_1}(c_2) \setminus \{c_1, c_3\})| &\geqslant 2. \quad (2) \end{aligned}$$

Analogously, from $\mathrm{subd}_{F_0}(c_1, c_2) \leq 2$, it follows that

$$\begin{aligned} |N_{F_0}(c_1) \cap (N_{F_1}(c_3) \setminus \{c_1, c_2\})| &\geqslant 2 \quad (3) \\ |N_{F_0}(c_2) \cap (N_{F_1}(c_3) \setminus \{c_1, c_2\})| &\geqslant 2. \quad (4) \end{aligned}$$

Therefore, $d_{F_0}(c_3) + d_{F_1}(c_3) = 11$, hence $d_{F_2}(c_3) + d_{F_3}(c_3) = 0$ and this is a contradiction.

2.2.2) $d(c_1) = 4$, $d(c_2) = 5$, $d(c_3) = 6$.

Analogously, as in the previous case, we have relations $(1) - (4)$. Therefore, $d_{F_0}(c_3) + d_{F_1}(c_3) = 10$, hence $d_{F_2}(c_3) + d_{F_3}(c_3) = 1$ and this is a contradiction.

2.2.3) $d(c_1) = 5$, $d(c_2) = 5$, $d(c_3) = 5$.

In this case $F_1 \cong G_{12}$.

3) There is a cycle of length at least 4.

Denote the set of vertices of the cycle by C. All vertices in $V(F_1) \setminus C$ are adjacent (in F_1) to exactly one vertex in C. There are two adjacent vertices $x, y \in C$ such that each vertex in $V(F_1) \setminus C$ is adjacent to at least one of them. Note that $|V(F_1) \setminus (N_{F_1}(x) \cup N_{F_1}(y))| \leq 1$, because the length of the unique cycle is at most 5. Therefore, $\mathrm{subd}_{F_0}(x, y) > 2$, which is a contradiction.

Hence, all the cases are exhausted and the claim is proved. □

Without loss of generality, we may assume that $e(F_1) \leq e(F_2) \leq e(F_3)$. Since $e(F_0) \geqslant 29$, it follows that $e(F_1) + e(F_2) + e(F_3) \leq 37$, hence $e(F_1) = e(F_2) = 12$. Therefore $F_1 \cong F_2 \cong G_{12}$. Denote the set of vertices of triangle in F_1 by $T_1 = \{t_{11}, t_{12}, t_{13}\}$ and denote the set of vertices of triangle in F_2 by $T_2 = \{t_{21}, t_{22}, t_{23}\}$. Let us prove

Claim 5. *Let $\{i, j\} \in \{1, 2\}$ and $k \in \{1, 2, 3\}$. Then $d_{F_i}(t_{ik}) = 5$, $d_{F_j}(t_{ik}) = 1$, $d_{F_3}(t_{ik}) = 1$ and $d_{F_0}(t_{ik}) = 4$. For each vertex $v \in V(F_0)$ adjacent (in F_0) to t_{ik} we have $d_{F_0[\{v\} \cup T_i]}(v) > 1$.*

Proof. It is obvious that $d_{F_i}(t_{ik}) = 5$. Denote $\{k, l, m\} = \{1, 2, 3\}$. Since $\mathrm{subd}_{F_0}(t_{ik}, t_{il}) \leq 2$, it follows that

$$|N_{F_0}(t_{ik}) \cap N_{F_0}(t_{il}) \cap (N_{F_i}(t_{im}) \setminus \{t_{ik}, t_{il}\})| \geqslant 2. \quad (1)$$

Analogously, since $\mathrm{subd}_{F_0}(t_{ik}, t_{im}) \leq 2$, it follows that

$$|N_{F_0}(t_{ik}) \cap N_{F_0}(t_{im}) \cap (N_{F_i}(t_{il}) \setminus \{t_{ik}, t_{im}\})| \geqslant 2. \quad (2)$$

This implies that $d_{F_0}(t_{ik}) \geqslant 4$. Since $d_{F_j}(t_{ik}) \geqslant 1$ and $d_{F_3}(t_{ik}) \geqslant 1$, it follows that $d_{F_0}(t_{ik}) = 4$, $d_{F_j}(t_{ik}) = 1$ and $d_{F_3}(t_{ik}) = 1$. Therefore, inequalities (1) and (2) are in fact equalities and

$$\begin{aligned} N_{F_0}(t_{ik}) \cap (N_{F_i}(t_{im}) \setminus \{t_{ik}, t_{il}\}) &\subseteq N_{F_0}(t_{il}) \\ N_{F_0}(t_{ik}) \cap (N_{F_i}(t_{il}) \setminus \{t_{ik}, t_{im}\}) &\subseteq N_{F_0}(t_{im}). \end{aligned}$$

Since

$$N_{F_0}(t_{ik}) \subseteq (N_{F_i}(t_{il}) \setminus \{t_{ik}, t_{im}\}) \cup (N_{F_i}(t_{im}) \setminus \{t_{ik}, t_{il}\}),$$

it follows that

$$N_{F_0}(t_{ik}) \subseteq N_{F_0}(t_{il}) \cup N_{F_0}(t_{im}),$$

which proves the claim. □

From the last claim, it easily follows that there are no vertices of degree one in $F_0[T_1 \cup T_2]$. Note that $e_{F_1}(T_1, T_2) \geqslant 3$ and $e_{F_2}(T_1, T_2) \geqslant 3$, hence $e_{F_0}(T_1, T_2) \leq 3$. Distinguish two cases:

1) $e_{F_0}(T_1, T_2) > 0$.

Note that $F_0[T_1 \cup T_2]$ is the spanning subgraph of $K_{3,3}$. But each spanning subgraph of $K_{3,3}$ with at least one edge and at most three edges has at least one vertex of degree 1 and this is a contradiction.

2) $e_{F_0}(T_1, T_2) = 0$.

Denote $S = V(K_{12}) \setminus (T_1 \cup T_2)$. Since $e_{F_0}(T_1, T_2) = 0$, it follows that

$$|N_{F_0}(t_{ij}) \cap S| = 4, \ i = 1, 2; \ j = 1, 2, 3,$$

hence $e_{F_0}(T_1 \cup T_2, S) = 24$. Also, we have

$$e_{F_i}(T_i, \{s\}) = 1, \ \ s \in S, \ i = 1, 2,$$

hence $e_{F_1}(T_1, S) = 6$ and $e_{F_2}(T_2, S) = 6$. Therefore, $e_{F_3}(T_1 \cup T_2, S) = 6 \cdot 6 - 24 - 6 - 6 = 0$. But, then F_3 is disconnected and this is a contradiction. Hence, we have proved that $\phi(3) \geqslant 12$.

The opposite inequality follows from the following *Figure*:

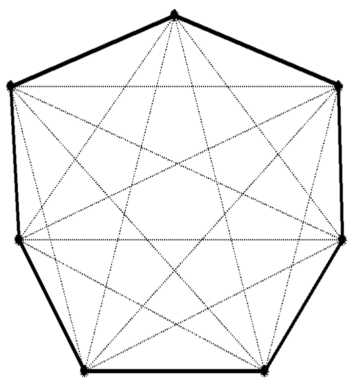

Figure 8. *The edges of the factor of subdiameter 2 are drawn with a dashed line and edges of factors of diameter 3 are drawn with a bold, a bold dashed and a bold dotted line, respectively*

Denote the factorization in this *Figure* by $\mathcal{D}'$.

6. The lower and upper bounds for the values of ϕ

First, we give a weaker lower bound that we need to prove the stronger one:

Lemma 2. *For each $k \geqslant 4$, we have $\phi(k) \geqslant 2k+3$.*

Proof. Let K_v be factorized into k factors of diameter 3 and one factor of subdiameter 2. From *Lemma 1*, it follows that each of the factors of diameter 3 has at least v edges, hence

$$k \cdot v + v \leq \binom{v}{2}.$$

Solving this, we get $v \geqslant 2k+3$, so the claim is proved. □

Lemma 3. *For each $k \geqslant 4$, we have $\phi(k) \geqslant 2k+6$.*

Proof. Let K_v be factorized into k factors of diameter 3 and one factor of subdiameter 2. As in the previous Lemma, each of the factors of diameter 3 has at least v edges. Now, we shall estimate the number of edges of F_0. Note that $\delta(F_0) \geqslant 3$. Let x be a vertex such that $d_{F_0}(x) = \delta(F_0)$. There are no isolated vertices in $F_0[N_{F_0}(x)]$ and each vertex in $V(F_0) \setminus (N_{F_0}(x) \cup \{x\})$ has at least two neighbors (in F_0) in $N_{F_0}(x)$, hence

$$\begin{aligned}\sum_{v\in V(F_0)} d(v) &= \sum_{v\in N_{F_0}(x)} d(v) + \sum_{v\in V(F_0)\setminus N_{F_0}(x)} d(v)\\ &= \left(\delta(F_0) + 2\cdot\left\lceil\frac{\delta(F_0)}{2}\right\rceil + 2\cdot(v-\delta(F_0)-1)\right)\\ &\quad + (v-\delta(F_0))\cdot\delta(F_0).\end{aligned}$$

Therefore,

$$e(F_0) \geqslant \frac{v-\delta(F_0)+1}{2}\cdot\delta(F_0) + \left\lceil\frac{\delta(F_0)}{2}\right\rceil + (v-\delta(F_0)-1).$$

Also note that

$$e(F_0) \geqslant \frac{\delta(F_0)\cdot v}{2}.$$

Therefore,

$$e(F_0) \geqslant \min_{\delta\geqslant 3}\left\{\max\left\{\begin{array}{c}\frac{v-\delta(F_0)+1}{2}\cdot\delta(F_0) + \left\lceil\frac{\delta(F_0)}{2}\right\rceil + (v-\delta(F_0)-1),\\ \frac{\delta(F_0)\cdot v}{2}\end{array}\right\}\right\}.$$

From the last Lemma, it follows that $v \geqslant 11$, hence from the last expression, it follows that

$$e(F_0) \geqslant \frac{5v-10}{2}.$$

Therefore,

$$\frac{5v-10}{2} + kv \leq \binom{v}{2} \text{ and } v \geqslant 11.$$

Solving the last inequalities, we get

$$v \geqslant \max\left\{11, \frac{(2k+6)+\sqrt{(2k+6)^2-40}}{2}\right\} > 2k+5,$$

which proves the claim. □

Now, we shall factorize K_{18} into five factors of diameter 3 and one factor of subdiameter 2. We shall denote this factorization by $\mathcal{D}$, its factors of diameter 2 by $F_{\mathcal{D},1}, F_{\mathcal{D},2}, ..., F_{\mathcal{D},5}$ and its factor of subdiameter 2 by $F_{\mathcal{D},0}$. Denote $V(K_{18}) = \{v_1, ..., v_{18}\}$. factorization $\mathcal{D}$ is given by the following table

099199299399499599
909919929939949959
990991992993994995
199011129139114599
919101929939941151
991110211311994519
299192022239499225
929221202939224959
992991220322492952
399193293033499353
939331332303349959
993991992330433539
499199424434044995
949149929943404445
994414942993440594
599515299395945055
959951255553949505
995919592399554550

where $T(i,j) = k$ denotes $v_i v_j \in F_{\mathcal{D},k}$, $k = 1, ..., 5$ and $T(i,j) = 9$ denotes $v_i v_j \in F_{\mathcal{D},0}$. A simple check shows that this factorization has required properties. Now, we can prove

Lemma 4. *Let* $k \geqslant 5$. *Then* $\phi(k) \leq 3 \cdot k + 3$.

Proof. We shall explicitly give a factorization of K_{3k+3} into factors $F_1, ..., F_k$ of diameter 3 and a factor F_0 of subdiameter 2. Denote $V(K_{3k+3}) = \{v_1, v_2, ..., v_{3k+3}\}$.

The edges of factor F_i, $1 \leq i \leq 4$, are given by:

1) $v_a v_b$ such that $v_a v_b \in F_{\mathcal{D},i}$.

2) $v_a v_b$ such that $4 \leq a \leq 15$, $b \geqslant 19$ and there is c such that $16 \leq c \leq 18$, $v_a v_c \in F_{\mathcal{D},i}$ and $b \equiv c \bmod 3$.

The edges of the factor F_i, $i \geqslant 5$ are given by

1) $v_a v_b$ such that $a \leq 15$, $3 \cdot i + 1 \leq b \leq 3 \cdot i + 3$ and there is c such that $16 \leq c \leq 18$, $v_a v_c \in F_{\mathcal{D},5}$ and $b = c \bmod 3$.

2) $v_a v_b$ such that $3 \cdot i + 1 \leq a \leq 3 \cdot i + 3$, $b \geqslant 3i + 4$, and $a \equiv b \bmod 3$.

3) $v_a v_b$ such that $3 \cdot i + 1 \leq a \leq 3 \cdot i + 3$, $16 \leq b \leq 3 \cdot i$ and $a \not\equiv b \bmod 3$.

4) $v_a v_b$ such that $3 \cdot i + 1 \leq a, b \leq 3 \cdot i + 3$ and $a \neq b$.

The remaining edges are the edges of factor F_0.

Let us prove that $\operatorname{diam} F_i = 3$, $1 \leq i \leq k$. Vertices v_{3i+1}, v_{3i+2} and $v_{3 \cdot i+3}$ form a triangle (in F_i) and every other vertex is adjacent (in F_i) to at least one of the vertices v_{3i}, v_{3i+1} and $v_{3 \cdot i+2}$, hence $\operatorname{diam} F_i = 3$, $i \leq 4$.

What remains to be proved is that $\operatorname{subdiam} F_0 = 2$. We need to prove that for each two vertices v_x and v_y there are two paths of length at most 2 connecting them. Without loss of generality, we may assume that $x < y$. Distinguish five cases:

1) $x \leq 18$, $y \leq 18$.

Note that $F_0[v_1, ..., v_{18}] \cong F_{\mathcal{D},0}[v_1, ..., v_{18}]$, hence there are two paths of length 2 that connect v_x and v_y in $F_{\mathcal{D},0}[v_1, ..., v_{18}]$.

2) $x \leq 15$, $y > 18$.

Note that

$$F_0\left[v_1, ..., v_{15}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+1}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+2}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+3}\right] \cong F_{\mathcal{D},0}[v_1, ..., v_{18}],$$

hence there are two paths of length at most 2 that connect v_x and v_y in $F_0\left[v_1, ..., v_{15}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+1}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+2}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+3}\right]$.

3) $x > 15, y > 15$, $x \not\equiv y \bmod 3$.

Let $16 \leq z \leq 3 \cdot k + 3$ be any number such that $z \not\equiv x \bmod 3$ and $z \not\equiv y \bmod 3$. Note that

$$F_0[v_1, ..., v_{15}, v_x, v_y, v_z] \cong F_{\mathcal{D},0}[v_1, ..., v_{18}],$$

hence there are two paths of length at most 2 that connect v_x and v_y in $F_{\mathcal{D},0}\left[v_1, ..., v_{15}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+1}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+2}, v_{3\cdot\left\lfloor \frac{y-1}{3}\right\rfloor+3}\right]$.

4) $x > 15$, $y > 15$, $x \equiv y \bmod 3$.

Let $1 \leq p, q \leq 3$ be two numbers such that $p \not\equiv x \bmod 3$ and $q \not\equiv x \bmod 3$. Note that xpy and xqy are paths in F_0.

Therefore, we have exhausted all the cases and we have proved the theorem. □

Using a similar technique to that in the last lemma and factorization $\mathcal{D}'$, it can be proved that $\phi(4) \leq 16$.

Summarizing our results, we get:

Theorem 1.

$$\begin{array}{rclrcl} \phi(1) & = & 7 & \phi(2) & = & 10 \\ \phi(3) & = & 13 & 14 & \leq & \phi(4) \leq 16 \end{array}$$

$$2k + 6 \leq \phi(k) \leq 3k + 3, \; k \geqslant 5. \tag{$*$}$$

The relation $(*)$ gives rather good bounds for small values of k. These bounds are not so good when k is large. Note that, from $(*)$, we can conclude only

$$2 \leq \underline{\lim}\frac{\phi(k)}{k} \leq \overline{\lim}\frac{\phi(k)}{k} \leq 3.$$

In fact, we have

$$\lim_{k\to\infty} \frac{\phi(k)}{k} = 2.$$

Let us prove:

Lemma 5. *If K_v can be factorized into k factors of diameter 3 and two factors of diameter 2, then it can be factorized into k factors of diameter 3 and one factor of subdiameter 2.*

Proof. Let K_v be factorized into k factors $F_1, ..., F_k$ of diameter 3 and two factors G_1 and G_2 of diameter 2. Denote by F_0 a graph such that $V(F_0) = V(K_v)$

and $E(F_0) = E(G_1) \cup E(G_2)$. Factors $F_0, F_1, ..., F_k$ form a factorization with the required properties. □

Denote by $f\left(\underbrace{2,...,2}_{k\text{-times}},\underbrace{3,...,3}_{p\text{-times}}\right)$ the smallest number v such that K_v can be factorized into k factors of diameter 2 and p factors of diameter 3. Note that from the last Lemma, it follows

$$\phi(k) \leq f\left(2,2,\underbrace{3,...,3}_{p\text{-times}}\right).$$

In [10], it is proved that

Theorem 2. $\lim_{k\to\infty} \frac{f\left(\underbrace{2,2,...,2}_{p\text{-}times},\underbrace{3,3...,3}_{k\text{-}times}\right)}{k} = 2,$ *where p is a fixed natural number.*

Combining the last Lemma and the last Theorem (taking $p = 2$), we get

Theorem 3.

$$\lim_{k\to\infty} \frac{\phi(k)}{k} = 2.$$

More precisely, by the construction analogous to the one given in [10], it follows that

Proposition 1. *For sufficiently large $k \in \mathbb{N}$, we have*

$$\phi(k) \leq 2k + 5 \cdot \left\lceil \sqrt{k} \right\rceil. \tag{3}$$

Proof. Let t be the smallest natural number such that

$$\binom{2t-1}{t-1} \geqslant k.$$

We will construct a factorization of K_n, $n = 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t + 2$, into factors $F_0, F_1, F_2, ..., F_k$ such that $\operatorname{subdiam}(F_0) = 2$ and $\operatorname{diam}(F_i) = 3$, $1 \leq i \leq k$. Let

$$V(K_n) = L \cup D \cup W \cup Z \cup U \cup U' \cup A \cup A' \cup B \cup B',$$

where

$$\begin{aligned} L &= \{l_1, ..., l_k\}, D = \{d_1, ..., d_k\}, W = \left\{w_0, ..., w_{\lceil\sqrt{k}\rceil-1}\right\}, Z = \left\{z_0, ..., z_{\lceil\sqrt{k}\rceil-1}\right\}, \\ U &= \left\{u_1, ..., u_{\lceil\sqrt{k}\rceil}\right\}, U' = \left\{u'_1, ..., u'_{\lceil\sqrt{k}\rceil}\right\}, A = \{a_1, a_2\}, A' = \{a'_1, a'_2\}, \\ B &= \{b_1, ..., b_{2t-1}\}, B' = \{b'_1, ..., b'_{2t-1}\}. \end{aligned}$$

Let $\mathcal{B}$ be the set of all $t-1$ element subsets of the set $\{1, 2, ..., 2t-1\}$. Let f be any injection

$$f : \{1, ..., k\} \to \mathcal{B}.$$

Let us notice that for each $j \in \{1, ..., k\}$ there are unique numbers q_j and r_j such that

$$j = q_j \cdot \left\lceil \sqrt{k} \right\rceil + r_j,\ 0 \leq q_j \leq \left\lceil \sqrt{k} \right\rceil - 1,\ 1 \leq r_j \leq \left\lceil \sqrt{k} \right\rceil .$$

The edges of the factor $F_i, 1 \leq i \leq k$ are

1) $l_i d_i$
2) $l_i l_j,\ 1 \leq j < i \leq k$
3) $d_i l_j,\ 1 \leq j < j \leq k$
4) $d_i d_j,\ 1 \leq j < i \leq k$
5) $l_i d_j,\ 1 \leq i < j \leq k$
6) $l_i a_1, l_i a_2, d_i a_1', d_i a_2'$
7) $l_i b_j, l_i b_j',\ j \in f(i)$
8) $d_i b_j, d_i b_j',\ j \in \{1, 2, ..., 2t-1\} \setminus f(i)$
9) $l_i w_j,\ 1 \leq j \leq \left\lceil \sqrt{k} \right\rceil - 1$
10) $d_i z_j,\ 1 \leq j \leq \left\lceil \sqrt{k} \right\rceil - 1$
11) $w_{q_i} u_{r_i}, u_{ri}'$
12) $z_{q_i} u_{r_i}, u_{ri}'$
13) $d_i u_j, d_i u_j'$ $1 \leq j \leq k,\ j \neq r_i$.

The other edges are edges of factor F_0.

In each factor $F_i, 1 \leq i \leq k$ all vertices are adjacent to either l_i or d_i, except u_{r_i} and $u_{r_i'}$ which have two common neighbors and which are connected by a path of length 2 to both, l_i and d_i, and also l_i and d_i are adjacent, hence we have $\operatorname{diam}(F_i) \leq 3,\ 1 \leq i \leq k$.

Now, let us prove that $\operatorname{diam}(F_i) \geqslant 3,\ 1 \leq i \leq k$. Let i be an arbitrary number such that $1 \leq i \leq k$. Let j be an element of the set $\{1, 2, ..., 2t-1\} \setminus f(i)$. Note that $d_{F_i}(a_1, b_j) = 3$, so the claim is proved.

It remains to prove that $\operatorname{subdiam}(F_0) = 2$. It is sufficient to prove that every two vertices x and y have two common neighbors. Without loss of generality, we may assume that we have one of the following cases:

1) $x \notin L \cup D$.

Distinguish two possibilities:

1a) $y \in L \Longrightarrow N_{F_0}(x) \cap N_{F_0}(y) = \{a_1', a_2'\}$.
1b) $y \notin L \Longrightarrow N_{F_0}(x) \cap N_{F_0}(y) = \{a_1, a_2\}$.
2) $x, y \in L \Longrightarrow N_{F_0}(x) \cap N_{F_0}(y) = \{a_1', a_2'\}$.
3) $x, y \in D \Longrightarrow N_{F_0}(x) \cap N_{F_0}(y) = \{a_1, a_2\}$.
4) $x \in L, y \in D$. We distinguish two cases.
4a) $x = l_i, y = d_i, 1 \leq i \leq k \Rightarrow u_{r_i}, u_{r_i}' \in N_{F_0}(l_i) \cap N_{F_0}(d_i)$.
4b) $x = l_i,\ y = d_j,\ 1 \leq i, j \leq k, i \neq j$. We have

$$|N_{F_0}(l_i) \cap B| + |N_{F_0}(d_j) \cap B| = t - 1 + t = |B|,$$

so either there is a vertex $b \in B$ element of $N_{F_0}(l_i) \cap N_{F_0}(d_j)$ or

$$N_{F_0}(l_i) \cap B = B \setminus N_{F_0}(d_j) = N_{F_0}(l_j) \cap B$$

which is impossible. Completely analogously we show that there is a vertex $b' \in B'$ element of $N_{F_0}(l_i) \cap N_{F_0}(d_j)$.

Therefore,

$$\phi(k) \leq 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t + 2.$$

For sufficiently large k, we have

$$\phi(k) \leq 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t + 2 \leq 2k + 5\left\lceil \sqrt{k} \right\rceil.$$

□

References

[1] B. Bollobás, *Extremal Graph Theory,* Academic Press, London, 1978.

[2] J. Bosák, *Decompositions of Graphs,* Kluwer Academic Publishers, Dordecht, 1989.

[3] J. Bosák, *Disjoint factors of diameter two in complete graphs*, Journal Of Combinatorial Theory (B) **16**(1974), 57–63.

[4] J. Bosák, A. Rosa, Š. Znám, *On decompositions of complete graphs into factors with given diameters,* Proc. colloq. Tihany (Hung) 1968, 37–56.

[5] D. Palumbíny, *On decompositions of complete graphs into factors with equal diameters*, Bollettino U.M.I. **4**(1973), 420–428.

[6] L. Stacho, E. Urland, *Factorizing the complete graph K-12 by computer*, Computers and Artificial Intelligence **13**(1994), 595–601.

[7] L. Stacho, E. Urland, *The nonexistence of a decomposition of complete graph K12 into three factors with diameter two*, Journal of Comb. Math. and Com. Computing **21**(1996), 147–159.

[8] P. Hrnčiar, *On decomposition of complete graphs into three factors with given diameters*, Czechoslovak Mathematical Journal **40**(1990), 388–396.

[9] N. Sauer *On factorization of complete graphs into factors of diameter two*, Journal of Combinatorial Theory **9**(1970), 423–426.

[10] D. Vukičević, *Decomposition of complete graph into factors of diameter two and three*, Discussiones Mathematicae Graph Theory, to appear

[11] Š. Znám, *Decomposition of complete graphs into factors of diameter two*, Mathematica Slovaca **30**(1980), 373–378.

[12] Š. Znám, *On a conjecture of a Bollobás and Bosák*, Journal of Graph Theory **6**(1962), 139–146.