Factorizations of the complete graphs into factor of subdiameter two and factors of diameter three

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Abstract. We search for the minimal number of vertices of the complete graph that can be decomposed into one factor of subdiameter 2 and k factors of diameter 3. We find as follows: exact values for $k \leq 3$, upper and lower bounds for small values of $k$ and

$$\lim_{k \to \infty} \frac{\varphi(k)}{k} = 2.$$ 

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1. Introduction and main results

Factorizations of graphs into factors with given diameters have been extensively studied. An excellent book [2] about this topic has been written and there exist numerous papers about this subject. The problem of factorization into the factors of equal diameters, where diameter of each factor is at least three has been solved in [5]. The problem of factorization into factors of diameter two is much harder and a lot of attention has been given to that problem. Denote by $f(k)$ the smallest natural number such that the complete graph with $n$ vertices can be factorized into $k$ factors of diameter 2. In [9], it was proved that

$$f(k) \leq 7k.$$ 

Then, in [3], this was improved to

$$f(k) \leq 6k.$$ 

In [11], it was proved that this upper bound is quite close to the exact value of $f(k)$ since

$$f(k) \geq 6k - 7, \text{ for } k \geq 664$$

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and in [12] the correct value of $f (k)$ is given for large values of $k$, namely

$$f (k) = 6k, \quad \text{for } k \geq 10^{17}.$$  

To conclude, factorization of the graph into factors with a small diameter is very interesting.

Factorizations into a small number of factors have been extensively studied as well. The case of factorization of the complete graph into two factors with given diameters was solved completely in [4]; and the case of factorization of the complete graph into three factors with given diameters was partially solved in [8]. One of the hardest problems was determining the exact value of $f (3)$; this problem was attacked and settled by a computer in [6] and [7]. The development of the fast computer gave a new boost to this area of mathematics, because computers can be of great assistance in attacking some very hard problems where a graph is factorized into a small number of factors. Therefore, it is very interesting to observe factorizations into a small number of factors.

This paper is a kind of a sequel of the paper [10]. In that paper factorization of the complete graph into $k$ factors of diameter 3 and one factor of diameter 2 was observed. In this paper, we observe factorization of the complete graph into $k$ factors of diameter 3 and one factor of diameter 2 in such a way that deletion of any of its edges does not increase its diameter. Hence, the results given here are more complicated than the results given there.

This is the problem we want to model. We have a system of $n$ devices and $k + 1$ communicational networks. The first $k$ communicational networks can have diameter 3. The remaining network is privileged and should work even if one of the links fails to work and even then it should have diameter 2.

We want to find out for what values of $n$ and $k$ this is possible.

Let $\phi (k)$ be the smallest number such that $K_{\phi (k)}$ can be factorized into $k$ factors of diameter 3 and one factor of subdiameter 2.

It can be easily proved, that for each $l \geq \phi (k)$, $K_l$ can be factorized into $k$ factors of diameter 3 and one factor of subdiameter 2.

In this paper we prove that

$$\begin{align*}
\phi (1) &= 7 \\
\phi (2) &= 10 \\
\phi (3) &= 13 \\
14 &\leq \phi (4) \leq 16 \\
2k + 6 &\leq \phi (k) \leq 3k + 3, \quad k \geq 5 \\
\lim_{k \to \infty} \frac{\phi (k)}{k} &= 2.
\end{align*}$$

2. Preliminaries and basic definitions

Let $G$ be a graph. By $V (G)$ we denote a set of vertices of $G$ and by $E (G)$ a set of edges of $G$. By $v (G)$ we denote the number of vertices of $G$ and by $e (G)$ the number of edges in $G$. By $d_G (x)$ we denote a degree of vertex $x$ (in $G$), by $\delta (G)$
the minimal degree of $G$ and by $\Delta (G)$ the maximal degree of $G$. We say that $G$ is $k$-uniform if $d_G (x) = k$, for each $x \in V (G)$. By $N_G (x)$ we denote the set of neighbors of $x$ (in $G$). We say that two vertices $x, y \in V (G)$ are adjacent (in $G$) if $xy \in E (G)$. In this case we also say that $x$ is a neighbor of $y$ (in $G$).

By $d_G (x, y)$, we denote the distance (in $G$) of vertices $x$ and $y$. The subdistance (in $G$) of vertices $x$ and $y$, denoted by $\text{subd}_G (x, y)$, is given by

$$\text{subd}_G (x, y) = \max \left\{ d_{G'} (x, y) : G' \text{ is obtained from } G \text{ by deletion of the single edge} \right\}.$$

Define diameter of $G$ and subdiameter of $G$ by

$$\text{diam } G = \max_{x,y \in V} \{ d_G (x, y) \}$$
$$\text{subdiam } G = \max_{x,y \in V} \{ \text{subd}_G (x, y) \}.$$

Let $A \subseteq V (G)$. By $G [A]$ we denote the subgraph of $G$ spanned by the set of vertices $A$. Let $a_1, ..., a_k \in V (G)$. By $G [a_1, ..., a_k]$ we denote $G [(a_1, ..., a_k)]$.

Let $A, B \subseteq V (G)$. By $E_G (A, B)$ we denote the set of edges that have one incident vertex in $A$ and the other in $B$. Let $e_G (A, B) = \left| E_G (A, B) \right|$ and $e_G (A) = e (G [A])$.

The factor of graph $G$ is any spanning subgraph of $G$. We say that the set of factors $F_0, F_1, ..., F_k$ of $G$ is factorization (or decomposition) of $G$ if each edge of $G$ is contained in exactly one of factors $F_0, F_1, ..., F_k$. Then we say that $G$ is factorized (or decomposed) into factors $F_0, F_1, ..., F_k$.

By $K_n$ we denote the complete graph with $n$ vertices and by $K_{n,n}$ the complete bipartite graph with $n$ vertices in each class.

We shall need a simple, but very useful Lemma given in [10] :

**Lemma 1.** Let $K_n$ have a factorization with a factor of diameter two. Then any factor (of that factorization) of diameter three has at least $n$ edges.

### 3. The value of $\phi (1)$

First, we prove that $\phi (1) \geq 7$. Suppose, to the contrary, that $K_6$ can be factorized into two factors $F_0$ and $F_1$ such that $\text{subdiam } F_0 = 2$ and $\text{diam } F_1 = 3$. It can be easily shown that $\delta (F_0) \geq 3$. Let us prove this.

**Claim 1.** $F_0$ is not a 3-uniform graph.

**Proof.** If none of the vertices has degree 3, the claim is trivial. Denote by $x$ an arbitrary vertex such that $d_{F_0} (x) = 3$. There are no isolated vertices in $F_0 [N_{F_0} (x)]$, hence $e (F_0 [N_{F_0} (x)]) \geq 2$. Note that each of vertices in $V (F_0) \setminus (N_{F_0} (x) \cup \{ x \})$ has at least two neighbors in $N_{F_0} (x)$. Therefore

$$\sum_{v \in N_{F_0} (x)} d_{F_0} (v) \geq 3 + 4 + 2 \cdot 2 = 11.$$

Hence, $F_0$ is not a 3-uniform graph. \qed
It follows that \( e (F_0) \geq 10 \). From Lemma 1, it follows that \( e (F_1) \geq 6 \), but this is in contradiction with \( e (K_6) = 15 \). Hence, indeed \( \phi (1) \geq 7 \). The opposite inequality follows from the following Figure.

\[
\begin{align*}
\text{Figure 1. The edges of } F_0 \text{ are drawn with a dashed line and of } F_1 \text{ by a bold line.}
\end{align*}
\]

4. The value of \( \phi (2) \)

Let us prove that \( \phi (2) \geq 10 \). Suppose to the contrary that \( K_9 \) can be factorized into factors \( F_0, F_1 \) and \( F_2 \) such that subdiam \( F_0 = 2 \) and diam \( F_1 = \text{diam } F_2 = 3 \). Note that \( e (K_9) = 36 \). It can be easily shown that \( \delta (F_0) \geq 3, \delta (F_1) \geq 1 \) and \( \delta (F_2) \geq 1 \), hence \( \Delta (F_0) \leq 6, \Delta (F_1) \leq 4 \) and \( \Delta (F_2) \leq 4 \).

Claim 2. \( e (F_1) \geq 10 \) and \( e (F_2) \geq 10 \).

Proof. Suppose to the contrary that \( e (F_1) < 10 \) or \( e (F_2) < 10 \). Without loss of generality, we may assume that \( e (F_1) < 10 \). From Lemma 1, it follows that \( e (F_1) \geq 9 \), hence \( e (F_1) = 9 \).

Denote the unique cycle in \( F_1 \) by \( C \). Cycle \( C \) has less than 8 vertices, because \( \text{diam } F_1 \leq 3 \). Hence, there are vertices in \( V (F_1) \setminus C \). Now, however, it follows that \( C \) is of length at most 5. Distinguish three subcases:

3.1) \( C \) has three vertices.

We have

\[
\sum_{v \in C} d_{F_0} (v) \geq 2 \cdot 3 + (9 - 3) = 12,
\]

hence \( d_{F_0} (v) = 4 \), for each \( v \in C \). Therefore, \( F_1 \) is isomorphic to the graph in Figure 2.

Since \( \text{subd}_{F_0} (x, y) \leq 2 \), it follows that \( xc, yc, xd, yd \in E (F_0) \) and since \( \text{subd}_{F_0} (x, z) \leq 2 \), it follows that \( xf, zf, xc, ze \in E (F_0) \). But then \( d_{F_2} (x) = 0 \), which is a contradiction.
3.2) $C$ has four vertices.

There are two adjacent vertices in $C$ such that each vertex in $V(F_1) \setminus C$ is adjacent to one of them. Denote these two vertices by $x$ and $y$. Note that

$$16 = d_{K_9}(x) + d_{K_9}(y) = (d_{F_1}(x) + d_{F_1}(y)) + (d_{F_2}(x) + d_{F_2}(y)) + (d_{F_0}(x) + d_{F_0}(y)) \geq (2 + 2 + (9 - 4)) + (1 + 1) + (3 + 3) = 17,$$

but this is a contradiction.

3.3) $C$ has five vertices.

There are two adjacent vertices in $C$ such that each vertex in $V(F_1) \setminus C$ is adjacent to one of them. Denote these two vertices by $x$ and $y$. There is a single vertex that is not in $N_{F_1}(x) \cup N_{F_1}(y)$, but this is in contradiction with subdiam $F_0 = 2$.

We have exhausted all the cases and we have proved the claim. □

From this claim, it follows that $\epsilon(F_0) \leq 16$. Hence, $\delta(F_0) = 3$. Denote by $x$ any vertex such that $d_{F_0}(x) = 3$. There are no isolated vertices in $F_0[N_{F_0}(x)]$, hence $\epsilon(F_0[N_{F_0}(x)]) = 2$. Hence, each vertex in $V(F_0) \setminus (N_{F_0}(x) \cup \{x\})$ has at least two neighbors (in $F_0$) in $N_{F_0}(x)$, therefore

$$\sum_{v \in V(F_0)} d(v) \geq \sum_{v \in N_{F_0}} d(v) + 6 \cdot 3 = (3 + 4 + 2 \cdot 6) + 3 \cdot 6 = 37,$$

but this is in contradiction with $\epsilon(F_0) \leq 16$. So, it is indeed, $\phi(2) \geq 10$. The opposite inequality follows from the following Figure.
5. The value of $\phi (3)$

First, we shall prove that $\phi (3) \geq 13$. Suppose, to the contrary that $K_{12}$ can be factorized into factors $F_0, F_1, F_2$ and $F_3$, such that subdiam $F_0 = 2$ and diam $F_1 =$ diam $F_2 =$ diam $F_3 = 3$. Note that $\delta (F_1) \geq 1$, $\delta (F_2) \geq 1$ and $\delta (F_3) \geq 1$, hence $\Delta (F_0) \leq 8$.

Claim 3. $e (F_0) \geq 29$

Proof. Distinguish four cases:
1) $\delta (F_0) \leq 2$
It can be easily shown that this case is impossible.
2) $\delta (F_0) = 3$
Distinguish two subcases:
2.1) There are two vertices $x_1$ and $x_2$ such that $d (x_1) = d (x_2) = 3$ and $|N_{F_0} (x_1) \cap N_{F_0} (x_2)| = 2$.
Denote $N_{F_0} (x_1) \cap N_{F_0} (x_2) = \{y_1, y_2\}$, $N_{F_0} (x_1) \setminus N_{F_0} (x_2) = \{u\}$ and $N_{F_0} (x_2) \setminus N_{F_0} (x_1) = \{v\}$. Note that $d_{F_0[y_1, y_2, u, v]} (u) \geq 2$, because subdiam $F_0 (x_2, u) \leq 2$. Also, note that $d_{F_0[y_1, y_2, u, v]} (v) \geq 2$, because subdiam $F_0 (x_1, v) \leq 2$. A simple analysis shows that $d_{F_0[y_1, y_2, u, v]} (y_1) + d_{F_0[y_1, y_2, u, v]} (y_2) \geq 3$, $d_{F_0[y_1, y_2, u, v]} (y_1) \geq 1$ and $d_{F_0[y_1, y_2, u, v]} (y_2) \geq 1$. Therefore, $e (F_0 [y_1, y_2, u, v]) \geq 4$.

Denote

\[
S = \{x_1, x_2, y_1, y_2, u, v\}
\]
\[
A = \{x \in V (F_0) \setminus S : \{y_1, y_2, u, v\} = N_{F_0} (x) \cap \{y_1, v, u, v\}\}
\]
\[
B_1 = \{x \in V (F_0) \setminus S : \{y_1, y_2, u\} = N_{F_0} (x) \cap \{y_1, y_2, u\}\}
\]
\[
B_2 = \{x \in V (F_0) \setminus S : \{y_1, y_2, v\} = N_{F_0} (x) \cap \{y_1, y_2, v\}\}
\]
\[
C = \{x \in V (F_0) \setminus S : \{y_1, y_2\} = N_{F_0} (x) \cap \{y_1, y_2, u\}\}
\]
\[
D_1 = \{x \in V (F_0) \setminus S : \{y_1, u, v\} = N_{F_0} (x) \cap \{y_1, y_2, u\}\}
\]
\[
D_2 = \{x \in V (F_0) \setminus S : \{y_2, u, v\} = N_{F_0} (x) \cap \{y_1, y_2, u\}\}
\]
\[
a = |A|, b_1 = |B_1|, b_2 = |B_2|, c = |C|, d_1 = |D_1|, d_2 = |D_2|.
\]
Note that $A \cup B_1 \cup B_2 \cup C \cup D_1 \cup D_2 = V (F_0) \setminus \{y_1, y_2, u, v\}$ and that $A, B_1, B_2, C, D_1$ and $D_2$ are pairwise disjoint sets, hence
\[ a + b_1 + b_2 + c + d_1 + d_2 = 6. \quad (1) \]

Since, $\Delta (F_0) \leq 8$, it follows that
\[ a + b_1 + b_2 + c + d_1 \leq 5 \quad (2) \]
\[ a + b_1 + b_2 + c + d_2 \leq 5 \quad (3) \]
\[ 2a + 2b_1 + 2b_2 + 2c + d_1 + d_2 \leq 9. \quad (4) \]

From (1) and (2), it follows that $d_2 \geq 1$ from (1) and (3), it follows that $d_1 \geq 1$ and from (1) and (4), it follows that $d_1 + d_2 \geq 3$. Without loss of generality, we may assume that $d_1 \geq d_2$, hence $d_1 \geq 2$.

We have
\[ e_{F_0} (S, V (F_0) \setminus S) = 4a + 3b_1 + 3b_2 + 2c + 3d_1 + 3d_2. \]

Distinguish two subsubcases:

2.1.1) $c \neq 0$.

Note that subdiam $F_0 = 2$ implies that all vertices in $C$ are connected with all vertices in $D_1$ in the graph $F_0 [V (F_0) \setminus S]$ and, also, that all vertices in $C$ are connected with all vertices in $D_2$ in the graph $F_0 [V (F_0) \setminus S]$, hence
\[ e (F_0 [V (F_0) \setminus S]) \geq c + d_1 + d_2 - 1. \]

Therefore,
\[ e (F_0) \geq 6 + e (F_0 [y_1, y_2, u, v]) + (4a + 3b_1 + 3b_2 + 2c + 3d_1 + 3d_2) + (c + d_1 + d_2 - 1) \geq 9 + 3(a + b_1 + b_2 + c + d_1 + d_2) + (d_1 + d_2) \geq 30. \]

2.1.2) $c = 0$.

Suppose to the contrary that $e (F_0) \leq 28$. Note that
\[ e (F_0) \geq 6 + e (F_0 [y_1, y_2, u, v]) + (4a + 3b_1 + 3b_2 + 3d_1 + 3d_2) \geq 10 + 3(a + b_1 + b_2 + d_1 + d_2) = 28, \]
hence $e (F_0) = 28$, $a = 0$, $e (F_0 [y_1, y_2, u, v]) = 4$ and $e (F_0 [V (F_0) \setminus S]) = 0$. Note, that
\[ d_{F_0} (y_1) + d_{F_0} (y_2) + d_{F_0} (u) + d_{F_0} (v) = 6 + 8 + 3 \cdot 6 = 32. \]

From $\Delta (F_0) = 8$, it follows that
\[ d_{F_0} (y_1) = d_{F_0} (y_2) = d_{F_0} (u) = d_{F_0} (v) = 8, \]
hence
\[ d_{F_1} (y_1) = d_{F_1} (y_2) = d_{F_1} (u) = d_{F_1} (v) = 1, \quad i = 1, 2, 3. \]
Since \( F_0 [y_1, y_2, u, v] \) is not a complete graph, there are two adjacent vertices of degree 1 in one of the factors \( F_1, F_2 \) and \( F_3 \) and that is a contradiction.

2.2) \( \delta (F_0) = 3 \) and every two vertices of degree 3 are adjacent or have the same set of neighbors.

Denote by \( x_1 \) an arbitrary vertex such that \( d_{F_0} (x_1) = 3 \). Denote its neighbors by \( Y = \{ y_1, y_2, y_3 \} \) and denote by \( X = \{ x_1, x_2, \ldots, x_p \} \) the set of vertices that have the same set of neighbors as \( x_1 \). Also, denote \( Z = V (F_0) \setminus (X \cup Y) = \{ z_1, \ldots, z_{9-p} \} \).

Note that \( e (F_0 [Y]) \geq 2 \) and \( e_{F_0} (X, Y) = 3p \), hence

\[
d_{F_0} (y_1) + d_{F_0} (y_2) + d_{F_0} (y_3) \geq 4 + 3p + 2 \cdot (9 - p). \tag{5}
\]

We also have

\[
\sum_{i=1}^{p} d_{F_0} (x_i) = 3p \tag{6}
\]

\[
\sum_{i=1}^{9-p} d_{F_0} (z_i) \geq 4 \cdot (9 - p). \tag{7}
\]

Adding up (5), (6) and (7), we get

\[
\sum_{v \in V(F_0)} e_{F_0} (v) \geq 58,
\]

hence \( e (F_0) \geq 29 \).

3) \( \delta (F_0) = 4 \). Suppose to the contrary that \( e (F_0) \leq 28 \). Distinguish two cases:

3.1) There is a vertex \( x \) of degree 4 such that \( d_{F_0} (y) \leq 7 \), for each \( y \in N_{F_0} (x) \).

Denote \( S = V (F_0) \setminus (\{ x \} \cup N_{F_0} (x)) \). From sub\( e_{F_0} (x, y) \leq 2 \), for each \( y \in V \), it follows that none of the vertices in \( N_{F_0} (x) \) can be an isolated vertex in \( F_0 [N_{F_0} (x)] \) and that each vertex in \( S \) has at least two neighbors (in \( F_0 \)) in \( x \cup N_{F_0} (x) \). Therefore

\[
e (F_0 [x \cup N_{F_0}]) \geq 6 \tag{8}
\]

\[
e_{F_0} (N_{F_0} (x), S) \geq 14 \tag{9}
\]

\[
|N_{F_0} (v) \cap S| \leq 5, \text{ for each } v \in N_{F_0} (x). \tag{10}
\]

From (8), it follows that

\[
e_{F_0} (S) + e_{F_0} (N_{F_0} (x), S) \leq 22
\]

or equivalently

\[
\sum_{v \in N_{F_0} (x)} |N_{F_0} (v) \cap S| + \frac{1}{2} \sum_{v \in S} d_{F_0 (S)} (v) \leq 22. \tag{11}
\]

Suppose that \( e_{F_0} (N_{F_0} (x), S) \geq 18 \), then \( \sum_{v \in N_{F_0} (x)} d (x) \geq 8 + 18 = 26 \), hence

\[
\sum_{v \in V(F_0)} d (x) \geq 58, \text{ which is in contradiction with } e (F_0) \leq 28. \text{ Therefore,}
\]

\[
14 \leq e_{F_0} (N_{F_0} (x), S) \leq 17
\]
or equivalently
\[ 14 \leq \sum_{v \in N_{F_0}(x)} |N_{F_0}(v) \cap S| \leq 17. \quad (12) \]

For each two vertices \( s_1, s_2 \in S \), there are two disjoint paths of length at most 2 connecting \( s_1 \) and \( s_2 \), hence
\[ \sum_{v \in N_{F_0}(x)} \left( \frac{|N_{F_0}(v) \cap S|}{2} \right) + e_{F_0}(S) + \sum_{v \in S} \left( \frac{d_{F_0[S]}(v)}{2} \right) \geq 2 \cdot \left( \frac{7}{2} \right). \quad (13) \]

From \( \delta(F_0) \geq 4 \) and \( |N_{F_0}(v) \cap N_{F_0}(x)| \geq 2 \), for each \( v \in S \), it follows that
\[ \sum_{v \in S} \max \{0, 2 - d_{F_0[S]}(v)\} \leq e_{F_0}(N_{F_0}(x), S) - 14 \]
or equivalently
\[ \sum_{v \in S} \max \{0, 2 - d_{F_0[S]}(v)\} \leq \sum_{v \in N_{F_0}(x)} |N_{F_0}(v) \cap S| - 14. \quad (14) \]

Also, from \( \delta(F_0) \geq 4 \), follows that
\[ \sum_{v \in N_{F_0}(x)} \max \{3 - |N_{F_0}(v) \cap S|\} \leq 2 \cdot e(F_0[N_{F_0}(x)]) \]
\[ \max_{v \in N_{F_0}(x)} \{0, 3 - |N_{F_0}(v) \cap S|\} \leq e(F_0[N_{F_0}(x)]) \quad (15) \]

and that
\[ \max_{v \in N_{F_0}(x)} \{0, 3 - |N_{F_0}(v) \cap S|\} \leq e(F_0[N_{F_0}(x)]) + 1 \text{ or } e(F_0[N_{F_0}(x)]) \geq 3. \quad (17) \]

Now let us observe multisets \( Deg_1 = \{|N_{F_0}(v) \cap S| : v \in N_{F_0}(x)\} \) and \( Deg_2 = \{d_{F_0[S]}(v) : V \in S\} \). A tedious check shows that relations (10) – (17) are satisfied only if
\[
Deg_1 = \{2, 2, 5, 5\} \quad (18)
\]
\[
Deg_2 = \{2, 2, 2, 2, 2, 4\}. \quad (19)
\]

In this case relation (13) is actually an equality, hence there are exactly two paths of length at most 2 connecting each two vertices in \( S \). From (19), it follows that \( F_0[S] \) is one of the following two graphs:

![Figure 4](image)
Denote by $s_1$ a vertex in $S$ such that $d_{F_0[S]}(s_1) = 4$. There are, in both cases, exactly 8 paths of length at most 2 in $F_0[S]$ starting from the vertex $s_1$. Therefore, there are exactly four paths of the form $s_1uw$, where $u \in N_{F_0}(x)$ and $v \in S$, hence

$$\sum_{w \in N_{F_0}(x) \cap N_{F_0}(s_1)} |N_{F_0}(w) - 1| = 4. \quad (20)$$

Recall that

$$|N_{F_0}(x) \cap N_{F_0}(s_1)| \geq 2. \quad (21)$$

Relations (18), (20) and (21) are inconsistent, hence the claim is proved in this case.

3.2) For each vertex $x \in V(F_0)$ such that $d_{F_0}(x) = 4$, there is a vertex $y$ adjacent to $x$ such that $d_{F_0}(y) = 8$.

Distinguish three subcases:

3.2.1) There are vertices $x, y$ and $z$ such that $d_{F_0}(x) = 4, d_{F_0}(y) = 8, d_{F_0}(z) = 4, xy \in E(F_0), xz, yz \notin E(F_0)$.

There are no isolated vertices in $F_0 \{N_{F_0}(z)\}$, hence $e(F_0 \{N_{F_0}(z)\}) \geq 2$. Each vertex in $V(F_0) \setminus \{N_{F_0}(z) \cup \{z\}\}$ has at least two neighbors (in $F_0$) in $N_{F_0}(z)$, therefore

$$\sum_{v \in V(F_0)} d_{F_0}(v) = \sum_{v \in N_{F_0}(z)} d_{F_0}(v) + d(y) + \sum_{v \in V(F_0) \setminus \{(N_{F_0}(z) \cup \{y\})\}} d_{F_0}(v) \leq (4 + 2 \cdot 2 + 7 \cdot 2) + 8 + 7 \cdot 4 = 58,$$

but this is in contradiction with $e(F_0) \leq 28$, so the claim is proved in this case.

3.2.2) There are vertices $x, y$ and $z$ such that $d_{F_0}(x) = 4, d_{F_0}(y) = 8, d_{F_0}(z) = 5, xy \in E(F_0), xz, yz \notin E(F_0)$.

There are no isolated vertices in $F_0 \{N_{F_0}(z)\}$, hence $e(F_0 \{N_{F_0}(z)\}) \geq 3$. Each vertex in $V(F_0) \setminus \{N_{F_0}(z) \cup \{z\}\}$ has at least two neighbors (in $F_0$) in $N_{F_0}(z)$ and vertex $y$ has at least three neighbors (in $F_0$) in $N_{F_0}(z)$, hence

$$\sum_{v \in V(F_0)} d_{F_0}(v) = \sum_{v \in N_{F_0}(z)} d_{F_0}(v) + d(y) + d(z) + \sum_{v \in V(F_0) \setminus \{(N_{F_0}(z) \cup \{y,z\})\}} d_{F_0}(v) \leq (5 + 2 \cdot 3 + 3 + 2 \cdot 5) + 8 + 5 + 5 \cdot 4 = 57.$$

This is in contradiction with $e(F_0) \leq 28$, so the claim is proved in this case.

3.2.3) For each two adjacent vertices $x$ and $y$ such that $d_{F_0}(x) = 4$ and $d_{F_0}(y) = 8$ and each vertex $z$ such that $xz, yz \notin E(F_0)$, we have $d_{F_0}(z) \geq 6$.

Let $x$ be an arbitrary vertex such that $d_{F_0}(x) = 4$. Denote

$$S = V(F_0) \setminus (N_{F_0}(x) \cup \{x\})$$

$$p = e_{F_0}(N_{F_0}(x))$$

$$q = e_{F_0}(N_{F_0}(x), S)$$

$$r = e_{F_0}(S).$$
Note that
\[
4 + p + q + r \leq 28 \quad (22)
\]
\[
p \geq 2 \quad (23)
\]
\[
q \geq 14 \quad (24)
\]
\[
q + 2r = \sum_{v \in S} d_{F_0}(v) \geq 6 \cdot 4 + 6 \quad (25)
\]

Solving (22) – (25), we get \( p = 2, q = 14, r = 8 \) and all inequalities (22) – (25) are, in fact, equalities. Hence,\
\[
d_{F_0}(v) = 4, \text{ for each } x \in S \setminus \{z\}
\]
\[
\sum_{v \in N_{F_0}(x)} d_{F_0}(x) = 14 + 2 \cdot 2 + 4 = 22.
\]

Since each vertex of degree 4 is adjacent to a vertex of degree 8, we may conclude that \( F_0 \) is the supergraph of the graph given in the following Figure:

Figure 5.

where \( d_{F_0}(n_1) = d_{F_0}(s_1) = d_{F_0}(s_2) = \ldots = d_{F_0}(s_6) = 4 \) and \( d_{F_0}(n_3) = d_{F_0}(n_3) = 5 \). Since \( x \) was an arbitrary vertex of degree four, it follows that each vertex of degree 4 has to be adjacent to two vertices of degree 5, but there are only two vertices of degree five and eight vertices of degree 4. This is a contradiction, so the claim is proved in this case.

4) \( \delta(F_0) \geq 5 \).

The claim is trivial in this case.

Hence, we have exhausted all the cases and we have proved our claim. □
Denote by $G_{12}$ the graph in the following Figure.

Let us prove Claim 4. If $e(F_i) \leq 12$, then $F_i \cong G_{12}$, for each $i = 1, 2, 3$.

**Proof.**

Suppose the contrary. Without loss of generality, we may assume that $e(F_1) \leq 12$ and $F_1 \not\cong G_{12}$. Note that $F_1$ has at most one cycle. The length of that cycle (if it exists) is at most 5. Also, note that $\Delta(F_1) \leq 11 - 3 - 1 - 1 = 6$. Distinguish the following cases:

1) There are no cycles in $F_1$.
   Then $e(F_1) < 12$, but this is in contradiction with Lemma 1.

2) There is a triangle in $F_1$.
   Denote vertices of triangle by $c_1, c_2$ and $c_3$ and denote $C = \{c_1, c_2, c_3\}$. Distinguish two subcases:

   2.1) Only one of the vertices in $C$ has neighbors that are not in $C$.
   From $\Delta(F_1) \leq 6$, it follows that $F_1$ is given by the following Figure.

   But now, $d_{F_0}(b, c) > 2$ and this is a contradiction.

   2.2) More than one vertex in $C$ has neighbors that are not in $C$.
   It follows that all vertices in $V(F_1) \setminus C$ are adjacent to exactly one vertex in $C$, because $F_1$ contains a single cycle and $\text{diam} F_1 = 3$. If there is $c_i$, $i = 1, \ldots, 3$ such that $d(c_i) \leq 3$, then $\text{subd}_{F_0}(c_j, c_k) > 2$, where $\{i, j, k\} = \{1, 2, 3\}$, which is a contradiction. Therefore, without loss of generality, we may assume that $4 \leq \Delta(F_1) \leq 6$.
Note that \( d(c_1) + d(c_2) + d(c_3) = 15 \). Distinguish three subsubcases:

1. \( d(c_1) = 4, d(c_2) = 4, d(c_3) = 7 \).

From \( \text{subd}_{F_0}(c_1, c_3) \leq 2 \), it follows that

\[
\begin{align*}
|N_{F_0}(c_1) \cap (N_{F_1}(c_2) \setminus \{c_1, c_3\})| &\geq 2, \\
|N_{F_0}(c_3) \cap (N_{F_1}(c_2) \setminus \{c_1, c_3\})| &\geq 2.
\end{align*}
\]

(1)

(2)

Analogously, from \( \text{subd}_{F_0}(c_1, c_2) \leq 2 \), it follows that

\[
\begin{align*}
|N_{F_0}(c_1) \cap (N_{F_1}(c_3) \setminus \{c_1, c_2\})| &\geq 2, \\
|N_{F_0}(c_2) \cap (N_{F_1}(c_3) \setminus \{c_1, c_2\})| &\geq 2.
\end{align*}
\]

(3)

(4)

Therefore, \( d_{F_0}(c_3) + d_{F_1}(c_3) = 11 \), hence \( d_{F_0}(c_3) + d_{F_1}(c_3) = 0 \) and this is a contradiction.

2. \( d(c_1) = 4, d(c_2) = 5, d(c_3) = 6 \).

Analogously, as in the previous case, we have relations (1) – (4). Therefore, \( d_{F_0}(c_3) + d_{F_1}(c_3) = 10 \), hence \( d_{F_0}(c_3) + d_{F_1}(c_3) = 1 \) and this is a contradiction.

3. \( d(c_1) = 5, d(c_2) = 5, d(c_3) = 5 \).

In this case \( F_1 \cong G_{12} \).

3) There is a cycle of length at least 4.

Denote the set of vertices of the cycle by \( C \). All vertices in \( V(F_1) \setminus C \) are adjacent to exactly one vertex in \( C \). There are two adjacent vertices \( x, y \in C \) such that each vertex in \( V(F_1) \setminus C \) is adjacent to at least one of them. Note that \( |V(F_1) \setminus (N_{F_1}(x) \cup N_{F_1}(y))| \leq 1 \), because the length of the unique cycle is at most 5. Therefore, \( \text{subd}_{F_0}(x, y) > 2 \), which is a contradiction.

Hence, all the cases are exhausted and the claim is proved.

Without loss of generality, we may assume that \( e(F_1) \leq e(F_2) \leq e(F_3) \). Since \( e(F_0) \geq 29 \), it follows that \( e(F_1) + e(F_2) + e(F_3) \leq 37 \), hence \( e(F_1) = e(F_2) = e(F_3) = 12 \). Therefore \( F_1 \cong F_2 \cong F_3 \). Denote the set of vertices of triangle in \( F_1 \) by \( T_1 = \{t_{11}, t_{12}, t_{13}\} \). Denote the set of vertices of triangle in \( F_2 \) by \( T_2 = \{t_{21}, t_{22}, t_{23}\} \).

Let us prove

**Claim 5.** Let \( \{i, j\} \in \{1, 2\} \) and \( k \in \{1, 2, 3\} \). Then \( d_{F_1}(t_ik) = 5, d_{F_2}(t_ik) = 1, d_{F_3}(t_ik) = 1 \) and \( d_{F_0}(t_ik) = 4 \). For each vertex \( v \in V(F_0) \) adjacent (in \( F_0 \)) to \( t_ik \) we have \( d_{F_0}(v) \leq 2 \), \( d_{F_0}(v) \leq 2 \) and \( d_{F_0}(v) \leq 2 \).

**Proof.** It is obvious that \( d_{F_1}(t_ik) = 5 \). Denote \( \{k, l, m\} = \{1, 2, 3\} \). Since \( \text{subd}_{F_0}(t_ik, t_{im}) \leq 2 \), it follows that

\[
|N_{F_0}(t_ik) \cap N_{F_0}(t_{im}) \cap (N_{F_1}(t_{il}) \setminus \{t_ik, t_{il}\})| \geq 2.
\]

(1)

Analogously, since \( \text{subd}_{F_0}(t_ik, t_{il}) \leq 2 \), it follows that

\[
|N_{F_0}(t_ik) \cap N_{F_0}(t_{il}) \cap (N_{F_1}(t_{im}) \setminus \{t_ik, t_{im}\})| \geq 2.
\]

(2)

This implies that \( d_{F_0}(t_ik) \geq 4 \). Since \( d_{F_1}(t_ik) \geq 1 \) and \( d_{F_3}(t_ik) \geq 1 \), it follows that

\[
\begin{align*}
&d_{F_0}(t_ik) = 4, d_{F_1}(t_ik) = 1, d_{F_3}(t_ik) = 1. \text{ Therefore, inequalities (1) and (2) are in fact equalities and} \\
&N_{F_0}(t_ik) \cap (N_{F_1}(t_{im}) \setminus \{t_ik, t_{il}\}) \subseteq N_{F_0}(t_{il}) \\
&N_{F_0}(t_ik) \cap (N_{F_1}(t_{il}) \setminus \{t_ik, t_{im}\}) \subseteq N_{F_0}(t_{im}).
\end{align*}
\]
Since
\[ N_{F_0}(t_{ik}) \subseteq (N_{F_i}(t_{il}) \setminus \{t_{ik}, t_{im}\}) \cup (N_{F_i}(t_{im}) \setminus \{t_{ik}, t_{il}\}), \]
it follows that
\[ N_{F_0}(t_{ik}) \subseteq N_{F_0}(t_{il}) \cup N_{F_0}(t_{im}), \]
which proves the claim.

From the last claim, it easily follows that there are no vertices of degree one in \( F_0[T_1 \cup T_2] \). Note that \( e_{F_1}(T_1, T_2) \geq 3 \) and \( e_{F_2}(T_1, T_2) \geq 3 \), hence \( e_{F_0}(T_1, T_2) \leq 3 \).

Distinguish two cases:

1) \( e_{F_0}(T_1, T_2) > 0 \).

Note that \( F_0[T_1 \cup T_2] \) is the spanning subgraph of \( K_{3,3} \). But each spanning subgraph of \( K_{3,3} \) with at least one edge and at most three edges has at least one vertex of degree 1 and this is a contradiction.

2) \( e_{F_0}(T_1, T_2) = 0 \).

Denote \( S = V(K_{12}) \setminus (T_1 \cup T_2) \). Since \( e_{F_0}(T_1, T_2) = 0 \), it follows that
\[ |N_{F_0}(t_{ij}) \cap S| = 4, \quad i = 1, 2; \quad j = 1, 2, 3, \]
hence \( e_{F_0}(T_1 \cup T_2, S) = 24 \). Also, we have
\[ e_{F_i}(T_i, \{s\}) = 1, \quad s \in S, \quad i = 1, 2, \]
hence \( e_{F_1}(T_1, S) = 6 \) and \( e_{F_2}(T_2, S) = 6 \). Therefore, \( e_{F_3}(T_1 \cup T_2, S) = 6 \cdot 6 - 24 - 6 - 6 = 0 \). But, then \( F_3 \) is disconnected and this is a contradiction. Hence, we have proved that \( \phi(3) \geq 12 \).

The opposite inequality follows from the following Figure:

![Figure 8. The edges of the factor of subdiameter 2 are drawn with a dashed line and edges of factors of diameter 3 are drawn with a bold, a bold dashed and a bold dotted line, respectively.](image)

Denote the factorization in this Figure by \( D' \).
6. The lower and upper bounds for the values of $\phi$

First, we give a weaker lower bound that we need to prove the stronger one:

**Lemma 2.** For each $k \geq 4$, we have $\phi(k) \geq 2k + 3$.

**Proof.** Let $K_v$ be factorized into $k$ factors of diameter 3 and one factor of subdiameter 2. From Lemma 1, it follows that each of the factors of diameter 3 has at least $v$ edges, hence

$$k \cdot v + v \leq \left(\frac{v}{2}\right).$$

Solving this, we get $v \geq 2k + 3$, so the claim is proved. \qed

**Lemma 3.** For each $k \geq 4$, we have $\phi(k) \geq 2k + 6$.

**Proof.** Let $K_v$ be factorized into $k$ factors of diameter 3 and one factor of subdiameter 2. As in the previous Lemma, each of the factors of diameter 3 has at least $v$ edges. Now, we shall estimate the number of edges of $F_0$. Note that $\delta(F_0) \geq 3$. Let $x$ be a vertex such that $d_{F_0}(x) = \delta(F_0)$. There are no isolated vertices in $F_0[N_{F_0}(x)]$ and each vertex in $V(F_0) \setminus (N_{F_0}(x) \cup \{x\})$ has at least two neighbors (in $F_0$) in $N_{F_0}(x)$, hence

$$\sum_{v \in V(F_0)} d(v) = \sum_{v \in N_{F_0}(x)} d(v) + \sum_{v \in V(F_0) \setminus N_{F_0}(x)} d(v) = \left(\delta(F_0) + 2 \cdot \left\lfloor \frac{\delta(F_0)}{2} \right\rfloor + 2 \cdot (v - \delta(F_0) - 1)\right) + (v - \delta(F_0)) \cdot \delta(F_0).$$

Therefore,

$$e(F_0) \geq \frac{v - \delta(F_0) + 1}{2} \cdot \delta(F_0) + \left\lfloor \frac{\delta(F_0)}{2} \right\rfloor + (v - \delta(F_0) - 1).$$

Also note that

$$e(F_0) \geq \frac{\delta(F_0) \cdot v}{2}.$$ 

Therefore,

$$e(F_0) \geq \min_{\delta \geq 3} \left\{ \max \left\{ \frac{v - \delta(F_0) + 1}{2} \cdot \delta(F_0) + \left\lfloor \frac{\delta(F_0)}{2} \right\rfloor + (v - \delta(F_0) - 1), \frac{\delta(F_0) \cdot v}{2} \right\} \right\}. $$

From the last Lemma, it follows that $v \geq 11$, hence from the last expression, it follows that

$$e(F_0) \geq \frac{5v - 10}{2}.$$ 

Therefore,

$$\frac{5v - 10}{2} + kv \leq \left(\frac{v}{2}\right) \text{ and } v \geq 11.$$ 

Solving the last inequalities, we get

$$v \geq \max \left\{ 11, \frac{(2k + 6) + \sqrt{(2k + 6)^2 - 40}}{2} \right\} > 2k + 5,$$
which proves the claim. \hfill \qed 

Now, we shall factorize \(K_{18}\) into five factors of diameter 3 and one factor of subdiameter 2. We shall denote this factorization by \(D\), its factors of diameter 2 by \(F_{D,1}, F_{D,2}, \ldots, F_{D,5}\) and its factor of subdiameter 2 by \(F_{D,0}\). Denote \(V(K_{18}) = \{v_1, \ldots, v_{18}\}\). factorization \(D\) is given by the following table

\[
\begin{align*}
099199299399499599 \\
909919929939949959 \\
990919929939949959 \\
99011129139114599 \\
91910199399411151 \\
9111021131194519 \\
299192022239499225 \\
929221292939224959 \\
929291220322492952 \\
399193293033499353 \\
93933133303349959 \\
93991992330433539 \\
499199424343444995 \\
4914929943404445 \\
99414929943404594 \\
59951529939545055 \\
95995125553949505 \\
99591950239955450 \\
\end{align*}
\]

where \(T(i,j) = k\) denotes \(v_iv_j \in F_{D,k}\), \(k = 1, \ldots, 5\) and \(T(i,j) = 9\) denotes \(v_iv_j \in F_{D,0}\). A simple check shows that this factorization has required properties.

Now, we can prove

**Lemma 4.** Let \(k \geq 5\). Then \(\phi(k) \leq 3 \cdot k + 3\).

**Proof.** We shall explicitly give a factorization of \(K_{3k+3}\) into factors \(F_1, \ldots, F_k\) of diameter 3 and a factor \(F_0\) of subdiameter 2. Denote \(V(K_{3k+3}) = \{v_1, v_2, \ldots, v_{3k+3}\}\).

The edges of factor \(F_i, 1 \leq i \leq 4\), are given by:

1) \(v_av_b\) such that \(v_av_b \in F_{D,i}\).
2) \(v_av_b\) such that \(4 \leq a \leq 15\), \(b \geq 19\) and there is \(c\) such that \(16 \leq c \leq 18\), \(v_av_c \in F_{D,i}\), and \(b \equiv c \mod 3\).

The edges of the factor \(F_i, i \geq 5\) are given by

1) \(v_av_b\) such that \(a \leq 15\), \(3 \cdot i + 1 \leq b \leq 3 \cdot i +3\) and there is \(c\) such that \(16 \leq c \leq 18\), \(v_av_c \in F_{D,5}\), and \(b \equiv c \mod 3\).
2) \(v_av_b\) such that \(3 \cdot i + 1 \leq a \leq 3 \cdot i +3\), \(b \geq 3i + 4\), and \(a \equiv b \mod 3\).
3) \(v_av_b\) such that \(3 \cdot i + 1 \leq a \leq 3 \cdot i +3\), \(16 \leq b \leq 3 \cdot i + 3\), and \(a \neq b \mod 3\).
4) \(v_av_b\) such that \(3 \cdot i + 1 \leq a, b \leq 3 \cdot i + 3\) and \(a \neq b\).

The remaining edges are the edges of factor \(F_0\). Let us prove that \(\text{diam } F_i = 3, 1 \leq i \leq k\). Vertices \(v_{3i+1}, v_{3i+2}\) and \(v_{3i+3}\) form a triangle (in \(F_i\)) and every other vertex is adjacent (in \(F_i\)) to at least one of the vertices \(v_{3i}, v_{3i+1}\) and \(v_{3i+2}\), hence \(\text{diam } F_i = 3, i \leq 4\).

What remains to be proved is that \(\text{subdiam } F_0 = 2\). We need to prove that for each two vertices \(v_x\) and \(v_y\) there are two paths of length at most 2 connecting them. Without loss of generality, we may assume that \(x < y\). Distinguish five cases:
1) \(x \leq 18, y \leq 18\).
Note that \(F_0 [v_1, \ldots, v_{18}] \cong F_{D,0} [v_1, \ldots, v_{18}]\), hence there are two paths of length 2 that connect \(v_x\) and \(v_y\) in \(F_{D,0} [v_1, \ldots, v_{18}]\).
2) \(x \leq 15, y > 18\).
Note that

\[
F_0 \left[ v_1, \ldots, v_{15}, v_3 \frac{x+y}{3} + 1, v_3 \frac{x+y}{3} + 2, v_3 \frac{x+y}{3} + 3 \right] \cong F_{D,0} [v_1, \ldots, v_{18}],
\]

hence there are two paths of length at most 2 that connect \(v_x\) and \(v_y\) in
\(F_0 \left[ v_1, \ldots, v_{15}, v_3 \frac{x+y}{3} + 1, v_3 \frac{x+y}{3} + 2, v_3 \frac{x+y}{3} + 3 \right]\).
3) \(x > 15, y > 15, x \not\equiv y \mod 3\).
Let \(16 \leq z \leq 3 \cdot k + 3\) be any number such that \(z \not\equiv x \mod 3\) and \(z \not\equiv y \mod 3\).
Note that
\(F_0 \left[ v_1, \ldots, v_{15}, v_z, v_x, v_y \right] \cong F_{D,0} [v_1, \ldots, v_{18}]\),

hence there are two paths of length at most 2 that connect \(v_x\) and \(v_y\) in
\(F_{D,0} \left[ v_1, \ldots, v_{15}, v_z, v_3 \frac{x+y}{3} + 1, v_3 \frac{x+y}{3} + 2, v_3 \frac{x+y}{3} + 3 \right]\).
4) \(x > 15, y > 15, x \equiv y \mod 3\).
Let \(1 \leq p, q \leq 3\) be two numbers such that \(p \not\equiv x \mod 3\) and \(q \not\equiv x \mod 3\).
Note that \(xpy\) and \(xyq\) are paths in \(F_0\).

Therefore, we have exhausted all the cases and we have proved the theorem. \(\square\)

Using a similar technique to that in the last lemma and factorization \(D'\), it can be proved that \(\phi (4) \leq 16\).

Summarizing our results, we get:

Theorem 1.

\[
\begin{align*}
\phi (1) &= 7 & \phi (2) &= 10 \\
\phi (3) &= 13 & 14 \leq \phi (4) \leq 16 \\
2k + 6 \leq \phi (k) \leq 3k + 3, & \quad k \geq 5.
\end{align*}
\]

The relation (*) gives rather good bounds for small values of \(k\). These bounds are not so good when \(k\) is large. Note that, from (*), we can conclude only

\[
2 \leq \lim_{k \to \infty} \frac{\phi (k)}{k} \leq \lim_{k \to \infty} \frac{\phi (k)}{k} \leq 3.
\]

In fact, we have

\[
\lim_{k \to \infty} \frac{\phi (k)}{k} = 2.
\]

Let us prove:

Lemma 5. If \(K_v\) can be factorized into \(k\) factors of diameter 3 and two factors of diameter 2, then it can be factorized into \(k\) factors of diameter 3 and one factor of subdiameter 2.

Proof. Let \(K_v\) be factorized into \(k\) factors \(F_1, \ldots, F_k\) of diameter 3 and two factors \(G_1\) and \(G_2\) of diameter 2. Denote by \(F_0\) a graph such that \(V (F_0) = V (K_v)\)
and \( E(F_0) = E(G_1) \cup E(G_2) \). Factors \( F_0, F_1, ..., F_k \) form a factorization with the required properties. \( \square \)

Denote by \( f \left( 2, 2, 3, ..., 3 \right) \) the smallest number \( v \) such that \( K_v \) can be factorized into \( k \) factors of diameter 2 and \( p \) factors of diameter 3. Note that from the last Lemma, it follows

\[
\phi(k) \leq f \left( 2, 2, 3, ..., 3 \right).
\]

In [10], it is proved that

**Theorem 2.** \( \lim_{k \to \infty} \frac{f \left( 2, 2, 3, ..., 3 \right)}{k} = 2 \), where \( p \) is a fixed natural number.

Combining the last Lemma and the last Theorem (taking \( p = 2 \)), we get

**Theorem 3.**

\[
\lim_{k \to \infty} \phi(k) = 2.
\]

More precisely, by the construction analogous to the one given in [10], it follows that

**Proposition 1.** For sufficiently large \( k \in \mathbb{N} \), we have

\[
\phi(k) \leq 2k + 5 \cdot \left\lceil \sqrt{k} \right\rceil.
\]  \hspace{1cm} (3)

**Proof.** Let \( t \) be the smallest natural number such that

\[
\binom{2t-1}{t-1} \geq k.
\]

We will construct a factorization of \( K_n, n = 2k + 4 \left\lceil \sqrt{k} \right\rceil + 4t + 2 \), into factors \( F_0, F_1, F_2, ..., F_k \) such that subdiam \( (F_0) = 2 \) and diam \( (F_i) = 3, 1 \leq i \leq k \). Let

\[
V(K_n) = L \cup D \cup W \cup Z \cup U \cup U' \cup A \cup A' \cup B \cup B',
\]

where

\[
L = \{l_1, ..., l_k\}, D = \{d_1, ..., d_k\}, W = \{w_0, ..., w_{\left\lceil \sqrt{k} \right\rceil - 1}\}, Z = \{z_0, ..., z_{\left\lceil \sqrt{k} \right\rceil - 1}\},
\]

\[
U = \{u_1, ..., u_{\left\lceil \sqrt{k} \right\rceil}\}, U' = \{u'_1, ..., u'_{\left\lceil \sqrt{k} \right\rceil}\}, A = \{a_1, a_2\}, A' = \{a'_1, a'_2\},
\]

\[
B = \{b_1, ..., b_{2t-1}\}, B' = \{b'_1, ..., b'_{2t-1}\}.
\]

Let \( B \) be the set of all \( t - 1 \) element subsets of the set \( \{1, 2, ..., 2t - 1\} \). Let \( f \) be any injection

\[
f : \{1, ..., k\} \to B.
\]
Let us notice that for each \( j \in \{1, \ldots, k\} \) there are unique numbers \( q_j \) and \( r_j \) such that
\[
j = q_j \cdot \sqrt[8]{k} + r_j, \quad 0 \leq q_j \leq \sqrt[8]{k} - 1, \quad 1 \leq r_j \leq \sqrt[8]{k}.
\]
The edges of the factor \( F_i, 1 \leq i \leq k \) are

1) \( l_id_i \)
2) \( l_il_j, \quad 1 \leq j < i \leq k \)
3) \( d_id_j, \quad 1 \leq j < i \leq k \)
4) \( d_id_j, \quad 1 \leq j < i \leq k \)
5) \( l_il_j, \quad 1 \leq i < j \leq k \)
6) \( l_ia_1, l_ia_2, d_ia'_1, d_ia'_2 \)
7) \( l_ib_j, l_ib'_j, \quad j \in f(i) \)
8) \( d_ib_j, d_ib'_j, \quad j \in \{1, 2, \ldots, 2t - 1\} \setminus f(i) \)
9) \( l_iw_j, \quad 1 \leq j \leq \lceil \sqrt[8]{k} \rceil - 1 \)
10) \( d_iz_j, \quad 1 \leq j \leq \lceil \sqrt[8]{k} \rceil - 1 \)
11) \( w_iu_{r_i}, w'_{r_i} \)
12) \( z_iu_{r_i}, u'_{r_i} \)
13) \( d_iw_j, d_ib'_j, \quad 1 \leq j \leq k, \quad j \neq r_i \).

The other edges are edges of factor \( F_0 \).

In each factor \( F_i, 1 \leq i \leq k \) all vertices are adjacent to either \( l_i \) or \( d_i \), except \( u_{r_i} \) and \( u'_{r_i} \) which have two common neighbors and which are connected by a path of length 2 to both, \( l_i \) and \( d_i \), and also \( l_i \) and \( d_i \) are adjacent, hence we have \( \text{diam}(F_i) \leq 3, 1 \leq i \leq k \).

Now, let us prove that \( \text{diam}(F_i) \geq 3, 1 \leq i \leq k \). Let \( i \) be an arbitrary number such that \( 1 \leq i \leq k \). Let \( j \) be an element of the set \( \{1, 2, \ldots, 2t - 1\} \setminus f(i) \). Note that \( d_{F_i}(a_1, b_j) = 3 \), so the claim is proved.

It remains to prove that \( \text{subdiam}(F_0) = 2 \). It is sufficient to prove that every two vertices \( x \) and \( y \) have two common neighbors. Without loss of generality, we may assume that we have one of the following cases:

1) \( x \notin L \cup D \).

Distinguish two possibilities:

1a) \( y \in L \implies N_{F_0}(x) \cap N_{F_0}(y) = \{a'_1, a'_2\} \).
1b) \( y \notin L \implies N_{F_0}(x) \cap N_{F_0}(y) = \{a_1, a_2\} \).

2) \( x, y \in L \implies N_{F_0}(x) \cap N_{F_0}(y) = \{a'_1, a'_2\} \).
3) \( x, y \in D \implies N_{F_0}(x) \cap N_{F_0}(y) = \{a_1, a_2\} \).
4) \( x \in L, y \in D \). We distinguish two cases.

4a) \( x = l_i, y = d_j, 1 \leq i \leq k \Rightarrow u_{r_i}, u'_{r_i} \in N_{F_0}(l_i) \cap N_{F_0}(d_j) \).
4b) \( x = l_i, y = d_j, 1 \leq i, j \leq k, i \neq j \). We have
\[
|N_{F_0}(l_i) \cap B| + |N_{F_0}(d_j) \cap B| = t - 1 + t = |B|,
\]
so either there is a vertex \( b \in B \) element of \( N_{F_0}(l_i) \cap N_{F_0}(d_j) \) or

\[
N_{F_0}(l_i) \cap B = B \setminus N_{F_0}(d_j) = N_{F_0}(l_i) \cap B
\]
which is impossible. Completely analogously we show that there is a vertex \( b' \in B' \) element of \( N_{F_0}(l_i) \cap N_{F_0}(d_j) \).
Therefore, 
\[ \phi(k) \leq 2k + 4 \left\lceil \sqrt{k} \right\rceil + 4t + 2. \]
For sufficiently large \( k \), we have 
\[ \phi(k) \leq 2k + 4 \left\lceil \sqrt{k} \right\rceil + 4t + 2 \leq 2k + 5 \left\lceil \sqrt{k} \right\rceil. \]
\[\square\]

References


[10] D. Vukičević, *Decomposition of complete graph into factors of diameter two and three*, Discussiones Mathematicae Graph Theory, to appear
