GS–trapezoids in GS–quasigroups

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Abstract. *In this paper the concept of a GS–trapezoid in a GS–quasigroup is defined and some characterizations of that are proved and geometrical representation of the properties of the quaternary relation GST in the GS–quasigroup* $C(\frac{1}{2}(1+\sqrt{5}))$ *is given.*

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In [1] a GS–quasigroup is defined as a quasigroup which satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c=b, \qquad a\cdot(a\cdot bc)c=b \tag{1, 1'}$$

and moreover the identity of idempotency

$$aa=a. \tag{2}$$

Remark 1. *Any groupoid with two identities (1) and (1)′ is necessary a quasigroup since the identity (1) implies left solvability and left cancellation of the considered groupoid* $(Q,\cdot)$. *Really, for every* $a,b\in Q$ *there is* $y\in Q$ *such that* $ya=b$. *Indeed, we can take* $y=a(ab\cdot a)$ *because of (1). Further, from* $ax_1=ax_2$ *it follows* $a(ax_1\cdot a)\cdot a=a(ax_2\cdot a)\cdot a$ *and according to (1), we have* $x_1=x_2$. *Analogously, the identity (1)′ implies right solvability and right cancellation of the considered groupoid.*

The considered GS–quasigroup $(Q,\cdot)$ satisfies the mediality, elasticity, left and right distributivity i.e. we have the identities

$$ab\cdot cd=ac\cdot bd \tag{3}$$

$$a\cdot ba=ab\cdot a \tag{4}$$

$$a\cdot bc=ab\cdot ac, \qquad ab\cdot c=ac\cdot bc \tag{5, 5'}$$

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Further, the identities

$$(6) \qquad a(ab \cdot b) = b, \qquad\qquad (b \cdot ba)a = b \qquad (6)'$$

$$(7) \qquad a(ab \cdot c) = b \cdot bc, \qquad\qquad (c \cdot ba)a = cb \cdot b \qquad (7)'$$

and equivalencies

$$(8) \qquad ab = c \Leftrightarrow a = c \cdot cb, \qquad\qquad ab = c \Leftrightarrow b = ac \cdot c \qquad (8)'$$

also hold.

Example. Let C be the set of points of the Euclidean plane. For any two different points a, b we define $ab = c$ if the point b or a divides the pair a, c (*Figure 1*) or the pair b, c (*Figure 2*), respectively, in the ratio of the golden section.

Figure 1.

Figure 2.

In [1] it is proved that $(Q, \cdot)$ is a GS–quasigroup in both cases. We shall denote these two quasigroups by $C(\frac{1}{2}(1+\sqrt{5}))$ and $C(\frac{1}{2}(1-\sqrt{5}))$ because we have $c = \frac{1}{2}(1+\sqrt{5})$ or $c = \frac{1}{2}(1-\sqrt{5})$ if $a = 0$ and $b = 1$.
The relations in any GS–quasigroup will be illustrated geometrically by figures which represent relations in the GS–quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$.

The considered two quasigroups are equivalent because of the following lemma.

Lemma 1. *If the operations $\cdot$ and $\bullet$ on the set Q are connected with the identity $a \bullet b = ba$, then $(Q, \bullet)$ is a GS–quasigroup if and only if $(Q, \cdot)$ is a GS–quasigroup.*

From now on, let $(Q, \cdot)$ be any GS–quasigroup. The elements of the set Q are called points.

We shall say that points a, b, c, d form a *parallelogram* and write $Par(a, b, c, d)$ if the following identity

$$a \cdot b(ca \cdot a) = d$$

holds (*Figure 3*).

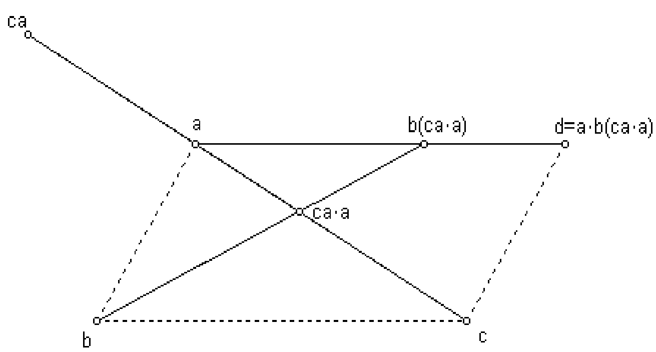

Figure 3.

In [1] the different properties of the quaternary relation Par on the set Q are proved. We shall mention only the two following properties which we shall use afterwards.

Lemma 2. *If* (e, f, g, h) *is any cyclic permutation of* (a, b, c, d) *or of* (d, c, b, a), *then* $Par(a, b, c, d)$ *implies* $Par(e, f, g, h)$.

Lemma 3. *From* $Par(a, b, c, d)$ *and* $Par(c, d, e, f)$ *follows* $Par(a, b, f, e)$.

The points a, b, c, d successively are said to be the vertices of the *golden section trapezoid* and it is denoted by $GST(a, b, c, d)$ if the identity

$$a \cdot ab = d \cdot dc$$

holds (*Figure 4*).

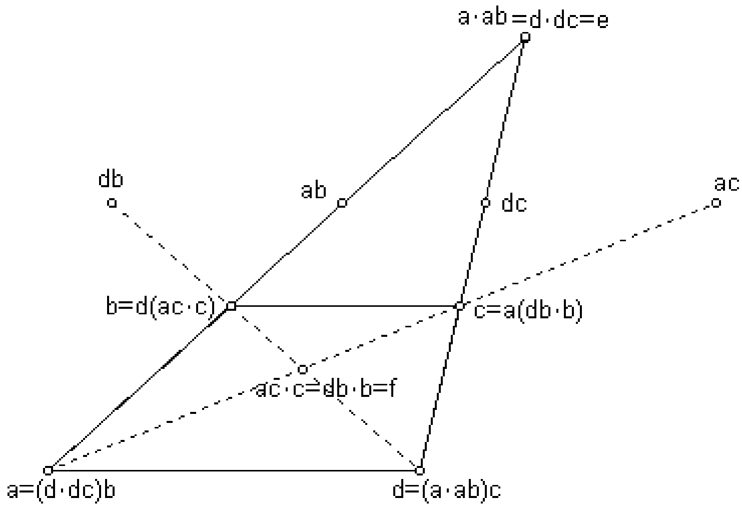

Figure 4.

Obviously, the following theorem holds.

Theorem 1. $GST(a, b, c, d)$ *implies* $GST(d, c, b, a)$.

If the relation $GST(a, b, c, d)$ holds, we shall say that the points c, a, d, b form a *GS–trapezoid of the second kind* and write $\overline{GST}(c, a, d, b)$. It means that the statements $GST(a, b, c, d)$ and $\overline{GST}(c, a, d, b)$ are equivalent. Because of that, the statements $GST(b, d, a, c)$ and $\overline{GST}(a, b, c, d)$ are equivalent and according to *Theorem 1*, the statements $\overline{GST}(a, b, c, d)$ and $GST(c, a, d, b)$ are also equivalent. Indeed, it means that the relations GST and $\overline{GST}$ are mutually symmetric.

Let us prove the next lemma now.

Lemma 4. *The statement* $\overline{GST}(a, b, c, d)$ *is equivalent to the equality* $ba \cdot a = cd \cdot d$.

Proof. We have successively

$$d \overset{(1)'}{=} c \cdot (c \cdot da)a \overset{(6)'}{=} c \cdot [(c \cdot ca)a \cdot da]a \overset{(5)'}{=} c \cdot [(c \cdot ca)d \cdot a]a,$$

wherefrom it is obvious that the equalities $b = (c \cdot ca)d$ and $d = c(ba \cdot a)$ are equivalent. However, by (8) the first equality is equivalent to $c \cdot ca = b \cdot bd$ i.e. $GST(c, a, d, b)$ i.e. $\overline{GST}(a, b, c, d)$. Analogously, by $(8)'$ the second equality is equivalent to $ba \cdot a = cd \cdot d$. □

Since the equality $ba \cdot a = cd \cdot d$ is equivalent to the equality $a \bullet (a \bullet b) = d \bullet (d \bullet c)$ in the quasigroup $(Q, \bullet)$ where the operation $\bullet$ is defined by $a \bullet b = ba$ by *Lemma 1* and *Lemma 4*, the next theorem follows immediately.

Theorem 2 [theorem about duality for GS–trapezoids]. *From every theorem about GS–trapezoids we get an analogous theorem about GS–trapezoids of the second kind (and vice versa) if the roles of both factors are interchanged in all products which appear in the theorem.*

Corollary 1. *From every theorem about GS–trapezoids we get again a theorem about GS-trapezoids, if every statement of the form $GST(a, b, c, d)$ is interchanged by the corresponding statement $GST(c, a, d, b)$ and the roles of both factors are interchanged in all products.*

In the interchanges mentioned in *Theorem 2* and *Corollary 1* it is not necessary to make any interchange in possible statements about relation Par, since the equality $d = a \cdot b(ca \cdot a)$ is equivalent to the equality $d = (a \cdot ac)b \cdot a$.
Really, we get the following

$$\begin{aligned} a \cdot b(ca \cdot a) &\overset{(5)}{=} ab \cdot a(ca \cdot a) \overset{(4)}{=} ab \cdot (a \cdot ca)a \overset{(7)'}{=} ab \cdot (ac \cdot c) \\ &\overset{(3)}{=} (a \cdot ac) \cdot bc \overset{(5)}{=} (a \cdot ac)b \cdot (a \cdot ac)c \overset{(6)'}{=} (a \cdot ac)b \cdot a. \end{aligned}$$

From the interrelation of two quasigroups $C(\frac{1}{2}(1 + \sqrt{5}))$ and $C(\frac{1}{2}(1 - \sqrt{5}))$ and according to the theorem about duality, it follows that a GS–trapezoid in one of these two quasigroups will be a GS–trapezoid of the second kind in the other quasigroup and vice versa. Indeed, it means it is the matter of convention which of the two quadrangles (a, b, c, d) or (c, a, d, b) will be called a GS–trapezoid and which one a GS–trapezoid of the second kind, since we cannot differ them in the general GS–quasigroup.

Theorem 3. *The statement $GST(a, b, c, d)$ is equivalent to the equality $ac \cdot c = db \cdot b$. (*Figure 4*).*

Proof. It follows by *Corollary 1* from the fact that the statement $GST(b, d, a, c)$ is equivalent to the equality $c \cdot ca = b \cdot bd$. □

According to (8), the equality $a \cdot ab = d \cdot dc$ is equivalent to the equalities $(a \cdot ab)c = d$ and $(d \cdot dc)b = a$, and similarly according to $(8)'$, the equality $ac \cdot c = db \cdot b$ is equivalent to the equalities $c = a(db \cdot b)$ and $b = d(ac \cdot c)$. From here the next theorem follows straightforward.

Theorem 4. *The statement $GST(a, b, c, d)$ is equivalent with any of the four equalities $a = (d \cdot dc)b$, $b = d(ac \cdot c)$, $c = a(db \cdot b)$, $d = (a \cdot ab)c$ (*Figure 4*).*

Corollary 2. *A GS–trapezoid is uniquely determined by any three of its vertices.*

Theorem 5.

*(i) The statement $GST(a,b,c,d)$ holds iff there is a point e such that $eb = a$, $ec = d$ (*Figure 4*).*

*(ii) The statement $GST(a,b,c,d)$ holds iff there is a point f such that $af = c$, $df = b$ (*Figure 4*).*

Proof. By (8) the equality $eb = a$ is equivalent to $e = a \cdot ab$, and analogously $ec = d$ is equivalent to $e = d \cdot dc$, wherefrom according to the equivalency of the statement $GST(a,b,c,d)$ and equality $a \cdot ab = d \cdot dc$, the statement (i) of the theorem follows. The statement (ii) follows from (i) by *Corollary 1* and by the substitution of the points a,b,c,d,e with the points b,d,a,c,f, respectively. □

Remark 2. *From now on, like here, the statement (ii) of any theorem follows from the corresponding statement (i) applying* Corollary 1 *and some substitutions of the points.*

Let us prove now some interesting characterizations of the statement $GST(a,b,c,d)$.

Theorem 6.

(i) The statement $GST(a,b,c,d)$ holds iff for any point x the equality $xa \cdot b = xd \cdot c$ is valid.

(ii) The statement $GST(a,b,c,d)$ holds iff for any point x the equality $a \cdot cx = d \cdot bx$ is valid.

Proof. (i) Since we have successively

$$xa \cdot b \overset{(6)}{=} xa \cdot a(ab \cdot b) \overset{(5)}{=} xa \cdot (a \cdot ab)(ab) \overset{(3)}{=} x(a \cdot ab) \cdot (a \cdot ab) \overset{(7)'}{=} [x \cdot (a \cdot ab)c]c$$

the equality $xa \cdot b = xd \cdot c$ is equivalent to the equality $d = (a \cdot ab)c$ i.e. $GST(a,b,c,d)$ because of *Theorem 4*. □

Now, we shall prove some more simple properties of the quaternary relation GST on the set Q.

Theorem 7. *For any three points a,b,c the following statements hold*

*(i) $GST(ab,b,c,ac)$ (*Figure 5*),*

*(ii) $GST(b,ca,ba,c)$ (*Figure 5*).*

Proof. By (8) from $ab = d$ and $ac = e$ it follows $a = d \cdot db = e \cdot ec$ i.e. $GST(d,b,c,e)$ i.e. $GST(ab,b,c,ac)$. □

Corollary 3. *For any two points a,b the statements*

$$GST(a,b,b,a), \quad GST(a,a,b,ab) \quad \text{and} \quad GST(a,ba,a,b)$$

hold.

Proof. For any points a,b there is an element c such that $cb = a$ and the first statement follows from $GST(cb,b,b,cb)$. We get the other two statements from *Theorem 7* with $b = a$ and the substitution $c \to b$ because of (2). □

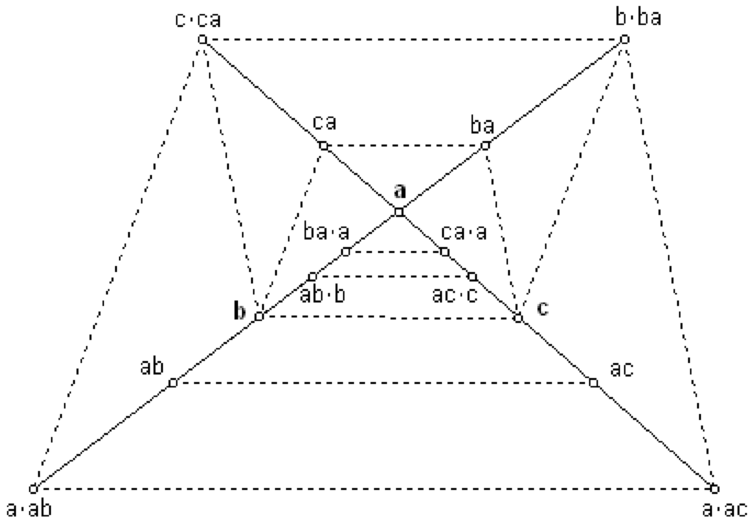

Figure 5.

Theorem 8. *For any three points a, b, c the following statements hold*

(i) $GST(b, ab \cdot b, ac \cdot c, c)$ (Figure 5),

(ii) $GST(b \cdot ba, c, b, c \cdot ca)$ (Figure 5).

Proof. (i) We have

$$b \cdot b(ab \cdot b) \overset{(4)}{=} b \cdot (b \cdot ab)b \overset{(1)'}{=} a \overset{(1)'}{=} c \cdot (c \cdot ac)c \overset{(4)}{=} c \cdot c(ac \cdot c).$$ □

Theorem 9. *For any three points a, b, c the following statements hold*

(i) $GST(a \cdot ab, c \cdot ca, b \cdot ba, a \cdot ac)$ (Figure 5),

(ii) $GST(ab \cdot b, ba \cdot a, ca \cdot a, ac \cdot c)$ (Figure 5).

Proof. (i) We have

$$\begin{aligned} [(a \cdot ab) \cdot (a \cdot ab)(c \cdot ca)](b \cdot ba) &\overset{(3)}{=} (a \cdot ab)b \cdot [(a \cdot ab)(c \cdot ca) \cdot ba] \\ &\overset{(3)}{=} (a \cdot ab)b \cdot [(a \cdot ab)b \cdot (c \cdot ca)a] \\ &\overset{(6)'}{=} a \cdot ac. \end{aligned}$$

□

Theorem 10. *The statement $GST(a, b, c, d)$ implies the statement $GST(c, d, ad, b \cdot bc)$* (Figure 6).

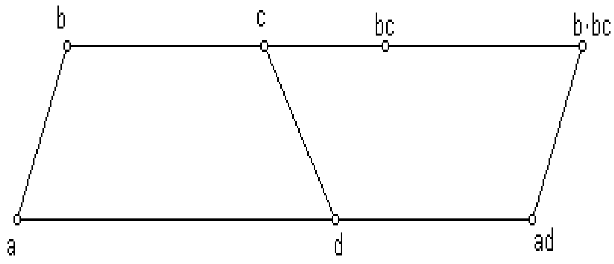

Figure 6.

Proof. As we have the equality $d = (a \cdot ab)c$, we obtain the following

$$\begin{aligned}(c \cdot cd) \cdot ad &\overset{(3)}{=} ca \cdot (cd \cdot d) = ca \cdot [c \cdot (a \cdot ab)c][(a \cdot ab)c] \overset{(4)}{=} ca \cdot [c(a \cdot ab) \cdot c][(a \cdot ab)c] \\ &\overset{(5)}{=} ca \cdot [c(a \cdot ab) \cdot (a \cdot ab)]c \overset{(3)}{=} c[c(a \cdot ab) \cdot (a \cdot ab)] \cdot ac \overset{(6)}{=} (a \cdot ab) \cdot ac \\ &\overset{(5)}{=} a(ab \cdot c) \overset{(7)}{=} b \cdot bc.\end{aligned}$$

□

Theorem 11.

(i) The statements

$$GST(a_1, b_1, b_2, a_2), GST(a_2, b_2, b_3, a_3), \ldots, GST(a_{n-1}, b_{n-1}, b_n, a_n)$$

imply the statement $GST(a_n, b_n, b_1, a_1)$.

(ii) The statements

$$GST(b_2, a_1, a_2, b_1), GST(b_3, a_2, a_3, b_2), \ldots, GST(b_n, a_{n-1}, a_n, b_{n-1})$$

imply the statement $GST(b_1, a_n, a_1, b_n)$.

Proof. (i) It follows straightforward from the equality

$$a_1 \cdot a_1 b_1 = a_2 \cdot a_2 b_2 = a_3 \cdot a_3 b_3 = \cdots = a_{n-1} \cdot a_{n-1} b_{n-1} = a_n \cdot a_n b_n.$$

□

Putting in the previous theorem $n = 2$ and introducing new labels we get the following result.

Corollary 4 [“golden” affine Desargues theorem].

(i) The statements $GST(a, b, c, d)$ *and* $GST(a, b, c', d')$ *imply the statement* $GST(d, c, c', d')$ *(*Figure 7*).*

(ii) The statements $GST(a, b, c, d)$ *and* $GST(a, b', c, d')$ *imply the statement* $GST(d, b', b, d')$.

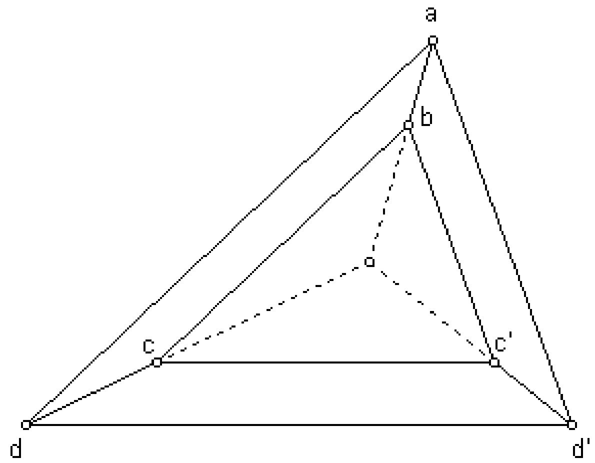

Figure 7.

Theorem 12.

(i) Any two of three statements $GST(a, b, c, d)$, $GST(b, c, d, e)$, $GST(c, d, e, a)$ *imply the remaining statement (*Figure 8*).*

(ii) Any two of three statements $GST(a, b, c, d)$, $GST(b, c, d, e)$, $GST(d, e, a, b)$ *imply the remaining statement (*Figure 8*).*

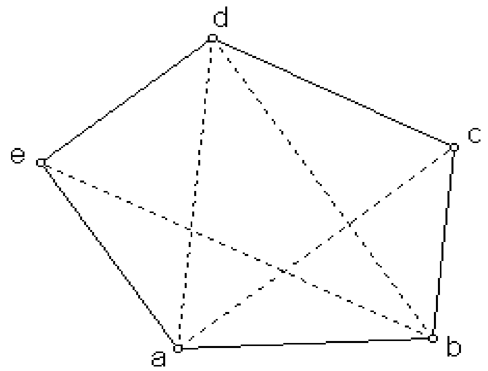

Figure 8.

Proof. According to *Theorem 1* we have symmetry $b \leftrightarrow e$, $c \leftrightarrow d$, so it is sufficient to prove with the assumption $GST(a, b, c, d)$ i.e. $d = (a \cdot ab)c$ the equivalency of the statements $GST(b, c, d, e)$ and $GST(c, d, e, a)$ i.e. the equivalency of the equalities $e = (b \cdot bc)d$ and $(c \cdot cd)e = a$. However, we obtain

$$\begin{aligned}(c \cdot cd) \cdot (b \cdot bc)d &= c\,[c \cdot (a \cdot ab)c] \cdot [(b \cdot bc) \cdot (a \cdot ab)c] \\ &\overset{(4)}{=} c\,[c(a \cdot ab) \cdot c] \cdot [(b \cdot bc) \cdot (a \cdot ab)c] \\ &\overset{(7)}{=} [(a \cdot ab) \cdot (a \cdot ab)c]\,[(b \cdot bc) \cdot (a \cdot ab)c] \\ &\overset{(5)'}{=} (a \cdot ab)(b \cdot bc) \cdot (a \cdot ab)c \overset{(5)}{=} (a \cdot ab) \cdot (b \cdot bc)c \\ &\overset{(6)'}{=} (a \cdot ab)b \overset{(6)'}{=} a.\end{aligned}$$

□

Theorem 13.

(i) Any three of the four statements

$$GST(a,b,c,d), \quad GST(a,b',c',d), \quad GST(b,a,b',e) \quad \textit{and} \quad GST(c,d,c',e)$$

imply the remaining statement (Figure 9)*.*

(ii) Any three of the four statements

$$GST(a,b,c,d), \quad GST(a',b,c,d'), \quad GST(a',a,e,c) \quad \textit{and} \quad GST(d',d,e,b)$$

imply the remaining statement.

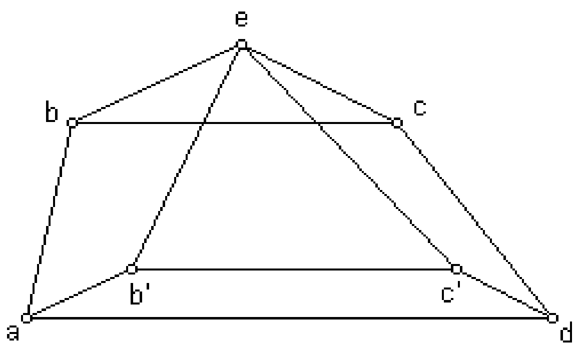

Figure 9.

Proof. Because of the assumption $GST(b,a,b',e)$ i.e. $(b \cdot ba)b' = e$ and as we have the symmetry $a \leftrightarrow d$, $b \leftrightarrow c$, $b' \leftrightarrow c'$, it is sufficient to prove that any two of three statements $GST(a,b,c,d)$, $GST(a,b',c',d)$, $GST(c,d,c',e)$ imply the remaining statement i.e. any two of three equalities

(9) $$d(ac \cdot c) = b,$$

(10) $$d(ac' \cdot c') = b',$$

(11) $$(c \cdot cd)c' = e$$

imply the remaining equality. We have successively

$$
\begin{aligned}
&[d(ac\cdot c)]\,[d(ac\cdot c)\cdot a]\cdot d(ac'\cdot c') \overset{(3)}{=} [d\cdot d(ac\cdot c)]\,[(ac\cdot c)a]\cdot d(ac'\cdot c')\\
\overset{(3)}{=}\ &[d\cdot d(ac\cdot c)]\,d\cdot[(ac\cdot c)a\cdot(ac'\cdot c')] \overset{(3)}{=} [d\cdot d(ac\cdot c)]\,d\cdot[(ac\cdot c)(ac')\cdot ac']\\
\overset{(3)}{=}\ &[d\cdot d(ac\cdot c)]\,d\cdot[(ac\cdot a)(cc')\cdot ac'] \overset{(3)}{=} [d\cdot d(ac\cdot c)]\,d\cdot[(ac\cdot a)a\cdot(cc'\cdot c')]\\
\overset{(4)}{=}\ &d\,[d(ac\cdot c)\cdot d]\cdot[(a\cdot ca)a\cdot(cc'\cdot c')] \overset{(7),(7)'}{=} [(ac\cdot c)\cdot(ac\cdot c)d]\cdot(ac\cdot c)(cc'\cdot c')\\
\overset{(5)}{=}\ &(ac\cdot c)\,[(ac\cdot c)d\cdot(cc'\cdot c')] \overset{(7)}{=} d\cdot d(cc'\cdot c') \overset{(5)}{=} d\cdot(d\cdot cc')(dc') \overset{(7)}{=} cc'\cdot(cc'\cdot dc')\\
\overset{(5)'}{=}\ &(c\cdot cd)c'
\end{aligned}
$$

then because of $(b\cdot ba)b'=e$, the implications $(9),(10)\Rightarrow(11)$ and $(9),(11)\Rightarrow(10)$ are obvious and because of (10) and (11) canceling by b', the equality $d(ac\cdot c)\cdot[d(ac\cdot c)\cdot a]=b\cdot ba$ follows, then multiplying on the right-hand side by a because of $(6)'$ there follows (9). □

Theorem 14. *If the statements*

$$GST(a,a',b',b),\quad GST(b,b',c',c),\quad GST(c,c',d',d),\quad GST(d,d',a',a)$$

hold (see Theorem 11 (i)*) then the statements $GST(a,b,c,d)$ and $GST(a',b',c',d')$ are equivalent.*

Proof. We have

$$a\cdot aa'=b\cdot bb'=c\cdot cc'=d\cdot dd'=o,$$

wherefrom by (8) it follows

$$a=oa',\quad b=ob',\quad c=oc',\quad d=od',$$

so we get

$$(a\cdot ab)c=(oa')(oa'\cdot ob')\cdot oc' \overset{(5)}{=} o\cdot(a'\cdot a'b')c'$$

and it is obvious that the equations $(a\cdot ab)c=d$ and $(a'\cdot a'b')c'=d'$ are equivalent. □

Theorem 15. *Any two of three statements $GST(a,b,c,d)$, $GST(b,e,f,c)$, $ae=df$ imply the remaining statement (*Figure 10*).*

Proof. We have

$$(a\cdot ab)c\cdot b(ce\cdot e) \overset{(3)}{=} (a\cdot ab)b\cdot c(ce\cdot e) \overset{(6)}{=} (a\cdot ab)b\cdot e \overset{(6)'}{=} ae,$$

and it is obvious that each of three equalities $(a\cdot ab)c=d$, $b(ce\cdot e)=f$, $ae=df$ is the consequence of the remaining two equalities. □

Theorem 16. *Any three of four statements $GST(a,b,c,d)$, $GST(b,e,f,c)$, $GST(a,e,f,g)$, $ag=d$ imply the remaining statement (*Figure 10*).*

Proof. As we have

$$
\begin{aligned}
a\,[(a\cdot ae)\cdot b(ce\cdot e)] &\overset{(3)}{=} a\,[ab\cdot(ae)(ce\cdot e)] \overset{(5)'}{=} a\,[ab\cdot(a\cdot ce)e)]\\
&\overset{(7)'}{=} a\cdot(ab)(ac\cdot c) \overset{(5)}{=} (a\cdot ab)\cdot a(ac\cdot c)\\
&\overset{(6)}{=} (a\cdot ab)c,
\end{aligned}
$$

so it follows that each of the equalities $(a \cdot ab)c = d$, $b(ce \cdot e) = f$, $(a \cdot ae)f = g$, $ag = d$ is the consequence of the remaining three equalities. $\square$

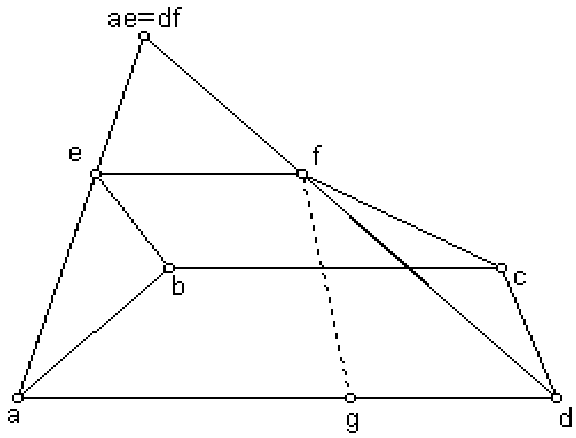

Figure 10.

Theorem 17.

*(i) Any three of the four statements $GST(a,b,c,d)$, $GST(b,e,f,c)$, $ab = e$, $dc = f$ imply the remaining statement (*Figure 11*).*

*(ii) Any three of the four statements $GST(a,b,c,d)$, $GST(h,d,a,g)$, $db = g$, $ac = h$ imply the remaining statement (*Figure 11*).*

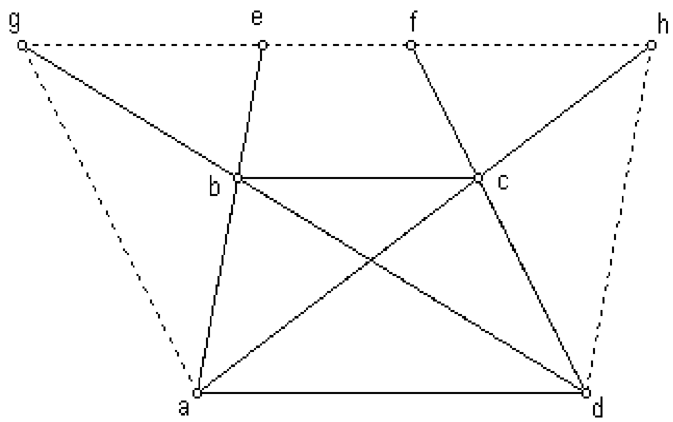

Figure 11.

Proof. (i) We have

$$\{c\cdot c\,[(a\cdot ab)c\cdot c]\}\cdot ab \overset{(4)}{=} \{c\cdot[c\cdot(a\cdot ab)c]\,c\}\cdot ab \overset{(1)'}{=} (a\cdot ab)\cdot ab \overset{(5)}{=} a(ab\cdot b) \overset{(6)}{=} b,$$

and each of the equalities $(a{\cdot}ab)c=d$, $(c{\cdot}cf)e=b$, $ab=e$, $dc=f$ is the consequence of the remaining three equalities. □

For the proof of some more statements about the relation GST we need some more lemmas.

Lemma 5.

*(i) Any two of the three statements $GST(a,b,c,d)$, $Par(a,b,c,e)$, $ae=d$ imply the remaining statement (*Figure 12*).*

*(ii) Any two of the three statements $GST(b,e,d,c)$, $Par(a,b,c,e)$, $ae=d$ imply the remaining statement (*Figure 12*).*

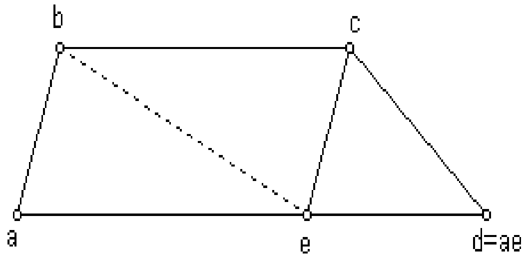

Figure 12.

Proof. (i) As we get

$$\begin{aligned} a\,[a\cdot b(ca\cdot a)] &\overset{(5)}{=} a\cdot[ab\cdot a(ca\cdot a)] \overset{(7)}{=} b\,[b\cdot a(ca\cdot a)] \overset{(4)}{=} b\,[b\cdot(a\cdot ca)a] \\ &\overset{(7)'}{=} b\cdot b(ac\cdot c) \overset{(5)}{=} b\cdot(b\cdot ac)(bc) \overset{(7)}{=} ac\cdot(ac\cdot bc) \overset{(5)'}{=} (a\cdot ab)c, \end{aligned}$$

it is obvious that each of the three equalities $(a\cdot ab)c=d$, $a\cdot b(ca\cdot a)=e$, $ae=d$ is the consequence of the remaining two equalities. □

Lemma 6. *Any two of the three statements $GST(a,b,c,d)$, $GST(a,b',c',d)$, $Par(b,b',c',c)$ imply the remaining statement (*Figure 13*).*

Proof. Let e be a point such that $ae=d$. Because of *Lemma 5 (i)*, the statement $GST(a,b,c,d)$ is equivalent to the statement $Par(a,b,c,e)$, and the statement $GST(a,b',c',d)$ to the statement $Par(a,b',c',e)$. However, because of symmetry $b\leftrightarrow b'$, $c\leftrightarrow c'$, it is sufficient because of *Lemma 2* to prove implications

$$Par(b,c,e,a),\ Par(e,a,b',c')\Rightarrow Par(b,c,c',b')$$

$$Par(e,a,b,c),\ Par(b,c,c',b')\Rightarrow Par(e,a,b',c'),$$

which follow by *Lemma 3.* □

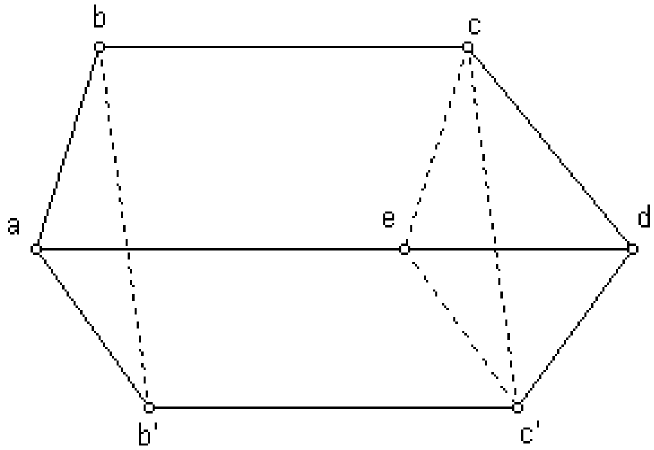

Figure 13.

Lemma 7. *Any two of the three statements $GST(a,b,c,d)$, $GST(a',b,c,d')$, $Par(a,a',d',d)$ imply the remaining statement (*Figure 14*).*

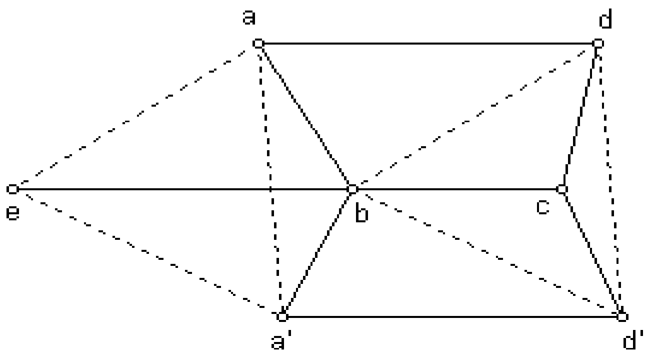

Figure 14.

Proof. The proof is analogous to the proof of *Lemma 6*, but instead of statement *(i)* of *Lemma 5* we use statement *(ii)* and we also use the point e such that $eb = c$ (*Figure 14*). □

Theorem 18. *Any of the statements*

$$GST(a,b,c,d), GST(a,b',c',d), GST(a',b,c,d'), GST(a',b',c',d')$$

*is the consequence of the remaining three statements (*Figure 15*).*

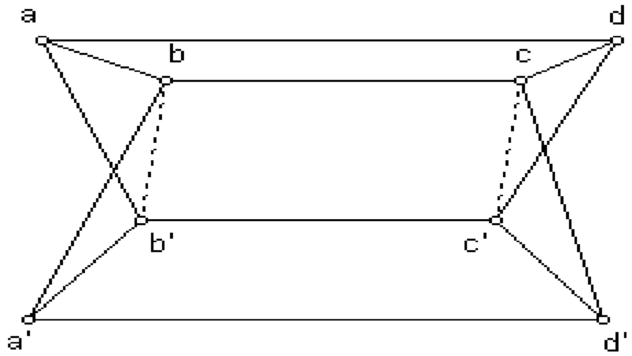

Figure 15.

Proof. Because of the symmetry $a \leftrightarrow a'$, $d \leftrightarrow d'$ and $b \leftrightarrow b'$, $c \leftrightarrow c'$, it is sufficient to prove that the fourth statement is the consequence of the first three statements. However, by *Lemma 6* we have implications

$$GST(a,b,c,d),\ GST(a,b',c',d) \Rightarrow Par(b,b',c',c)$$

$$GST(a',b,c,d'),\ Par(b,b',c',c) \Rightarrow GST(a',b',c',d').$$

□

Corollary 5. *Let the statements $GST(a,b,c,d)$, $GST(a,b',c',d)$ be valid and let a' be a given point. Then there is one and only one point d' such that $GST(a',b,c,d')$, $GST(a',b',c',d')$ are valid (*Figure 15*).*

Corollary 6. *Let the statements $GST(a,b,c,d)$, $GST(a',b,c,d')$ be valid and let b' be a given point. Then there is one and only one point c' such that $GST(a,b',c',d)$, $GST(a',b',c',d')$ are valid (*Figure 15*).*

Theorem 19. *Any of the statements*

$$GST(a,b,c,d), GST(a,b',c',d), GST(b,e,f,c), GST(b',e,f,c')$$

*is the consequence of the remaining three statements (*Figure 16*).*

Proof. By *Lemma 6* any of the statements $GST(a,b,c,d)$, $GST(a,b',c',d)$, $Par(b,b',c',c)$ follows from the two remaining statements, and by *Lemma 7* any of the statements $GST(b,e,f,c)$, $GST(b',e,f,c')$, $Par(b,b',c',c)$ follows from the remaining two statements, which proves our theorem. □

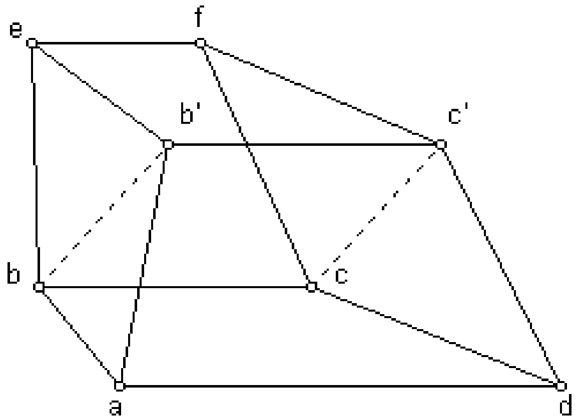

Figure 16.

Theorem 20. *Any four of five statements*

$$GST(a,b,c,d), GST(a',b',c',d'), GST(a,c,c',a'), GST(b,c,c',b'), GST(c,d,d',c')$$

*imply the remaining statement (*Figure 17).

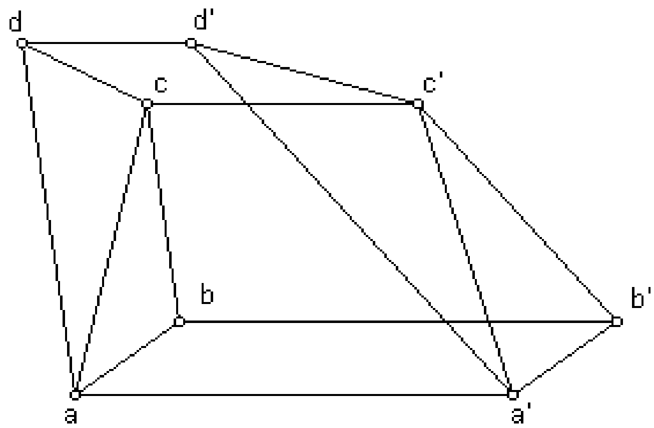

Figure 17.

Proof. First, let there hold $GST(a,c,c',a')$ i.e. $a' = (a \cdot ac)c'$. We will show that any three of four statements

$$GST(a,b,c,d),\ GST(a',b',c',d'),\ GST(b,c,c',b'),\ GST(c,d,d',c')$$

i.e.

$$(a \cdot ab)c = d,\ (a' \cdot a'b')c' = d',\ (b \cdot bc)c' = b',\ c \cdot cd = c' \cdot c'd'$$

imply the remaining statement.

However, it is obvious after the following conclusion

$$\begin{aligned} c\,[c\cdot(a\cdot ab)c] &\overset{(4)}{=} c\,[c(a\cdot ab)\cdot c] \overset{(7)}{=} (a\cdot ab)\cdot(a\cdot ab)c \overset{(5)'}{=} (a\cdot ab)\cdot(ac)(ac\cdot bc)\\ &\overset{(3)}{=} (a\cdot ab)\cdot(a\cdot ac)(c\cdot bc) \overset{(3)}{=} a(a\cdot ac)\cdot(ab)(c\cdot bc)\\ &\overset{(3)}{=} a(a\cdot ac)\cdot(ac)(b\cdot bc) \overset{(3)}{=} (a\cdot ac)\cdot(a\cdot ac)(b\cdot bc)\\ &\overset{(1)'}{=} c'\cdot\{c'\cdot[(a\cdot ac)\cdot(a\cdot ac)(b\cdot bc)]\,c'\}c'\\ &\overset{(4)}{=} c'\cdot c'\{[(a\cdot ac)\cdot(a\cdot ac)(b\cdot bc)]\,c'\cdot c'\}\\ &\overset{(5)'}{=} c'\cdot c'\{[(a\cdot ac)c']\,[(a\cdot ac)c'\cdot(b\cdot bc)c']\cdot c'\}\\ &= c'\cdot c'\{a'\,[a'\cdot(b\cdot bc)c']\cdot c'\}. \end{aligned}$$

We will now prove that statement $GST(a,c,c',a')$ i.e. $a\cdot ac = a'\cdot a'c'$ follows from the remaining statements $GST(a,b,c,d)$, $GST(a',b',c',d')$, $GST(b,c,c',b')$, $GST(c,d,d',c')$, i.e. $a=(d\cdot dc)b$, $a'=(d'\cdot d'c')b'$, $b'=(b\cdot bc)c'$, $c'=(c\cdot cd)d'$. Firstly, we get

$$\begin{aligned} a' &= (d'\cdot d'c')b' = (d'\cdot d'c')\cdot(b\cdot bc)c' = d'\,[d'\cdot(c\cdot cd)d']\cdot[(b\cdot bc)\cdot(c\cdot cd)d']\\ &\overset{(4)}{=} d'\,[d'(c\cdot cd)\cdot d']\cdot[(b\cdot bc)\cdot(c\cdot cd)d']\\ &\overset{(7)}{=} [(c\cdot cd)\cdot(c\cdot cd)d']\,[(b\cdot bc)\cdot(c\cdot cd)d']\\ &\overset{(5)'}{=} (c\cdot cd)(b\cdot bc)\cdot(c\cdot cd)d' \overset{(5)}{=} (c\cdot cd)\cdot(b\cdot bc)d', \end{aligned}$$

so it follows

$$\begin{aligned} a'\cdot a'c' &= [(c\cdot cd)\cdot(b\cdot bc)d']\cdot[(c\cdot cd)\cdot(b\cdot bc)d']\,[(c\cdot cd)d']\\ &\overset{(5)}{=} (c\cdot cd)\cdot[(b\cdot bc)d']\,[(b\cdot bc)d'\cdot d'] \overset{(5)'}{=} (c\cdot cd)\cdot[(b\cdot bc)\cdot(b\cdot bc)d']\,d'\\ &\overset{(6)'}{=} (c\cdot cd)(b\cdot bc) \overset{(7)}{=} d(dc\cdot d)\cdot(b\cdot bc) \overset{(4)}{=} (d\cdot dc)d\cdot(b\cdot bc)\\ &\overset{(3)}{=} (d\cdot dc)b\cdot(d\cdot bc) \overset{(6)'}{=} (d\cdot dc)b\cdot[(d\cdot dc)c\cdot bc]\\ &\overset{(5)'}{=} (d\cdot dc)b\cdot[(d\cdot dc)b\cdot c] = a\cdot ac. \end{aligned}$$

□

References

[1] V. Volenec, *GS-quasigroups*, Čas. pěst. mat. **115**(1990), 307–318.