On a class of module maps of Hilbert $C^*$-modules

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**Abstract.** The paper describes some basic properties of a class of module maps of Hilbert $C^*$-modules.

In Section 1 ideal submodules are considered and the canonical Hilbert $C^*$-module structure on the quotient of a Hilbert $C^*$-module over an ideal submodule is described. Given a Hilbert $C^*$-module $V$, an ideal submodule $V_z$, and the quotient $V/V_z$, canonical morphisms of the corresponding $C^*$-algebras of adjointable operators are discussed.

In the second part of the paper a class of module maps of Hilbert $C^*$-modules is introduced. Given Hilbert $C^*$-modules $V$ and $W$ and a morphism $\varphi : A \rightarrow B$ of the underlying $C^*$-algebras, a map $\Phi : V \rightarrow W$ belongs to the class under consideration if it preserves inner products modulo $\varphi$: $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ for all $x, y \in V$. It is shown that each morphism $\Phi$ of this kind is necessarily a contraction such that the kernel of $\Phi$ is an ideal submodule of $V$. A related class of morphisms of the corresponding linking algebras is also discussed.

**Key words:** $C^*$-algebra, Hilbert $C^*$-module, adjointable operator

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**Introduction**

A (right) Hilbert $C^*$-module over a $C^*$-algebra $A$ is a right $A$-module $V$ equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle$ which is $A$-linear in the second and conjugate linear in the first variable such that $V$ is a Banach space with the norm $\|v\| = \|(v, v)\|^{1/2}$. Hilbert $C^*$-modules are introduced and initially investigated in [3], [5] and [8].

The present paper is organized as an introduction to a study of extensions of Hilbert $C^*$-modules.

Section 1 contains a detailed discussion on ideal submodules. As their basic properties are already known (see [10] and [7]), some of the results are stated without proof. The starting point is Theorem 1.6 which states that the quotient of a
Hilbert $C^*$-module over an ideal submodule admits a natural Hilbert $C^*$-module structure. Considering a Hilbert $C^*$-module $V$, an ideal submodule $V_I \subseteq V$, and the quotient $V/V_I$, we describe canonical morphisms of the corresponding $C^*$-algebras of adjointable operators $B(V), B(V_I)$ and $B(V/V_I)$. Also, some properties of ideal submodules arising from essential ideals are obtained. In particular, we show in Theorem 1.12 that the canonical morphism $\alpha : B(V) \rightarrow B(V_I)$ sending each operator $T$ to its restriction $T|_{V_I}$ is an injection if and only if $I$ is an essential ideal in the underlying $C^*$-algebra $A$.

In Section 2 a class of module maps of Hilbert $C^*$-modules over possibly different $C^*$-algebras is introduced. We consider morphisms of Hilbert $C^*$-modules which are in a sense supported by morphisms of the underlying $C^*$-algebras. Their basic properties are collected and a couple of examples is provided. In Theorem 2.15 we establish a correspondence between the class of module maps under consideration and a class of morphisms of the corresponding linking algebras.

The present material provides a necessary tool for the later study of extensions of Hilbert $C^*$-modules. A related discussion will appear in our subsequent paper.

Throughout the paper we denote the $C^*$-algebras of all adjointable and "compact" operators on a Hilbert $C^*$-module $V$ by $B(V)$ and $K(V)$, respectively. We also use $B(\cdot, \cdot)$ and $K(\cdot, \cdot)$ to denote spaces of all adjointable, resp. "compact" operators acting between different Hilbert $C^*$-modules.

We denote by $\langle V, V \rangle$ the closed linear span of all elements in the underlying $C^*$-algebra $A$ of the form $(x, y), x, y \in V$. Obviously, $\langle V, V \rangle$ is an ideal in $A$. (Throughout the paper, an ideal in a $C^*$-algebra always means a closed two-sided ideal.) $V$ is said to be a full $A$-module if $\langle V, V \rangle = A$.

For this and other general facts concerned with Hilbert $C^*$-modules we refer to [4], [7] and [9].

1. Ideal submodules and quotients of Hilbert $C^*$-modules

We begin with the definition of an ideal submodule. A related discussion can be found in [10].

**Definition 1.1.** Let $V$ be a Hilbert $C^*$-module over $A$, and $I$ an ideal in $A$. The associated ideal submodule $V_I$ is defined by

$$V_I = [VI]^- = \{vb : v \in V, b \in I\}^-$$

(the closed linear span of the action of $I$ on $V$).

Clearly, $V_I$ is a closed submodule of $V$. It can be also regarded as a Hilbert $C^*$-module over $I$.

In general, there exist closed submodules which are not ideal submodules. For instance, if a $C^*$-algebra $A$ is regarded as a Hilbert $A$-module (with the inner product $(a, b) = a^*b$), then ideal submodules of $A$ are precisely ideals in $A$, while closed submodules of $A$ are closed right ideals in $A$.

We proceed with a couple of basic properties of ideal submodules. Our first proposition is already known ([10]).
Proposition 1.2. Let $V$ be a Hilbert $C^*$-module over $A$, and let $\mathcal{I}$ be an ideal in $A$. Then $V_\mathcal{I} = \{ v \mathcal{I} : v \in V, b \in \mathcal{I} \}$.

Proof. The associated ideal submodule $V_\mathcal{I}$ is by definition equal to $V_\mathcal{I} = [V \mathcal{I}]^- = \{ [vb] : v \in V, b \in \mathcal{I} \}$, where $[vb] = \{ vb + z \mathcal{I} : z \in V \mathcal{I} \}$. Regarding $V_\mathcal{I}$ as a Hilbert $\mathcal{I}$-module we may apply the Hewitt-Cohen factorization theorem ([6], Theorem 4.1, see also [7], Proposition 2.31): for each $x \in V_\mathcal{I}$ there exist $y \in V_\mathcal{I}$ and $b \in \mathcal{I}$ such that $x = yb$. This shows $V_\mathcal{I} \subseteq [V \mathcal{I}]^- = V_\mathcal{I} \subseteq V_\mathcal{I} \subseteq V_\mathcal{I}$, i.e. $V_\mathcal{I} = V_\mathcal{I}$.

Proposition 1.3. Let $V$ be a Hilbert $A$-module, $\mathcal{I}$ an ideal in $A$, and $V_\mathcal{I}$ the associated ideal submodule. Then

$$V_\mathcal{I} = \{ x : \langle x, x \rangle \in \mathcal{I} \} = \{ x : \langle x, v \rangle \in \mathcal{I}, \forall v \in V \}. $$

If $V$ is full, then $V_\mathcal{I}$ is full as a Hilbert $\mathcal{I}$-module.

Proof. $\langle vb, vb \rangle = b^* (v, v)b \in \mathcal{I}$, $\forall b \in \mathcal{I}, v \in V$. This shows $x = vb \in V_\mathcal{I} \Rightarrow \langle x, x \rangle \in \mathcal{I}$. A well known formula ([9], Lemma 15.2.9)

$$x = \lim_n \left( \langle x, x \rangle + \frac{1}{n} \right)^{-1} \langle x, x \rangle, \forall x \in V$$

implies the converse. The second equality is now an immediate consequence.

Suppose that $V$ is full as a Hilbert $C^*$-module over $A$. Then there is an approximate unit $(a_\mathcal{I})$ for $A$ such that each $a_\mathcal{I}$ is a finite sum of the form $a_\mathcal{I} = \sum_{i=1}^{n(\lambda)} \langle x_\mathcal{I}, x_\mathcal{I}^* \rangle$ ([1], Remark 1.9). Take any positive $b \in \mathcal{I}$, let $\varepsilon$ be given.

Since $(a_\mathcal{I})$ is an approximate unit for $A$, there exists $\lambda$ such that $\| b^{1/2} - a_\mathcal{I} b^{1/2} \|$ is small enough so that $\| b^{1/2} (b^{1/2} - a_\mathcal{I} b^{1/2}) \| < \varepsilon$. It remains to observe that the left-hand side of the above inequality can be rewritten in the form

$$\| b^{1/2} a_\mathcal{I} b^{1/2} \| = \| b - \sum_{i=1}^{n(\lambda)} \langle x_\mathcal{I} b^{1/2}, x_\mathcal{I} b^{1/2} \rangle \|.$$

This shows that $b$ can be approximated by inner products of elements from $V_\mathcal{I}$, i.e. $b \in (V_\mathcal{I}, V_\mathcal{I})$.

Now we introduce a natural Hilbert $C^*$-module structure on the quotient of a Hilbert $C^*$-module over an ideal submodule.

Definition 1.4. Let $V$ be a Hilbert $C^*$-module over $A$, $\mathcal{I}$ an ideal in $A$, and $V_\mathcal{I}$ the associated ideal submodule. Denote by $\pi : A \to A/\mathcal{I}$ and $q : V \to V/V_\mathcal{I}$ the quotient maps. A right action of $A/\mathcal{I}$ on the linear space $V/V_\mathcal{I}$ is defined by $q(v) \pi (a) = q(va)$.

The action of $A/\mathcal{I}$ on the quotient $V/V_\mathcal{I}$ given by $q(v) \pi (a) = q(va)$ is well defined precisely because $V_\mathcal{I}$ is an ideal submodule of $V$. Indeed, if $\pi (a) = \pi (a')$ then $q(v) \pi (a) = q(v) \pi (a')$ is ensured by definition of an ideal submodule: $vb \in V_\mathcal{I}$, $\forall b \in \mathcal{I}, \forall v \in V$.

If $X$ is an arbitrary closed submodule of $V$ one can also consider the quotient of linear spaces $V/X$. Further, denote by $\mathcal{I} = \langle X, X \rangle \subseteq A$ the closed linear span of
the set of all \((x, y), x, y \in X\). Since \(X\) is by assumption a closed submodule of \(V\), \(\mathcal{I}\) is an ideal in \(\mathcal{A}\).

Now an action of \(\mathcal{A}/\mathcal{I}\) on \(V/X\) given by \(q(x)\pi(a) = q(xa)\) will be unambiguously defined if and only if \(vb \in X\) is satisfied for each \(b \in \mathcal{I}\) and \(v \in V\); i.e. \(V\mathcal{I} \subseteq X\). Since \(X\) is a closed submodule, this implies \(V_\mathcal{I} \subseteq X\). Because the reverse inclusion is always satisfied, we conclude: the action of \(\mathcal{A}/\mathcal{I}\) on \(V/X\) is well defined if and only if \(X\) is the ideal submodule \(V_\mathcal{I}\) associated with \(\mathcal{I} = \langle X, X \rangle\).

Remark 1.5. The role of ideal submodules in the preceding discussion should be compared with Proposition 3.25 in [7]. Recall that each right Hilbert \(\mathcal{A}\)-module \(V\) is also equipped with a natural left Hilbert \(K(V)\)-module structure. Moreover, there is a standard Hilbert \(K(V)\)–\(\mathcal{A}\) bimodule structure on \(V\). Now one easily show the following assertions (which are stated without proofs):

1. Each ideal submodule \(V_\mathcal{I}\) of \(V\) is also an ideal submodule of the left Hilbert \(K(V)\)-module \(V\).
2. Let \(X\) be a closed submodule of a right Hilbert \(C^*\)-module \(V\). Then \(X\) is an ideal submodule of \(V\) if and only if \(X\) is a closed subbimodule of the Hilbert \(K(V)\)–\(\mathcal{A}\) bimodule \(V\).

The following theorem is known ([7], Proposition 3.25, [10], Lemma 3.1). We state it for the sake of completeness.

Theorem 1.6. Let \(V\) be a Hilbert \(\mathcal{A}\)-module, \(\mathcal{I}\) an ideal in \(\mathcal{A}\), and \(V_\mathcal{I}\) the associated ideal submodule. Then \(V/V_\mathcal{I}\) equipped with a right \(\mathcal{A}/\mathcal{I}\)-action from Definition 1.4 is a pre-Hilbert \(\mathcal{A}/\mathcal{I}\)-module with the inner product given by \(\langle q(v), q(w) \rangle = \pi((v, w))\). The resulting norm \(\| q(v) \| = \| \pi((v, v)) \|^{1/2}\) coincides with the quotient norm \(d(v, V_\mathcal{I})\) defined on the quotient of Banach spaces \(V/V_\mathcal{I}\). In particular, \(V/V_\mathcal{I}\) is complete, hence a Hilbert \(C^*\)-module over \(\mathcal{A}/\mathcal{I}\).

Remark 1.7. \(V/V_\mathcal{I}\) is a full \(\mathcal{A}/\mathcal{I}\)-module if and only if \(V\) is full. This follows at once from the evident equality \(\langle V/V_\mathcal{I}, V/V_\mathcal{I} \rangle = \pi((V, V))\).

Example 1.8. Let us briefly describe an application of Theorem 1.6. Consider a Hilbert \(C^*\)-module \(V\) over \(\mathcal{A}\) and a surjective morphism of \(C^*\)-algebras \(\varphi : \mathcal{A} \to \mathcal{B}\). Define

\[ N_\varphi = \{ x \in V : \varphi((x, x)) = 0 \}. \]

One easily shows that \(N_\varphi\) is a closed submodule of \(V\). There is a standard construction ([2], p. 19) which provides a pre-Hilbert \(\mathcal{B}\)-module structure on \(V/N_\varphi\); one defines \(q(v)\varphi(a) = q(va)\) and \(q(x), q(y) = \varphi((x, y))\). However, it seems to be overlooked that \(V/N_\varphi\) is already complete with respect to the resulting norm.

To prove this, first observe that \(\mathcal{A}/\text{Ker}\varphi\) and \(\mathcal{B}\) are isomorphic \(C^*\)-algebras. This enables us to regard \(V/N_\varphi\) as a Hilbert \(\mathcal{A}/\text{Ker}\varphi\)-module. Now, \(N_\varphi = \{ x \in V : (x, x) \in \text{Ker}\varphi \}\) (by Proposition 1.3) = \(V_{\text{Ker}\varphi}\); i.e. \(N_\varphi\) is the ideal submodule associated to the ideal \(\text{Ker}\varphi\). It remains to apply Theorem 1.6.

Theorem 1.6 also implies that a property of the Rieffel correspondence is that, assuming that two \(C^*\)-algebras are Morita equivalent, the corresponding ideals and
quotients are Morita equivalent themselves (Proposition 3.25 in [7]). We shall proceed in a different direction. Our goal is to compare the C*-algebras of all adjo
jointable and “compact” operators acting on a Hilbert C*-module V with the corresponding algebras of operators on an ideal submodule V⊥ in V and the quotient V/V⊥, respectively.

To fix our notation, we recall the definition of the ideal of all “compact” operators on a Hilbert C*-module V. Given v, w ∈ V, let θv,w : V → V denote the operator defined by θv,w(x) = v⟨w, x⟩. Each θv,w is an adjointable operator on V and the linear span

{[θv,w : v, w ∈ V]}

is a two-sided ideal in B(V). Its closure in the operator norm

K(V) = {[θv,w : v, w ∈ V]} ⊆ B(V)

is an ideal in B(V) and elements of K(V) are called “compact” operators.

Let V be a Hilbert A-module. Assume that I is an ideal in A, and let V⊥ be the associated ideal submodule. Observe that V⊥ is invariant for each T ∈ B(V); namely T(vb) = (Tv)b ∈ V⊥, ∀v, b ∈ I, ∀v ∈ V. Consequently, there is an operator T|V⊥ on V⊥ induced by T such that (T|V⊥)∗ = T∗|V⊥. This gives a well defined map α : B(V) → B(V⊥), α(T) = T|V⊥. Clearly, α is a morphism of C*-algebras.

We shall prove that the map α is an injection if and only if I is an essential ideal in A. (An ideal I in a C*-algebra A is said to be essential if its annihilator I⊥ = {a ∈ A : aI = {0}} is trivial: I⊥ = {0}.)

To do this, we need a few simple results on ideal submodules associated to essential ideals. We start with a property of essential ideals which is certainly known. Since we are unable to provide a reference, the proof is included.

Lemma 1.9. Let I be an ideal in a C*-algebra A. Then I is an essential ideal in A if and only if there exists a faithful representation ρ : A → B(H) of A on a Hilbert space H such that I acts non-degenerately on H.

Proof. Suppose I ⊆ A ⊆ B(H) such that I acts non-degenerately on H. Let (ui) be an approximate unit for I. Then ξ = limλ→∞ uλξ, ∀ξ ∈ H. Now a ∈ I⊥ implies auλ = 0, ∀λ, hence a = 0.

To prove the converse, suppose that I is an essential ideal in A. Taking any faithful representation of A we may write I ⊆ A ⊆ B(H). Define H0 = [TH]⊥. Clearly, I acts non degenerately on H0. Since I is an ideal in A, H0 reduces A. We shall show that a → a|H0 is also a faithful representation of A. Let aH0 = 0. Since H0 is invariant for each b ∈ I, this implies abH⊥ 0 = 0, ∀b ∈ I. On the other hand, ab ∈ I shows ab|H⊥ 0 = 0, ∀b ∈ I (observe H⊥ 0 = ∩b∈I Ker b). This gives ab = 0, ∀b ∈ I and, since I is essential, a = 0.

Lemma 1.10. Let I be an ideal in a C*-algebra A. The following conditions are mutually equivalent:

(a) I is an essential ideal in A.
(b) ∥a∥ = supb∈I,∥b∥≤1 ∥ab∥, ∀a ∈ A.
(c) ∥a∥ = supb∈I,∥b∥≤1 ∥ba∥, ∀a ∈ A.
(d) ∥a∥ = supb∈I,∥b∥≤1 ∥bab∥, ∀a ∈ A⊥.
Proof. $(a) \Rightarrow (b)$: By Lemma 1.9 we may assume $\mathcal{I} \subset \mathcal{A} \subseteq \mathcal{B}(H)$ such that $\mathcal{I}$ acts non-degenerately on $H$. Given $a \in \mathcal{A}$, we have to show $\|a\| \leq \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|ab\|$ (the opposite inequality is trivial). Let $(a_\lambda)$ be an approximate unit for $\mathcal{I}$. Then $\xi = \lim_{\lambda} a_\lambda \xi$, $\forall \xi \in H$. Take $\|\xi\| \leq 1$. Then

$$
\|a\xi\| = \lim_{\lambda} \|a_\lambda a_\lambda \xi\| \leq \lim_{\lambda} \sup_{\lambda} \|a_\lambda \xi\| \leq \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|ab\|.
$$

$(b) \Leftrightarrow (c)$ is obvious (by taking adjoints).

$(c) \Rightarrow (d)$: Let $a$ be positive. Then

$$
\|a\| = \|a^{1/2}\|^2 = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|ba^{1/2}\|^2 = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|bab^*\|.
$$

$(d) \Rightarrow (a)$: Take any $a \in \mathcal{I}^\perp$. Then $(d)$ applied to $a^*a$ gives $a^*a = 0$, thus $\mathcal{I}^\perp = \{0\}$.

Proposition 1.11. Let $V$ be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an essential ideal in $\mathcal{A}$, and $V_\mathcal{I}$ the associated ideal submodule. Then

$(1)$ $\|v\| = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|vb\|$, for each $v \in V$ and

$(2)$ $\|v\| = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|\langle v, y \rangle\|$, for each $v \in V$.

Conversely, if $V$ is a full $\mathcal{A}$-module in which $(1)$ or $(2)$ is satisfied with respect to (the ideal submodule associated with) some ideal $\mathcal{I}$ in $\mathcal{A}$, then $\mathcal{I}$ is an essential ideal in $\mathcal{A}$.

Proof. Take any $v \in V$. Using Lemma 1.10(d) we find

$$
\|v\|^2 = \|\langle v, v \rangle\| = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|b^* \langle v, v \rangle b\| = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|vb\|^2.
$$

To prove the second formula, take any $v \in V$ such that $\|v\| = 1$. Then

$$
\|v\| = \|v\|^2 = \|\langle v, v \rangle\| = \|\langle v, v \rangle\| = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|\langle v, v \rangle b\| = \sup_{b \in \mathcal{I}, \|b\| \leq 1} \|\langle v, vb \rangle\| \leq \sup_{y \in V, \|y\| \leq 1} \|\langle v, y \rangle\| \leq \|v\|.
$$

To prove the converse, suppose that $V$ is a full $\mathcal{A}$-module and $\mathcal{I}$ is not essential so that $\mathcal{I}^\perp \neq \{0\}$. Take any $c \in \mathcal{I}^\perp$, $c \neq 0$. Then there exists $v \in V$ such that $vc \neq 0$. Indeed, $vc = 0$, $\forall v \in V$ would imply $\langle v, vc \rangle = 0$, $\forall v \in V$ or $\langle v, v \rangle c = 0$, $\forall v \in V$. Since $V$ is full, it would follow $c^*c = 0$, thus $c = 0$.

After all, it remains to observe that $x = vc \neq 0$ with $c \in \mathcal{I}^\perp$ contradicts to $(1)$ and $(2)$, respectively.

Theorem 1.12. Let $V$ be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $V_\mathcal{I}$ the associated ideal submodule. If $\mathcal{I}$ is an essential ideal in $\mathcal{A}$, then the map $\alpha : B(V) \rightarrow B(V_\mathcal{I})$, $\alpha(T) = T|V_\mathcal{I}$ is an injection. Conversely, if $V$ is full and if $\alpha$ is injective, then $\mathcal{I}$ is an essential ideal in $\mathcal{A}$.
Proposition 1.11. \( T \) is an ideal submodule of \( A \) if and only if \( \theta \) is a surjection.  

Proof. Suppose \( \alpha(T) = T|_{V_2} = 0 \) for some \( T \). Observe that, since \( V_2 \) is an ideal submodule, \( \forall v \in V \), \( \forall v \in V \). Since by assumption \( T \) vanishes on \( V_2 \), this implies \( T(v)b) = 0 \), \( \forall b \in I, \forall v \in V \). Now, taking arbitrary \( v \in V \), we find

\[
\|Tv\|(b) = \sup \|Tv(b)\| = \sup_{b \in x, \|b\| \leq 1} \|Tv(b)\| = 0.
\]

To prove the converse, let \( V \) be full and \( \alpha \) injective. Assume that \( I \) is not essential. For \( c \in I^\perp \), \( c \neq 0 \), find \( v \in V \) such that \( vc \neq 0 \) (as in the preceding proof). Then \( \alpha(vc) = \alpha(vc)V_2 = 0 \) - a contradiction. \( \square \)

Remark 1.13. In general, \( \alpha \) is not surjective, even if \( I \) is an essential ideal in \( A \). As an example, consider a nonunital \( C^* \)-algebra \( A \) contained as an essential ideal in a unital \( C^* \)-algebra \( B \). Assume further that \( B \) is not the maximal unitalization of \( A \), i.e. that \( B \) is properly contained in the multiplier algebra \( M(A) \). Consider \( B \) as a Hilbert \( B \)-module. It is well known that, since \( B \) is unital, \( K(B) = B(B) = B \). Further, \( A \) is an ideal submodule of \( B \) associated with the essential ideal \( A \) of \( B \). We also know \( K(A) = A \) and \( B(A) = M(A) \). One easily concludes that the map \( \alpha : B(A) = B \rightarrow B(A) = M(A) \) from Theorem 1.12 acts as the inclusion \( B \hookrightarrow M(A) \); thus, by assumption, \( \alpha \) is not a surjection.

Consider again an arbitrary Hilbert \( A \)-module and an ideal \( I \) in \( A \). Using the map \( \alpha \) one can easily determine \( K(V_2) \). Our next proposition, in which \( K(V_2) \) is recognized as an ideal in \( K(V) \), is known; hence we state it without proof. For the proof we refer to [7], Theorem 3.22. (Alternatively, it can be deduced from Theorem 1.12 above after observing that for each ideal \( I \) in \( A \), we have \( V_2 \oplus V_2^* = V_2 \oplus \mathbb{C} \).

Proposition 1.14. Let \( V \) be a Hilbert \( A \)-module, \( I \) an ideal in \( A \), and \( V_2 \) be the associated ideal submodule. Then \( J = \{[\theta_{x,y} : x, y \in V_2]\} = K(V_2) \) is an ideal in \( K(V) \) and the restriction \( \alpha' = \alpha[J] : J \hookrightarrow K(V_2) \) is an isomorphism of \( C^* \)-algebras.

Remark 1.15. Using the same notation as above one easily concludes that \( V_2 \) is an ideal submodule of the left \( K(V) \)-module \( V \) (with the inner product \( [x, y] = \theta_{x,y} \)) associated with the ideal \( J = \{[\theta_{x,y} : x, y \in V_2]\} \subset K(V) \). As in Proposition 1.3 one obtains \( V_2 = \{x \in V : \theta_{x,v} \in J, \forall v \in V \} \).

Corollary 1.16. Let \( V \) be a full Hilbert \( A \)-module, \( I \) an ideal in \( A \), \( tV_2 \) the associated ideal submodule. Then:

(i) \( J = \{[\theta_{x,y} : x, y \in V_2]\} = K(V_2) \) is an essential ideal in \( K(V) \) if and only if \( I \) is an essential ideal in \( A \).

(ii) \( J = K(V) \) if and only if \( I = A \).

Proof. Assume that \( I \) is an essential ideal in \( A \) and take \( T \in K(V) \) such that \( T \perp J \). By the preceding remark for each \( v \in V \) and \( x \in V_2 \) the operator \( \theta_{v,x} \) belongs to \( J \), hence \( T\theta_{v,x} = \theta_{v,x} = 0 \). In particular, \( Tv(x,y) = 0, \forall x, y \in V_2 \). Since \( V \) is full, \( V_2 \) is a full \( I \)-module and now the first assertion of Proposition 1.11 implies \( Tv = 0 \).
The proof of the second assertion is similar, hence omitted.

We end this section with the corresponding result on quotients. Let $I$ be an ideal in $A$, and let $V_z$ be the associated ideal submodule. Since $V_z$ is invariant for each $T \in B(V)$, there is a well defined induced operator $\hat{T}$ on $V/V_z$ given by $\hat{T}(q(v)) = q(Tv)$. Moreover, $\hat{T}$ is adjointable because $(\hat{T})^* = \hat{T}^*$. This enables us to define $\beta : B(V) \to B(V/V_z)$, $\beta(T) = \hat{T}$. Obviously, $\beta$ is a morphism of $C^*$-algebras.

The following proposition is proved by applying $\beta$ to the ideal $K(V)$ of all "compact" operators on $V$. However, as the result is already known ([7], Proposition 3.25, see also [10]), we omit the proof.

**Proposition 1.17.** Let $V$ be a Hilbert $A$-module, $I$ an ideal in $A$, $V_z$ the associated ideal submodule, and let $J = \{[q(x) : x \in V_z]\}^\perp \subset K(V)$ be as in Proposition 1.14. Then $K(V)/J$ and $K(V/V_z)$ are isomorphic $C^*$-algebras.

**Corollary 1.18.** Let $V$ be a Hilbert $A$-module, $I$ an ideal in $A$, and $V_z$ the associated ideal submodule. Then the map $\beta : B(V) \to B(V/V_z)$, $\beta(T) = \hat{T}$ is the unique morphism of $C^*$-algebras satisfying $\beta(q(x)) = q(x), \forall x, y \in V$ and $\beta(K(V)) = K(V/V_z)$. If $V$ is countably generated, then $\beta$ is surjective.

**Proof.** The equality $\beta(q(x)) = q(x), \forall x, y \in V$ is verified by a direct calculation. Since $\beta$ is a morphism of $C^*$-algebras, this ensures $\beta(K(V)) = K(V/V_z)$. Now the small extension theorem applies (see [9], Propositions 2.2.16 and 2.3.7) because $B(V)$ and $B(V/V_z)$ are the multiplier algebras of $\beta(K(V))$, resp. $K(V/V_z)$. Thus $\beta : B(V) \to B(V/V_z)$ is uniquely determined as the extension of $\beta' = \beta|K(V)$.

The last assertion follows from Tietze's extension theorem. First, if $V$ is countably generated, then $K(V)$ is a $\sigma$-unital $C^*$-algebra ([4], Proposition 6.7]). Since $\beta' : K(V) \to K(V/V_z)$ is a surjection, Proposition 6.8 from [4] implies that $\beta$ is also a surjective map. □

## 2. Morphisms of Hilbert $C^*$-modules

In this section we introduce a class of module maps of Hilbert $C^*$-modules, not necessarily over the same $C^*$-algebra (cf. [2], p. 9, [4], p. 24 and also [7], p. 57). The motivating example is provided by the quotient map $q : V \to V/V_z$, taking values in the quotient module of $V$ over an ideal submodule $V_z$ satisfying $\langle q(x), q(y) \rangle = \pi(\langle x, y \rangle)$.

**Definition 2.1.** Let $V$ and $W$ be Hilbert $C^*$-modules over $C^*$-algebras $A$ and $B$, respectively. Let $\varphi : A \to B$ be a morphism of $C^*$-algebras. A map $\Phi : V \to W$ is said to be a $\varphi$-morphism of Hilbert $C^*$-modules if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ is satisfied for all $x, y \in V$.

Using polarization, one immediately concludes that $\Phi$ is a $\varphi$-morphism if and only if $\langle \Phi(x), \Phi(x) \rangle = \varphi(\langle x, x \rangle)$ is satisfied for each $x$ in $V$.

It is also easy to show that each $\varphi$-morphism is necessarily a linear operator and a module map in the sense $\Phi(\alpha \tau) = \Phi(\alpha) \Phi(\tau), \forall v \in V, \forall a \in A$. 
Further, let \( \varphi : A \to B \) and \( \psi : B \to C \) be morphisms of \( C^* \)-algebras and let \( V,W,Z \) be Hilbert \( C^* \)-modules over \( A,B,C \), respectively. If \( \Phi : V \to W \) is a \( \varphi \)-morphism and \( \Psi : W \to Z \) is a \( \psi \)-morphism, then obviously \( \Psi \Phi : V \to Z \) is a \( \varphi \psi \varphi \)-morphism of Hilbert \( C^* \)-modules.

**Example 2.2.** Consider a Hilbert \( C^* \)-module \( V \) over a \( C^* \)-algebra \( A \). Let \( I \) be an ideal in \( A \), and let \( V_I \) be the associated ideal submodule. Then we have an exact sequence of \( C^* \)-algebras \( I \to A \xrightarrow{\pi} A/I \) and the corresponding sequence of Hilbert \( C^* \)-modules \( V_I \to V \xrightarrow{\pi} V/V_I \). (Here \( i \) and \( j \) denote inclusions while \( x \) and \( q \) denote canonical quotient maps). Obviously, \( j \) is an \( i \)-morphism and \( q \) is a \( \pi \)-morphism in the sense of the above definition.

**Theorem 2.3.** Let \( V \) and \( W \) be Hilbert \( C^* \)-modules over \( C^* \)-algebras \( A \) and \( B \), respectively. Let \( \varphi : A \to B \) be a morphism of \( C^* \)-algebras and let \( \Phi : V \to W \) be a \( \varphi \)-morphism of Hilbert \( C^* \)-modules. Then \( \Phi \) is a contraction satisfying \( \text{Ker} \Phi = V_{\text{Ker} \varphi} \). If \( \varphi \) is an injection, then \( \Phi \) is an isometry, hence also injective. If \( V \) is a full \( A \)-module and if \( \Phi \) is injective, then \( \varphi \) is also an injection.

**Proof.** \( \langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle) \Rightarrow \|\Phi(x)\|^2 = \|\langle \Phi(x), \Phi(x) \rangle\| = \|\varphi(\langle x, x \rangle)\| \leq \|\langle x, x \rangle\| = \|x\|^2, \forall x \in V \). This proves that \( \Phi \) is a contraction. The same calculation also shows: if \( \varphi \) is an injection, then the inequality above is replaced by the equality, hence \( \Phi \) is also an isometry.

Obviously, \( \text{Ker} \Phi \) is a closed submodule of \( V \) such that \( V_{\text{Ker} \varphi} \subseteq \text{Ker} \Phi \).

Further, \( x \in \text{Ker} \Phi \Rightarrow \langle \Phi(x), \Phi(x) \rangle = 0 \Rightarrow \varphi(\langle x, x \rangle) = 0 \); i.e. \( \langle x, x \rangle \in \text{Ker} \varphi \). By Proposition 1.3 we conclude \( x \in V_{\text{Ker} \varphi} \) which gives \( \text{Ker} \Phi \subseteq V_{\text{Ker} \varphi} \).

Finally, suppose that \( \Phi \) is an injection. Then \( \text{Ker} \Phi = V_{\text{Ker} \varphi} = \{0\} \). Take any \( a \in \text{Ker} \varphi \). Then the last equality means \( xa = 0, \forall x \in V \). In particular, \( \langle y, xa \rangle = \langle y, x \rangle a = 0, \forall x, y \in V \). Since \( V \) is by hypothesis full, this implies \( a = 0 \).

**Lemma 2.4.** Let \( V \) and \( W \) be Hilbert \( C^* \)-modules over \( C^* \)-algebras \( A \) and \( B \), respectively. Let \( \varphi : A \to B \) be a morphism of \( C^* \)-algebras and let \( \Phi : V \to W \) be a \( \varphi \)-morphism of Hilbert \( C^* \)-modules. Denote by \( \hat{\varphi} \) and \( \hat{\Phi} \) the maps induced on the quotients by \( \varphi \) and \( \Phi \), respectively:

\[
\hat{\varphi} : A/\text{Ker} \varphi \to B, \quad \hat{\varphi}(\pi(a)) = \varphi(a), \quad \hat{\Phi} : V/\text{Ker} \Phi \to W, \quad \hat{\Phi}(\pi(v)) = \Phi(v).
\]

Then \( \hat{\Phi} \) is a well defined \( \hat{\varphi} \)-morphism of Hilbert \( C^* \)-modules \( V/\text{Ker} \Phi \) and \( W \).

**Proof.** First, by Theorem 2.3, \( \text{Ker} \Phi = V_{\text{Ker} \varphi} \). This ensures that \( V/\text{Ker} \Phi = V/V_{\text{Ker} \varphi} \) is a Hilbert \( A/\text{Ker} \varphi \)-module. Both maps are obviously well defined, so we only need to check that \( \hat{\Phi} \) is a \( \hat{\varphi} \)-morphism. Indeed:

\[
\langle \hat{\Phi}(q(v)), \hat{\Phi}(q(w)) \rangle = \hat{\Phi}(q(v)) = \varphi(\langle v, w \rangle) = \frac{1}{6}(\langle q(v), q(w) \rangle).
\]
Proposition 2.5. Let \( V \) and \( W \) be Hilbert \( C^* \)-modules over \( C^* \)-algebras \( A \) and \( B \), respectively. Let \( \varphi : A \to B \) be a morphism of \( C^* \)-algebras and let \( \Phi : V \to W \) be a \( \varphi \)-morphism of Hilbert \( C^* \)-modules. Then \( \operatorname{Im} \Phi \) is a closed subspace of \( W \). It is also a Hilbert \( C^* \)-module over the \( C^* \)-algebra \( \operatorname{Im} \varphi \subseteq B \) such that \( \langle \operatorname{Im} \Phi, \operatorname{Im} \Phi \rangle = \varphi(\langle V, V \rangle) \). If \( V \) is a full \( A \)-module, then \( \operatorname{Im} \Phi \) is a full \( \operatorname{Im} \varphi \)-module. In particular, if \( \Phi \) is surjective, and if \( W \) is a full \( B \)-module, then \( \varphi \) is also a surjection.

Proof. First suppose that \( \varphi \) is injective. Then by Theorem 2.3 \( \Phi \) is an isometry which implies that \( \operatorname{Im} \Phi \) is a closed subspace of \( W \). Also, \( \Phi(v)\varphi(a) = \Phi(va) \in \operatorname{Im} \Phi \) and \( \langle \Phi(v), \Phi(w) \rangle = \varphi(\langle v, w \rangle) \in \operatorname{Im} \varphi \). This shows that \( \operatorname{Im} \Phi \) is a Hilbert \( \varphi \)-module. The last equality also proves \( \langle \operatorname{Im} \Phi, \operatorname{Im} \Phi \rangle = \varphi(\langle V, V \rangle) \).

If \( V \) is full, this implies \( \langle \operatorname{Im} \Phi, \operatorname{Im} \Phi \rangle = \varphi(\langle A \rangle) \) which means that \( \operatorname{Im} \Phi \) is a full \( \varphi \)-module. If \( \Phi \) is a surjection and if \( W \) is full, we additionally get \( B = \langle W, W \rangle = \langle \operatorname{Im} \Phi, \operatorname{Im} \Phi \rangle = \varphi(\langle V, V \rangle) \), hence \( \varphi \) is also a surjection.

To prove the general case, take the maps \( \hat{\varphi} \) and \( \hat{\Phi} \) from Lemma 2.4. Since \( \hat{\varphi} \) is an injection, we may apply the first part of the proof.

To do this, one has only to observe \( \operatorname{Im} \varphi = \operatorname{Im} \hat{\varphi} \), \( \operatorname{Im} \Phi = \operatorname{Im} \hat{\Phi} \) and \( \langle \operatorname{Im} \Phi, \operatorname{Im} \Phi \rangle \). \( \hat{\varphi}(\langle V/V_{\operatorname{ker} \varphi}, V/V_{\operatorname{ker} \varphi} \rangle) = \varphi(\langle V, V \rangle) \).

(The equality \( \langle V/V_{\operatorname{ker} \varphi}, V/V_{\operatorname{ker} \varphi} \rangle \) is noted in Remark 1.7.) \( \square \)

Remark 2.6. Let us observe: if \( V \) is a full \( A \)-module and if \( \varphi \) and \( \Phi \) are surjective, then \( W \) is also a full \( B \)-module.

On the other hand, we cannot conclude that \( \Phi \) is a surjection if \( \varphi \) is surjective, even if \( V \) and \( W \) are full. As an example we may take \( V = A, W = A \oplus A, \varphi = \text{id}, \Phi(a) = (a, 0) \).

Example 2.7. Let \( A \) and \( B \) be \( C^* \)-algebras considered as Hilbert \( C^* \)-modules over \( A \) and \( B \), respectively. Let \( \varphi : A \to B \) be a morphism of \( C^* \)-algebras and let \( \Phi : V \to W \) be a \( \varphi \)-morphism of Hilbert \( C^* \)-modules \( A \) and \( B \). Then there exists an isometry \( m \) in the multiplier \( C^* \)-algebra of \( B \), \( m \in M(B) \), such that \( \Phi(a) = m\varphi(a), \forall a \in A \).

To prove this, let us take any approximate unit \( (e_j) \) for \( A \). We shall show that \( \Phi(e_j) \) is a net in \( B \) strictly convergent in \( M(B) \). First observe that \( A \) and \( B \) are full, so \( \varphi \) is also surjective.

For each \( b \in B \) there exists \( a \in A \) such that \( \varphi(a) = b \). Now, \( \Phi(e_j)b = \Phi(e_j)\varphi(a) = \Phi(e_j)a \) converges since \( (e_j) \) is an approximate unit for \( A \) and \( \Phi \) is continuous. On the other hand, since \( \Phi \) is by assumption a surjection, there exists \( c \in A \) such that \( \Phi(c) = a \). This implies \( b\Phi(e_j) = (\Phi(c))^{\ast}\Phi(e_j) = (\Phi(c), \Phi(e_j)) = \varphi((c, e_j)) = \varphi(c^{\ast}e_j) \), hence \( b\Phi(e_j) \) converges too.

Let \( m \in M(B) \) be the strict limit: \( m = (\text{stlim} \Phi(e_j)) \); i.e. \( mb = \lim_j \Phi(e_j)b, bm = \lim_j \Phi(e_j)b, \forall b \in B \). Using continuity of \( \Phi \) we get \( \Phi(a) = \Phi(\text{stlim} e_j a) = \lim_j \Phi(e_j) \varphi(a) = m \varphi(a), \forall a \in A \). It remains to show that \( m \) is an isometry. First, \( \langle \Phi(x), \Phi(y) \rangle = \langle m \varphi(x), m \varphi(y) \rangle = \varphi(x)^{\ast}m^{\ast}m \varphi(y) \). On the other hand, \( \langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle) = \varphi(x^{\ast}y) = \varphi(x)^{\ast} \varphi(y) \). Since \( \varphi \) is a surjection, this gives \( bm^{\ast}mc = bc, \forall b, c \in B \); i.e. \( bm^{\ast}m - b = 0, \forall b \in B \). Taking \( c = (bm^{\ast}m - b)^{\ast} \), we find \( bm^{\ast}m - b = 0, \forall b \in B \). The last equality can be written in the form
Definition 2.8. Let $A$ and $B$ be $C^*$-algebras, and let $V$ and $W$ be Hilbert $C^*$-modules over $A$ and $B$, respectively. A map $\Phi : V \to W$ is said to be a unitary operator if there exists an injective morphism of $C^*$-algebras $\varphi : A \to B$ such that $\Phi$ is a surjective $\varphi$-morphism.

Remark 2.9.

(a) Each unitary operator of Hilbert $C^*$-modules is necessarily (by Theorem 2.3) an isometry.

(b) Since $\Phi$ is a surjection, Proposition 2.5 implies $\langle W, W \rangle = \varphi((V, V)) \simeq \langle V, V \rangle$. If $W$ is additionally a full $B$-module, then $\varphi$ is also surjective, hence an isomorphism of $C^*$-algebras.

(c) If $V$ is a Hilbert $C^*$-module over a $C^*$-algebra $A$ and if $\varphi : A \to B$ is an isomorphism of $C^*$-algebras, then $V$ can also be regarded a Hilbert $B$-module and the identity map is obviously a unitary operator between these two versions of $V$.

Conversely, if $V$ and $W$ are full unitary equivalent Hilbert $C^*$-modules over $C^*$-algebras $A$ and $B$, respectively (in the sense that there exists a unitary operator $\Phi : V \to W$, then $A$ and $B$ are isomorphic $C^*$-algebras.

(d) Suppose that $V$ and $W$ are full Hilbert $C^*$-modules over $A$ and $B$, respectively. Let $\varphi : A \to B$ be an isomorphism of $C^*$-algebras. Then a surjective operator $\Phi : V \to W$ satisfying $\Phi(va)\Phi(v)\varphi(a), \forall v \in V, \forall a \in A$ is a unitary operator of Hilbert $C^*$-modules if and only if $\Phi$ is an isometry.

To see this, we have to show that $\Phi$, having the property $\|\Phi(v)\| = \|v\|, \forall v \in V$, also satisfies the condition from Definition 2.1. This can be done by repeating the nice argument from [4], Theorem 3.5.

Take $x \in V$ and $b \in B$. Then there exists $a \in A$ such that $\varphi(a) = b$ and

$$
\|\langle \Phi(x), \Phi(x) \rangle^{1/2}b\|^2 = \|b^*\langle \Phi(x), \Phi(x) \rangle b\| = \|\langle \Phi(x)b, \Phi(x)b \rangle\|
= \|\langle \Phi(x)\varphi(a), \Phi(x)\varphi(a) \rangle\| = \|\langle xa, xa \rangle\|
= \|\Phi(xa)\|^2 = \|xa\|^2 = \|\langle xa, xa \rangle\| = \|\varphi(\langle xa, xa \rangle)\|
= \|\varphi(\langle x, x \rangle)^{1/2}\varphi(a)\|^2 = \|\varphi(\langle x, x \rangle)^{1/2}b\|^2.
$$

By Lemma 3.4 from [4] this implies $\langle \Phi(x), \Phi(x) \rangle^{1/2} = \varphi(\langle x, x \rangle)^{1/2}$.

(e) Unitary equivalence of full Hilbert $C^*$-modules is an equivalence relation.

(f) Suppose that $V$ and $W$ are full Hilbert $C^*$-modules over $C^*$-algebras $A$ and $B$, respectively such that $\varphi : A \to B$ is an isomorphism and that $\Phi : V \to W$ is a unitary $\varphi$-morphism. Then $\Phi^{-1} : W \to V$ is a unitary $\varphi^{-1}$-morphism. Then we also have

$$
\langle w, \Phi(x) \rangle = \varphi(\langle \Phi^{-1}(w), x \rangle), \forall x \in V, w \in W.
$$
Indeed, putting \( w = \Phi(v) \), one obtains
\[
\langle w, \Phi(x) \rangle = \langle \Phi(v), \Phi(x) \rangle = \varphi((v, x)) = \varphi(\Phi^{-1}(w), x).
\]

**Example 2.10.** Consider an arbitrary \( C^* \)-algebra \( A \) regarded as a Hilbert \( A \)-module with \( \langle a, b \rangle = a^*b \). It is well known that the map \( \gamma : A \to K(A) \), \( \gamma(a) = T_a \), \( T_a(x) = ax \) is an isomorphism of \( C^* \)-algebras. Its unique extension to the corresponding multiplier algebras ([9], Proposition 2.2.16) \( \gamma : M(A) \to B(A) \) is also an isomorphism of \( C^* \)-algebras.

Let \( V \) be a Hilbert \( A \)-module, let us denote \( V_d = B(A, V) \). It is well known that \( V_d \) is a Hilbert \( B(A) \)-module with the \( B(A) \)-valued inner product \( \langle r_1, r_2 \rangle = r_1^*r_2 \) such that the resulting norm coincides with the operator norm on \( V_d \).

Further, each \( v \in V \) induces the map \( r_v : v \in V \) given by \( r_v(a) = va \). It is also known ([7], Lemma 2.32) that \( \{ r_v : v \in V \} = K(A, V) \subseteq V_d \).

Now one can easily verify the following assertions:

1. \( \Gamma : V \to V_d, \Gamma(v) = r_v \) is a \( \gamma \)-morphism of Hilbert \( C^* \)-modules.
2. \( \text{Im} \Gamma \) is the ideal submodule of \( V_d \) associated with the ideal \( K(A) \) of \( B(A) \).
3. \( \Gamma : V \to \text{Im} \Gamma = K(A, V) \) is a unitary \( \gamma \)-morphism of Hilbert \( C^* \)-modules.

**Proposition 2.11.** Let \( V \) and \( W \) be Hilbert \( C^* \)-modules over \( C^* \)-algebras \( A \) and \( B \) respectively, let \( \varphi : A \to B \) be an injective morphism and let \( \Phi : V \to W \) be a unitary \( \varphi \)-morphism. Then the map \( \Phi^+ : B(V) \to B(W) \), \( \Phi^+(T) = \Phi T \Phi^{-1} \) is an isomorphism of \( C^* \)-algebras. Moreover, \( \Phi^+((\theta_x, y)) = \theta_{\varphi(x)}, \varphi(y), \forall x, y \in V \) and \( \Phi^+(K(V)) = K(W) \).

**Proof.** First observe that \( \Phi^+(T) = \Phi T \Phi^{-1} \) is an adjointable operator, in fact we claim \( (\Phi T \Phi^{-1})^* = \Phi^* \Phi^{-1} \). Indeed,
\[
\langle w_1, \Phi T \Phi^{-1} w_2 \rangle = \varphi((\Phi^{-1} w_1, T \Phi^{-1} w_2)) = \varphi((T^* \Phi^{-1} w_1, \Phi^{-1} w_2)) = \langle \Phi^* T^* \Phi^{-1} w_1, w_2 \rangle.
\]

Now one easily verifies that \( \Phi^+ \) is an isomorphism of \( C^* \)-algebras. Further,
\[
\Phi^+(\theta_x, y)(w) = \Phi \theta_x y \Phi^{-1}(w) = \Phi \theta_{\varphi(x)}(\varphi(y)) = \Phi(x)(\varphi(y), \Phi(v)) = \theta_{\varphi(x)}, \varphi(y)(w).
\]

The last statement is an immediate consequence. \( \square \)

**Remark 2.12.**
(a) Since \( B(V) \) and \( B(W) \) are the multiplier \( C^* \)-algebras of \( K(V) \) and \( K(W) \),
we know that \( \Phi^+ \), satisfying \( \Phi^+((\theta_x, y)) = \theta_{\varphi(x)}, \varphi(y), \forall x, y \in V \), is uniquely determined.
(b) If one applies Proposition 2.11 to the case $V = \mathcal{A}$, $W = \mathcal{B}$, $\Phi = \varphi$, one obtains the uniquely determined extension of $\varphi$ ensured by the small extension theorem ([9], Proposition 2.2.16): $\varphi^+: \mathcal{B}(\mathcal{A}) \to \mathcal{B}(\mathcal{B})$, $\varphi^+(T)\varphi^{-1}$.

(c) Proposition 2.11 applied to $V \cong \Gamma(V) = K(\mathcal{A}, V)$ coincides with (a special case of) Proposition 7.1 in [4].

**Corollary 2.13.** Let $V$ and $W$ be Hilbert $C^*$-modules over $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, let $\varphi: \mathcal{A} \to \mathcal{B}$ be a surjective morphism of $C^*$-algebras and let $\Phi: V \to W$ be a surjective $\varphi$-morphism. There exists a morphism of $C^*$-algebras $\Phi^+: \mathcal{B}(V) \to \mathcal{B}(W)$ satisfying $\Phi^+((\theta_{x,y}) = \theta_{\varphi(x), \varphi(y)}$ and $\Phi^+(K(V)) = K(W)$.

**Proof.** Considering the quotient $V/\text{Ker} \Phi$ we first apply Proposition 1.17 and Corollary 1.18. The proof is completed by a direct application of Lemma 2.4 and the preceding proposition.

**Remark 2.14.** Let $V$ and $W$ be full (right) Hilbert $C^*$-modules over $\mathcal{A}$, resp. $\mathcal{B}$, let $\varphi: \mathcal{A} \to \mathcal{B}$ be a morphism of $C^*$-algebras and let $\Phi: V \to W$ be a surjective $\varphi$-morphism of Hilbert $C^*$-modules. We note that $\Phi$ is also a $\Phi^+$-morphism of left Hilbert $C^*$-modules $V$ and $W$ (when $V$ and $W$ are regarded as the left Hilbert $C^*$-modules over $K(V)$ and $K(W)$, respectively).

To show this, let us denote by $[,]$ the $K(V)$-inner product on $V$; i.e. $[x, y] = \theta_{x,y}$; the same notation will be used in $W$. Now the condition from Definition 2.1 is an immediate consequence of the preceding corollary: $[\Phi(x), \Phi(y)] = \theta_{\varphi(x), \varphi(y)} = \Phi^+([x, y])$.

Now we are able to describe morphisms of Hilbert $C^*$-modules in terms of the corresponding linking algebras.

Recall that, given a Hilbert $\mathcal{A}$-module $V$, the linking algebra $\mathcal{L}(V)$ may be written as the matrix algebra of the form

\[
\mathcal{L}(V) = \begin{bmatrix} K(\mathcal{A}) & K(V, \mathcal{A}) \\ K(\mathcal{A}, V) & K(V) \end{bmatrix}.
\]

(cf. [7], Lemma 2.32 and Corollary 3.21). Observe that $\mathcal{L}(V)$ is in fact the $C^*$-algebra of all "compact" operators acting on $\mathcal{A} \oplus V$. Keeping the notation from Example 2.10 we may write

\[
\mathcal{L}(V) = \begin{bmatrix} K(\mathcal{A}) & K(V, \mathcal{A}) \\ K(\mathcal{A}, V) & K(V) \end{bmatrix} = \left\{ \begin{bmatrix} T_{x,y} & l_y \\ r_x & T \end{bmatrix} : a \in \mathcal{A}, x, y \in V, T \in K(V) \right\}.
\]

Accordingly, we shall also identify the $C^*$-algebras of "compact" operators with the corresponding corners in the linking algebra: $K(\mathcal{A}) = K(\mathcal{A} \oplus 0) \subseteq K(\mathcal{A} \oplus V) = \mathcal{L}(V)$ and $K(V) = K(0 \oplus V) \subseteq K(\mathcal{A} \oplus V) = \mathcal{L}(V)$.

**Theorem 2.15.** Let $V$ and $W$ be full Hilbert $C^*$-modules over $\mathcal{A}$, resp. $\mathcal{B}$, let $\varphi: \mathcal{A} \to \mathcal{B}$ be a morphism of $C^*$-algebras and let $\Phi: V \to W$ be a surjective $\varphi$-morphism of Hilbert $C^*$-modules. Then the map $\rho_{\varphi, \Phi}: \mathcal{L}(V) \to \mathcal{L}(W)$ defined by

\[
\rho_{\varphi, \Phi}\left( \begin{bmatrix} T_{a} & l_y \\ r_x & T \end{bmatrix} \right) = \begin{bmatrix} T_{\varphi(a)} & l_{\varphi(y)} \\ r_{\varphi(x)} & \Phi^+(T) \end{bmatrix}.
\]
is a morphism of $C^*$-algebras. Conversely, let $\rho : \mathcal{L}(V) \to \mathcal{L}(W)$ be a morphism of $C^*$-algebras such that $\rho(\mathcal{K}(A)) \subseteq \mathcal{K}(B)$ and $\rho(\mathcal{K}(V)) \subseteq \mathcal{K}(W)$. Then there exist a morphism of $C^*$-algebras $\varphi : \mathcal{A} \to \mathcal{B}$ and a $\varphi$-morphism $\Phi : V \to W$ such that $\rho = \rho_{\varphi, \Phi}$.

**Proof.** Clearly, $\rho_{\varphi, \Phi}$ is a linear map. Further,

$$\rho_{\varphi, \Phi} \left( \begin{bmatrix} T_a l_v \\ r_w T \\ r_y S \end{bmatrix} \right) = \rho_{\varphi, \Phi} \left( \begin{bmatrix} T_{ab} + T(v, y) l_{xa^*} + l_{S^*v} \\ r_{wb} + r_{ty} \theta_{w, x} + TS \end{bmatrix} \right) = \begin{bmatrix} T_{\varphi(ab + (v, y))} l_{\Phi(xa^* + S^*v)} \\ r_{\Phi(wb + ty)} \Phi^*(\theta_{w, x} + TS) \end{bmatrix} = \text{(applying Remark 2.14 to)} \begin{bmatrix} T_{\varphi(a)} l_{\Phi(v)} \\ r_{\Phi(w)} \Phi^*(T) \end{bmatrix} \begin{bmatrix} T_{\varphi(b)} l_{\Phi(x)} \\ r_{\Phi(y)} \Phi^*(S) \end{bmatrix} = \rho_{\varphi, \Phi} \left( \begin{bmatrix} T_a l_v \\ r_w T \\ r_y S \end{bmatrix} \right).$$

To prove the converse, first observe that, by assumption, we may write

$$\rho = \rho_{\varphi, \Phi} \left( \begin{bmatrix} T_a \\ 0 \\ 0 \\ T \end{bmatrix} \right) = \begin{bmatrix} \rho_{\varphi(a)} \\ 0 \\ 0 \\ \Psi(T) \end{bmatrix}.$$  

It should be noted that the definition of $\varphi$ actually uses the standard identification $a \mapsto T_a$, $a \in \mathcal{A}, T_a \in \mathcal{K}(A)$ denoted by $\gamma$ in Example 2.10. Obviously, both $\varphi$ and $\Psi$ are morphisms of $C^*$-algebras.

Take any $x \in V$ and write $\rho \left( \begin{bmatrix} 0 \\ 0 \\ r_x \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \rho_{11}(x) \\ \rho_{12}(x) \\ \rho_{21}(x) \\ \rho_{22}(x) \end{bmatrix}$. Then

$$\rho \left( \begin{bmatrix} 0 \\ 0 \\ r_x \\ 0 \end{bmatrix} \right)^* \rho \left( \begin{bmatrix} 0 \\ 0 \\ r_x \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \rho_{11}(x)^* \rho_{21}(x)^* \\ \rho_{12}(x)^* \rho_{22}(x)^* \end{bmatrix} \begin{bmatrix} \rho_{11}(x) \\ \rho_{12}(x) \\ \rho_{21}(x) \\ \rho_{22}(x) \end{bmatrix} = \begin{bmatrix} \rho_{11}(x)^* \rho_{11}(x) + \rho_{21}(x)^* \rho_{21}(x) + \rho_{12}(x)^* \rho_{12}(x) + \rho_{22}(x)^* \rho_{22}(x) \\ \rho_{12}(x)^* \rho_{11}(x) + \rho_{22}(x)^* \rho_{21}(x) + \rho_{12}(x)^* \rho_{12}(x) + \rho_{22}(x)^* \rho_{22}(x) \end{bmatrix}.$$  

Observing $\begin{bmatrix} 0 \\ 0 \\ r_x \end{bmatrix}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and comparing the above result with

$$\rho \left( \begin{bmatrix} 0 \\ 0 \\ r_x \end{bmatrix} \right) = \rho \left( \begin{bmatrix} T_{(x, x)} \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} T_{\varphi(x, x)} \\ 0 \\ 0 \end{bmatrix}$$

we find $\rho_{12}(x) = \rho_{22}(x) = 0$. Similarly, calculating $\rho \left( \begin{bmatrix} 0 \\ 0 \\ r_x \end{bmatrix} \right) \rho \left( \begin{bmatrix} 0 \\ 0 \\ r_x \end{bmatrix} \right)^*$, one additionally gets $\rho_{11} = 0$. After all, we conclude that $\rho$ may be written in the form

$$\rho \left( \begin{bmatrix} T_a l_y \\ r_x T \end{bmatrix} \right) = \begin{bmatrix} T_{\varphi(a)} l_{\Phi(y)} \\ r_{\Phi(x)} \Psi(T) \end{bmatrix}.$$  

Obviously, the induced maps $\Phi_1$ and $\Phi_2$ are linear.
Let us show $\Phi_1 = \Phi_2$. Indeed, $\rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)^* = \rho\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$ implies $\begin{bmatrix} 0 \\ l_{\Phi_2(x)} \end{bmatrix} = \begin{bmatrix} 0 \\ l_{\Phi_1(x)} \end{bmatrix}$, thus $\Phi_1(x) = \Phi_2(x)$ which enables us to write $\Phi_1 = \Phi_2 = \Phi$.

Finally, $\rho\left(\begin{bmatrix} 0 & l_y \\ r_x & 0 \end{bmatrix}\begin{bmatrix} 0 & l_x \\ r_y & 0 \end{bmatrix}\right) =\rho\left(\begin{bmatrix} 0 & l_y \\ r_x & 0 \end{bmatrix}\right)\rho\left(\begin{bmatrix} 0 & l_x \\ r_y & 0 \end{bmatrix}\right) = \begin{bmatrix} T_{\phi(y)} & 0 \\ 0 & \theta_{\phi(x),\phi(x)} \end{bmatrix}$.

On the other hand, knowing that $\rho$ is multiplicative, one obtains

\[
\rho\left(\begin{bmatrix} 0 & l_y \\ r_x & 0 \end{bmatrix}\begin{bmatrix} 0 & l_x \\ r_y & 0 \end{bmatrix}\right) = \rho\left(\begin{bmatrix} 0 & l_y \\ r_x & 0 \end{bmatrix}\right)\rho\left(\begin{bmatrix} 0 & l_x \\ r_y & 0 \end{bmatrix}\right) = \begin{bmatrix} T_{\phi(y)} & 0 \\ 0 & \theta_{\phi(x),\phi(x)} \end{bmatrix}.
\]

This is enough (together with Corollary 2.13) to conclude $\rho = \rho_{\varphi,\Phi}$. Notice that the assumptions $\rho(K(A)) \subseteq K(B)$ and $\rho(K(V)) \subseteq K(W)$ cannot be dropped from the hypothesis of the second assertion in Theorem 2.15.

Let us also note an alternative description of $\rho_{\varphi,\Phi}$. First, define $\varphi \oplus \Phi : A \oplus V \rightarrow B \oplus W$, $(\varphi \oplus \Phi)(a, v) = (\varphi(a), \Phi(v))$.

One easily verifies that $\varphi \oplus \Phi$ is a $\varphi$-morphism of Hilbert $C^*$-modules. After applying Corollary 2.13 it turns out that $(\varphi \oplus \Phi)^+ = \rho_{\varphi,\Phi}$.

At the end let us mention a similar characterization of ideal submodules in terms of linking algebras: there is a natural bijective correspondence between the set of all ideal submodules of a Hilbert $C^*$-module $V$ and the set of all ideals of the corresponding linking algebra $L(V)$. Moreover, the ideal submodule associated with an essential ideal corresponds to an essential ideal in $L(V)$. The proof is an easy calculation similar to the preceding one, hence omitted.

Note added in proof: In Corollary 2.13 as well as in the subsequent Remark 2.14 and Theorem 2.15 the assumption that $\Phi$ is surjective is redundant. In fact, the map $\Phi^+ : B(V) \rightarrow B(W)$ satisfying $\Phi^*(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$ is always well defined. This can be seen using the identification $K(V) = V \otimes_{hA} V^*$ (cf. D. BLECHER, A new approach to Hilbert $C^*$-modules, Math. Ann. 307(1997), 253-290). We thank to the anonymous referee for this observation.

References


