On some subspaces of an FK-space

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Abstract. In this paper we study the subspaces $C_1S$, $C_1W$, $C_1F$ and $C_1B$ for a locally convex FK-space $X$ containing $\phi$, the space of finite sequences.

Key words: FK-space, AK-space, $\sigma K$-space, $\sigma B$-space, $C_1$-summability method

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1. Introduction and notation

Let $w$ denote the space of all complex-valued sequences. An FK-space is a locally convex vector subspace of $w$ which is also a Fréchet space (complete linear metric) with continuous coordinates. A BK-space is a normed FK-space. The basic properties of FK-spaces may be found in [7], [8] and [10]. We now define the Cesàro summability matrix which is used throughout this paper: The Cesàro mean is given by the matrix $C_1$ whose $nk$th entry is

$$C_1[n, k] = \begin{cases} \frac{1}{n+1}, & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n. \end{cases}$$

The sequence spaces

$$\sigma_0 = \left\{ x \in w : \lim_{n} \frac{1}{n} \sum_{j=1}^{n} x_j = 0 \right\},$$

$$\sigma b = \left\{ x \in w : \sup_{n} \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j \right| < \infty \right\}.$$
and
\[
\sigma s = \left\{ x \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_{j} \text{ exists} \right\}
\]
are BK-spaces with the norm
\[
\|x\|_{\sigma s} = \sup_{n} \left| \frac{1}{n} \sum_{k=1}^{n} x_{k} \right|
\]
and
\[
\|x\|_{\sigma b} = \sup_{n} \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_{j} \right|
\]
respectively ([1], [2] and [9]).

Throughout the paper \( \delta^j \), \((j = 1, 2, ...), \) the sequence \((0, 0, ..., 0, 1, 0, ...)\) with the one in the \( j \)-th position; \( \phi \) the linear span of the \( \delta^j \)'s. The topological dual of \( X \) is denoted by \( X' \). A sequence \( x \) in a locally convex sequence space \( X \) is said to have the property AK (respectively \( \sigma K \)) if \( x^{(n)} \to x \) (respectively \( \frac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x \)) in \( X \) where \( x^{(n)} = (x_1, x_2, ..., x_n, 0, ...) = \sum_{k=1}^{n} x_k \delta^k \). It is known that if an FK-space \( \phi \subset X \) is said to have \( \sigma B \) if \( \left\{ \frac{1}{n} \sum_{k=1}^{n} x^{(k)} \right\} \) is a bounded set in \( X \) for each \( x \in X \).

Also, an FK-space \( X \) is said to have \( F\sigma K \) (functional \( \sigma K \)) if \( X \subset C_1 F^+ \) i.e., \( X = C_1 F \) ([1], [2] and [4]).

We recall (see [3] and [4]) that the \( f, \sigma - \) and \( \sigma b - \) duals of a subset \( X \) of \( w \) are defined to be
\[
X^f = \{ \{ f(\delta^k) \} : f \in X' \},
\]
\[
X^\sigma = \left\{ x \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_{j} y_{j} \text{ exists for all } y \in X \right\}
\]
\[
= \{ x \in w : x.y \in \sigma s \text{ for all } y \in X \},
\]
\[
X^{\sigma b} = \left\{ x \in w : \sup_{n} \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_{j} \right| < \infty \text{ for all } y \in X \right\}
\]
\[
= \{ x \in w : x.y \in \sigma b \text{ for all } y \in X \},
\]
respectively, where \( x.y = (x_n y_n) \).
2. Some subspaces of $X$

Following [4] we recall some important subspaces of a locally convex FK-space $X$ containing $\phi$.

**Definition 1.** Let $X$ be an FK-space $\supset \phi$. Then

$$W := W(X) = \left\{ x \in X : x^{(k)} \to x$$(weakly) in $X$ $\right\}$$

$$C_1W := C_1W(X) = \left\{ x \in X : \frac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x$$(weakly) in $X$ $\right\}$$

$$= \left\{ x \in X : f(x) = \lim_{n} \frac{1}{n} \sum_{k=1}^{n} x \delta \{j\} \text{ for all } f \in X' \right\}$$

$$= \left\{ x \in X : x \text{ has } S\sigma K \text{ in } X \right\}$$

$$C_1S := C_1S(X) = \left\{ x \in X : \frac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x \right\}$$

$$= \left\{ x \in X : x \text{ has } \sigma K \text{ in } X \right\}$$

$$= \left\{ x \in X : x = \lim_{n} \frac{1}{n} \sum_{k=1}^{n} x \delta \right\}$$

$$C_1F^+ := C_1F^+(X) = \left\{ x \in w : \left( \frac{1}{n} \sum_{k=1}^{n} x^{(k)} \right) \text{ is weakly Cauchy in } X \right\}$$

$$= \left\{ x \in w : (x_n f (\delta^n)) \in \sigma s \text{ for all } f \in X' \right\}$$

$$C_1B^+ := C_1B^+(X) = \left\{ x \in w : \left( \frac{1}{n} \sum_{k=1}^{n} x^{(k)} \right) \text{ is bounded in } X \right\}$$

$$= \left\{ x \in w : (x_n f (\delta^n)) \in \sigma b \text{ for all } f \in X' \right\}$$

also

$$C_1F := C_1F^+ \cap X \text{ and } C_1B := C_1B^+ \cap X.$$

We note that subspaces $W$ and $C_1W$ are closely related to conullity and Cesàro conullity of the FK-space $X$ (see [5] and [6]).

We now study some inclusions which are analogous to those given in [8; Chapter 10].

**Theorem 2.** Let $X$ be an FK-space $\supset \phi$. Then

$$\phi \subset C_1S \subset C_1W \subset C_1F \subset C_1B \subset X \text{ and } \phi \subset C_1S \subset C_1W \subset \overline{\phi}.$$

**Proof.** The only non-trivial part is $C_1W \subset \overline{\phi}$. Let $f \in X'$ and $f = 0$ on $\phi$. The definition of $C_1W$ shows that $f = 0$ on $C_1W$. Hence, the Hahn-Banach theorem gives the result.

**Theorem 3.** The subspaces $E = C_1S, C_1W, C_1F, C_1F^+, C_1B, \text{ and } C_1B^+$ of $X$ FK-space are monotone i.e., if $X \subset Y$ then $E(X) \subset E(Y)$.

**Proof.** The inclusion map $i : X \to Y$ is continuous by Corollary 4.2.4 of [8], so $\frac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x$ in $X$ implies the same in $Y$. This proves the assertion for $C_1S$. For $C_1W$ it follows from the fact that $i$ is weakly continuous by (4.0.11) of [8]. Now
Let
\[ \text{Theorem 8.} \]
Let \( \sigma_0 \) be an \( AK \)-space. Then
\[ \sigma_0 \subset C_1 S \subset C_1 W. \]

**Proof.** By Definition 1, \( \sigma \) in \( C_1 B^+ \) if and only if \( \sigma = \sigma b \) for each \( \sigma \in \sigma b \).

**Theorem 5.** Let \( X \) be an FK-space \( \sigma \). Then \( C_1 B^+ = X^f \sigma. \)

**Proof.** By Definition 1, \( z \in C_1 B^+ \) if and only if \( z \in \sigma b \) for each \( \sigma \in \sigma b \).

**Theorem 6.** Let \( X \) be an FK-space \( \sigma \). Then \( C_1 B^+ \) is the same for all FK-spaces \( Y \) between \( \sigma \) and \( X \); i.e., \( \sigma \subset Y \subset X \) implies \( C_1 B^+(Y) \subset C_1 B^+(X) \). Here the closure of \( \sigma \) is calculated in \( X \).

**Proof.** By Theorem 3 we have \( C_1 B^+(\sigma) \subset C_1 B^+(Y) \subset C_1 B^+(X) \). By Theorem 5 and by (7.2.4) of [8] the first and the last are equal.

**Theorem 7.** Let \( X \) be an FK-space such that \( C_1 B \supset \sigma \). Then \( \sigma \) has \( \sigma K \) and \( C_1 S = C_1 W = \sigma. \)

**Proof.** Suppose first that \( X \) has \( \sigma K \). Define \( f_n : X \to X \) by
\[
   f_n(x) = x - \frac{1}{n} \sum_{k=1}^{n} x^{(k)}.
\]

Then \( \{f_n\} \) is pointwise bounded, hence equicontinuous by (7.0.2) of [8]. Since \( f_n \to 0 \) on \( \sigma \) then also \( f_n \to 0 \) on \( \sigma \) by (7.0.3) of [8]. This is the desired conclusion.

**Theorem 8.** Let \( X \) be an FK-space \( \sigma \). Then \( C_1 F^+ = X^f \sigma. \)

**Proof.** This may be proved as in Theorem 5, with \( \sigma K \) instead of \( \sigma b \).

**Theorem 9.** Let \( X \) be an FK-space \( \sigma \). Then \( C_1 F^+ \) is the same for all FK-spaces \( Y \) between \( \sigma \) and \( X \); i.e., \( \sigma \subset Y \subset X \) implies \( C_1 F^+(Y) \subset C_1 F^+(X) \). (The closure of \( \sigma \) is calculated in \( X \).)

The proof is similar to that of Theorem 6.

**Lemma 10.** Let \( X \) be an FK-space in which \( \sigma \) has \( \sigma K \). Then \( C_1 F^+ = (\sigma)^{\sigma} \).

**Proof.** Observe that \( C_1 F^+ = X^f \sigma \) by Theorem 8. Since \( X^f = (\sigma)^{f} \) by Theorem 7.2.4 of [8], we have \( X^f = (\sigma)^{f} \). Hence, by Theorem 1.9 of [4] the result follows.

**Theorem 11.** Let \( X \) be an FK-space \( \sigma \). Then \( X \) has \( F \sigma K \) if and only if \( \sigma \) has \( \sigma K \) and \( X \subset (\sigma)^{\sigma} \).

**Proof.** Necessity. \( X \) has \( \sigma K \) since \( C_1 F \subset C_1 B \) so \( \sigma \) has \( \sigma K \) by Theorem 7. The remainder of the proof follows from Lemma 10. Sufficiency is given by Lemma 10.

**Theorem 12.** Let \( X \) be an FK-space \( \sigma \). The following are equivalent:

(i) \( X \) has \( F \sigma K \),
(ii) $X \subset C_1 S^{\sigma}$, 
(iii) $X \subset C_1 W^{\sigma}$, 
(iv) $X \subset C_1 F^{\sigma}$, 
(v) $X^{\sigma} = C_1 S^{\sigma}$, 
(vi) $X^{\sigma} = C_1 F^{\sigma}$.

**Proof.** Observe that (ii) implies (iii) and (iii) implies (iv) and that they are trivial since 

$$C_1 S \subset C_1 W \subset C_1 F.$$ 

If (iv) is true, then $X^f \subset C_1 F^{\sigma} = X^{\sigma \sigma} \subset X^{\sigma}$ so (i) is true by Theorem 1.9 of [4]. If (i) holds, then Theorem 11 implies that $\overline{\phi} = C_1 S$ and that (ii) holds. The equivalence of (v), (vi) with the others is clear.

**Theorem 13.** Let $X$ be an FK-space $\supset \phi$. The following are equivalent:

(i) $X$ has $S\sigma K$,
(ii) $X$ has $\sigma K$,
(iii) $X^{\sigma} = X'$.

**Proof.** Clearly (ii) implies (i). Conversely if $X$ has $S\sigma K$ it must have AD for $C_1 W \subset \overline{\phi}$ by Theorem 2. It also has $\sigma B$ since $C_1 W \subset C_1 B$. Thus $X$ has $\sigma K$ by Theorem 7, this proves that (i) and (ii) are equivalent. Assume that (iii) holds. Let $f \in X'$, then there exists $u \in X^{\sigma}$ such that

$$f(x) = \lim_{n} \frac{1}{n} \sum_{k=1}^{\infty} \sum_{j=1}^{k} u_j x_j$$

for $x \in X$. Since $f(\hat{\delta}^j) = u_j$, it follows that each $x \in C_1 W$ which shows that (iii) implies (i). That (ii) implies (iii) is known (see [2], page 97).

**Theorem 14.** Let $X$ be an FK-space $\supset \phi$. The following are equivalent:

(i) $C_1 W$ is closed in $X$,
(ii) $\overline{\phi} \subset C_1 B$,
(iii) $\overline{\phi} \subset C_1 F$,
(iv) $\overline{\phi} = C_1 W$,
(v) $\overline{\phi} = C_1 S$,
(vi) $C_1 S$ is closed in $X$.

**Proof.** (ii) implies (v): By Theorem 7, $\overline{\phi}$ has $\sigma K$, i.e. $\overline{\phi} \subset C_1 S$. The opposite inclusion is Theorem 2. Note that (v) implies (iv), (iv) implies (iii) and (iii) implies (ii) because 

$$C_1 S \subset C_1 W \subset \overline{\phi}, C_1 W \subset C_1 F \subset C_1 B;$$

(i) implies (iv) and (vi) implies (v) since $\phi \subset C_1 S \subset C_1 W \subset \overline{\phi}$. Finally (iv) implies (i) and (v) implies (vi).
References


