Some relations concerning k-chordal and k-tangential polygons

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Abstract. *In papers [6] and [7] the k-chordal and the k-tangential polygons are defined and some of their properties are proved. In this paper we shall consider some of their other properties. Theorems 1-4 are proved.*

Key words: *geometrical inequalities, k-chordal polygon, k-tangential polygon*

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1. Preliminaries

A polygon with vertices $A_1 \dots A_n$ (in this order) will be denoted by $A_1 \dots A_n$ and the lengths of the sides of $A_1 \dots A_n$ will be denoted by $a_1, \dots, a_n$, where $a_i = \mid A_iA_{i+1} \mid$, $i = 1, 2, \dots, n$. For the interior angle at the vertex A_i we write α_i or $\angle A_i$, i.e. $\angle A_i = \angle A_{n-1+i}A_iA_{i+1}, \quad i = 1, \dots, n$. Of course, indices are calculated modulo n.

For convenience we list some definitions given in [6] and [7].

Definition 1. *Let $\underline{A} = A_1 \dots A_n$ be a chordal polygon and let C be its circumcircle. By S_{A_i} and $\widehat{S}_{A_i}$ we denote the semicircles of C such that*

$$S_{A_i} \cup \widehat{S}_{A_i} = C, \quad A_i \in S_{A_i} \cap \widehat{S}_{A_i}.$$

The polygon $\underline{A}$ is said to be of the first kind if the following is fulfilled:
1. all vertices $A_1 \dots A_n$ do not lie on the same semicircle,
2. for every three consecutive vertices A_i, A_{i+1}, A_{i+2} it holds

$$A_i \in S_{A_{i+1}} \Rightarrow A_{i+2} \in \widehat{S}_{A_{i+1}}$$

3. any two consecutive vertices A_i, A_{i+1} do not lie on the same diameter.

Definition 2. *Let $\underline{A} = A_1 \dots A_n$ be a chordal polygon and let k be a positive integer. The polygon $\underline{A}$ is said to be a k-chordal polygon if it is of the first kind and if there holds*

$$\sum_{i=1}^{n} \angle A_iCA_{i+1} = 2k\pi, \tag{1}$$

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where C is the centre of the circumcircle of the polygon $\underline{A}$.

Using (1) it is easy to see that the angles of a k-chordal polygon $A_1 \ldots A_n$ satisfy the relation:

$$\sum_{i=1}^{n} \angle A_i = (n-2k)\pi. \tag{2}$$

Definition 3. *Let $\underline{A} = A_1 \ldots A_n$ be a tangential polygon and let k be a positive integer so that $k \le \lfloor \frac{n-1}{2} \rfloor$, that is, $k \le \frac{n-1}{2}$ if n is odd, and $k \le \frac{n-2}{2}$ if n is even. The polygon $\underline{A}$ will be called a k-tangential polygon if any two of its consecutive sides have only one common point, and if there holds*

$$\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2}, \tag{3}$$

where $2\beta_i = \angle A_i, \quad i = 1, ..., n$.

Consequently, a tangential polygon $\underline{\mathrm{A}}$ is k-tangential if

$$\varphi_1 + \cdots + \varphi_n = 2k\pi, \tag{4}$$

where $\varphi_i = \angle A_i C A_{i+1}$ and C is the centre of the circle inscribed into the polygon $\underline{\mathrm{A}}$.

The integer k in relations (1)-(4) can be at most $\frac{n-1}{2}$ if n is odd and $\frac{n-2}{2}$ if n is even.

Remark 1. *In the following considerations we shall denote the angles $\beta_1, \ldots, \beta_n$ such that*

$$\beta_i = \angle C A_i A_{i+1}, \quad \textit{if it is a question of a chordal polygon},$$

$$\beta_i = \frac{1}{2}\angle A_i, \quad \textit{if it is a question of a tangential polygon.}$$

2. Some inequalities concerning the radius of k-chordal and k-tangential polygons

At first we prove some results concerning a k-chordal polygon.

Theorem 1. *Let $a_1, \ldots, a_n$ be the lengths of the sides of a k-chordal polygon $\underline{A} = A_1 \ldots A_n$ and let $a_1 = min\{a_1 \ldots a_n\}$. If there exist angles $\gamma_1, \ldots, \gamma_n$ such that*

$$\gamma_1 + \cdots + \gamma_n = (n-2k)\frac{\pi}{2}, \quad 0 < \gamma_i < \frac{\pi}{2}, \quad 1 = 1, \ldots, n, \tag{5}$$

$$a_1 \sin\gamma_1 = a_2 \sin\gamma_2 = \ldots = a_n \sin\gamma_n. \tag{6}$$

Then

$$2r > \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i - \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) a_1^2 \sin^2 (n-2k)\frac{\pi}{2n}}, \tag{7}$$

where r is the radius of the circumcircle of the polygon $\underline{A}$.

Proof. Since $\beta_i = \angle CA_iA_{i+1}, \quad i = 1, \ldots, n$, we have the following relations

$$\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2}, \quad 0 < \beta_i < \frac{\pi}{2}, \quad i = 1, \ldots, n \tag{8}$$

$$2r \cos \beta_i = a_i, \quad i = 1, \ldots, n \tag{9}$$

from which it follows

$$2ra_i \cos \beta_i = a_i^2, \quad i = 1, \ldots, n$$

$$2r = \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i \cos \beta_i}. \tag{10}$$

In addition to the angles $\beta_1, \ldots, \beta_n$ there are infinitely many angles $\gamma_1, \ldots, \gamma_n$ such that

$$\gamma_1 + \cdots + \gamma_n = (n-2k)\frac{\pi}{2}, \quad 0 < \gamma_i < \frac{\pi}{2}, \quad i = 1, \ldots, n.$$

We shall prove that $\sum\limits_{i=1}^{n} a_i \cos \gamma_i =$ maximum if the angles $\gamma_1, \ldots, \gamma_n$ satisfy

$$a_1 \sin \gamma_1 = a_2 \sin \gamma_2 = \ldots = a_n \sin \gamma_n.$$

First we shall prove the following lemma.

Lemma 1. *If a_1 and a_2 are positive numbers and*

$$\gamma_1 + \gamma_2 = s, \quad 0 < s < \pi, \quad 0 < \gamma_i < \frac{\pi}{2}, \quad i = 1, 2$$

then the function $f(\gamma_1, \gamma_2) = a_1 \cos \gamma_1 + a_2 \cos \gamma_2$ assumes maximum if $a_1 \sin \gamma_1 = a_2 \sin \gamma_2$.

Proof. Let $g(\gamma_1) = a_1 \cos \gamma_1 + a_2 \cos(s - \gamma_1)$, then

$$g'(\gamma_1) = -a_1 \sin \gamma_1 + a_2 \sin(s - \gamma_1),$$

$$g''(\gamma_1) = -a_1 \cos \gamma_1 - a_2 \cos(s - \gamma_1) < 0,$$

$$a_1 \sin\gamma_1 + a_2 \sin(s-\gamma_1) = 0 \quad \Rightarrow \quad a_1 \sin\gamma_1 = a_2 \sin(s-\gamma_2).$$

$\square$

From the above lemma it is clear that the sum $\sum_{i=1}^{n} a_i \cos\gamma_i$ assumes maximum if for each sum

$$a_i \cos\gamma_i + a_j \cos\gamma_j, \quad i,j \in \{1,\dots,n\}$$

there holds $a_i \sin\gamma_i = a_j \sin\gamma_j$, since we can put $\gamma_i + \gamma_j = s$.

Now, we are going to prove that the inequality (7) is valid if (6) is fulfiled. Based on the assumption that equations (6) exist, we can write

$$a_i \sin\gamma_i = \lambda, \quad i = 1,\dots,n$$

from which it follows

$$\cos\gamma_i = \sqrt{1-\left(\frac{\lambda}{a_i}\right)^2} < 1 - \frac{1}{2}\left(\frac{\lambda}{a_i}\right)^2, \quad i = 1,\dots,n,$$

$$\sum_{i=1}^{n} a_i \left[1 - \frac{1}{2}\left(\frac{\lambda}{a_i}\right)^2\right] > \sum_{i=1}^{n} a_i \cos\gamma_i \geq \sum_{i=1}^{n} a_i \cos\beta_i$$

so that instead of (10) we can write

$$2r > \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i - \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{a_i}\right)\lambda^2}. \tag{11}$$

Since $\gamma_i = \arcsin\frac{\lambda}{a_i}, \quad i = 1,\dots,n$ we have the equation

$$\sum_{i=1}^{n} \arcsin\frac{\lambda}{a_i} = (n-2k)\frac{\pi}{2}, \tag{12}$$

or

$$\left(\frac{\lambda}{a_1} + \cdots + \frac{\lambda}{a_n}\right) + \frac{1}{6}\left[\left(\frac{\lambda}{a_1}\right)^3 + \cdots + \left(\frac{\lambda}{a_n}\right)^3\right] + \cdots = (n-2k)\frac{\pi}{2}. \tag{13}$$

Since by assumption $a_1 = \min\{a_1,\dots,a_n\}$, from (13) it follows that

$$\left(\frac{\lambda}{a_1} + \cdots + \frac{\lambda}{a_1}\right) + \frac{1}{6}\left[\left(\frac{\lambda}{a_1}\right)^3 + \cdots + \left(\frac{\lambda}{a_1}\right)^3\right] + \cdots \geq (n-2k)\frac{\pi}{2}$$

or

$$\arcsin\frac{\lambda}{a_1} \geq (n-2k)\frac{\pi}{2n}.$$

Hence

$$\lambda \geq a_1 \sin(n-2k)\frac{\pi}{2n}. \tag{14}$$

Now using (11) and (14) we readily get (7). So, *Theorem 1* is proved. □

Before stating some of its corollaries here is an example. If $A_1 \dots A_5$ is a l-chordal pentagon as shown in *Figure 1*, then there are angles $\gamma_1, \dots, \gamma_5$ such that

$$\gamma_1 + \dots + \gamma_5 = (5-2)\frac{\pi}{2}, \qquad a_1 \sin\gamma_1 = \dots = a_5 \sin\gamma_5$$

if instead of the drawn circles these can be drawn greater such that the above equalities are valid. (For these drawn ones it is $\gamma_1 + \dots + \gamma_5 < \frac{3\pi}{2}$. Let us remark that in the case when a side is small enough, then there are no angles $\gamma_1, \dots, \gamma_5$ such that $\gamma_1 + \dots + \gamma_5 = \frac{3\pi}{2}$.)

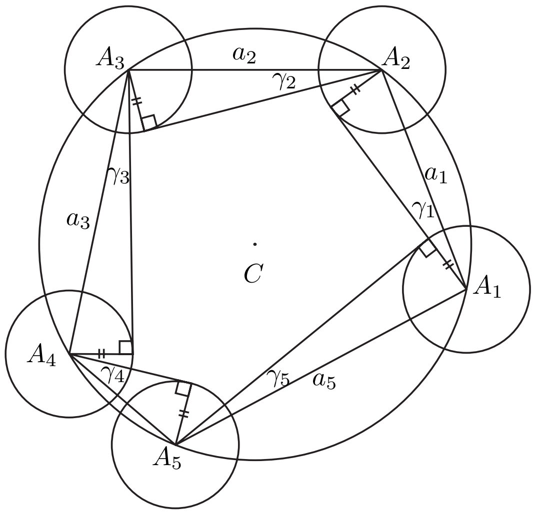

Figure 1.

Now we state some of the corollaries of *Theorem 1*.

Corollary 1. *There are angles $\gamma_1, \dots, \gamma_n$ such that (5) and (6) hold if and only if*

$$\frac{a_1}{H(a_1, \dots, a_n)} + \frac{1}{6}\frac{a_1^3}{H(a_1^3, \dots, a_n^3)} + \dots \geq (n-2k)\frac{\pi}{2n} \tag{15}$$

where $H(a_1^i,\dots,a_n^i)$ is the harmonic mean of $a_1^i,\dots,a_n^i$.

Proof. It is clear from (13) since λ may be at most a_1. □

Corollary 2. *A sufficient condition for the existence of the angles $\gamma_1,\dots,\gamma_n$ such that (5) and (6) hold is the inequality*

$$a_1 \geq H(a_1,\dots,a_n)\sin(n-2k)\frac{\pi}{2n} \tag{16}$$

Proof. If (16) holds, then obviously (15) holds, too. Namely, if

$$\frac{a_1}{H(a_1,\dots,a_n)} + \frac{1}{6}\left[\frac{a_1}{H(a_1,\dots,a_n)}\right]^3 + \cdots \geq (n-2k)\frac{\pi}{2n},$$

then certainly (15) is valid because of the property of the arithmetics mean. □

Corollary 3. *If there exists a k-chordal polygon whose sides have the lengths $\frac{1}{a_1},\dots,\frac{1}{a_n}$ and $\frac{2k}{n} \geq \sin(n-2k)\frac{\pi}{2n}$, then there exist angles $\gamma_1,\dots,\gamma_n$ such that (5) and (6) hold.*

Proof. We shall use Corollary 2 in [6]. If $a_1,\dots,a_n$ are the lengths of the sides of the k-chordal polygon $\underline{A}$, then

$$\sum_{i=1}^{n} a_i > 2ka_j, \quad j = 1,\dots,n. \tag{17}$$

If $\dfrac{1}{a_1},\dots,\dfrac{1}{a_n}$ are also the lengths of the sides of a k-chordal polygon, then

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} > \frac{2k}{a_1}$$

or

$$a_1 > \frac{2k}{n}H(a_1,\dots,a_n). \tag{18}$$

Accordingly, if $\dfrac{2k}{n} \geq \sin(n-2k)\dfrac{\pi}{2n}$ then (16) is valid. □

Corollary 4. *If n is odd and k is maximal, i.e. $k = \dfrac{n-1}{2}$, then there exist the angles $\gamma_1,\dots,\gamma_n$ such that (5) and (6) hold.*

Proof. If $k = \dfrac{n-1}{2}$, then equation (5) can be written as

$$\gamma_1 + \ldots + \gamma_n = \frac{\pi}{2},$$

and obviously there is λ such that $\displaystyle\sum_{i=1}^{n} \arcsin\frac{\lambda}{a_i} = \frac{\pi}{2}$. □

Corollary 5. *If $n = 3$ and a, b, c are the lengths of the sides of an acute triangle, then*

$$2r > \frac{a^2+b^2+c^2}{a+b+c-\dfrac{3}{8}\dfrac{a^2}{H(a,b,c)}} \tag{19}$$

where $a = \min\{a, b, c\}$. In connection with this, the following remarks may be interesting.

Remark 2. *Since*

$$\sqrt{1-\left(\frac{\lambda}{a}\right)^2} < 1 - \frac{1}{2}\left(\frac{\lambda}{a}\right)^2,$$

inequality (19) follows from the inequality

$$2r \geq \frac{a^2+b^2+c^2}{\sqrt{a^2-\lambda^2}+\sqrt{b^2-\lambda^2}+\sqrt{c^2-\lambda^2}}, \tag{20}$$

where $\lambda = a\sin\dfrac{\pi}{6}$. *Here the equality appears for* $a = b = c$.
Analogously holds for inequality (7).

Remark 3. *In the case when* $n = 3$, Corollary 4 *can be also proved as follows:*

$$\begin{aligned}
\gamma_1+\gamma_2+\gamma_3 &= \tfrac{\pi}{2},\\
\cos(\gamma_1+\gamma_2) &= \sin\gamma_3,\\
\cos\gamma_1\cos\gamma_2 &= \sin\gamma_1\sin\gamma_2+\sin\gamma_3,\\
\sqrt{1-\left(\tfrac{\lambda}{a}\right)^2}\sqrt{1-\left(\tfrac{\lambda}{b}\right)^2} &= \tfrac{\lambda}{a}\tfrac{\lambda}{b}+\tfrac{\lambda}{c},\\
2abc\lambda^3+(a^2b^2+b^2c^2+c^2a^2)\lambda^2-a^2b^2c^2 &= 0.
\end{aligned}$$

The above equation in λ *has one positive root and it lies between 0 and* a *since* $f(0) < 0, f(a) > 0$, *where* $f(\lambda) = 2abc\lambda^3 + (a^2b^2 + b^2c^2 + c^2a^2)\lambda^2 - a^2b^2c^2$. *For example, if* $a_1 = a = 7, \quad a_2 = b = 8, \quad a_3 = c = 10$ *(*Figure 2*), then* $\lambda = 4.063986$ *and* $\gamma_1 = 35.49060749, \quad \gamma_2 = 30.53058949, \quad \gamma_3 = 23.97880303$.

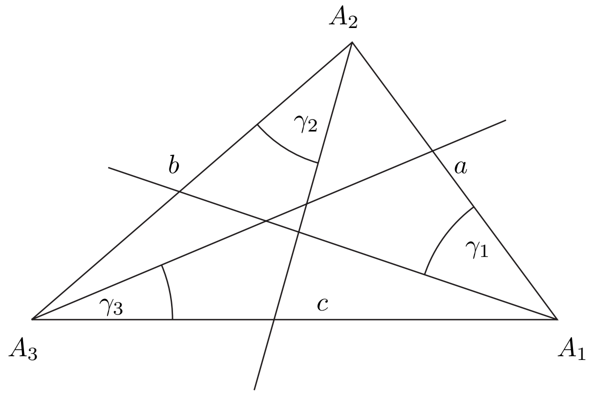

Figure 2.

Analogously holds in the case when $n > 3$. *But in this case it may be very difficult to solve the equation obtained in* λ. *So, if* $A_1 \ldots A_5$ *is a 2-chordal pentagon, then we have*

$$\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5 = \frac{\pi}{2},$$

$$\cos(\gamma_1+\gamma_2+\gamma_3+\gamma_4)=\sin\gamma_5,$$

$$\cos(\gamma_1+\gamma_2)\cos(\gamma_3+\gamma_4)-\sin(\gamma_1+\gamma_2)\sin(\gamma_3+\gamma_4)=\sin\gamma_5,$$

and so on. But it may be interesting that using the expressions

$$\sin\gamma_i=\frac{\lambda}{a_i},\quad \cos\gamma_i=\sqrt{1-\left(\frac{\lambda}{a_i}\right)^2},\quad i=1,\ldots,5$$

we obtain the equation which has a unique positive solution λ.

Corollary 6. *Let (for simplicity) in equation* (13) *in the case when* $n=4$ *there be written* a,b,c,d *instead of* a_1,a_2,a_3,a_4, *and let* $a=min\{a,b,c,d\}$. *Then there are angles* $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ *such that* (5) *and* (6) *hold in the case when* $n=4$ *if and only if*

$$\frac{a^2}{2}\le\frac{u}{v}\le a^2,$$

where

$$u=-\frac{1}{a^4}-\frac{1}{b^4}-\frac{1}{c^4}-\frac{1}{d^4}+\frac{2}{a^2b^2}+\frac{2}{a^2c^2}+\frac{2}{a^2d^2}+\frac{2}{b^2c^2}+\frac{2}{b^2d^2}+\frac{2}{c^2d^2}+\frac{8}{abcd},$$

$$v=\frac{4}{a^2b^2c^2}+\frac{4}{b^2c^2d^2}+\frac{4}{c^2d^2a^2}+\frac{4}{d^2a^2b^2}+\frac{4}{a^3bcd}+\frac{4}{ab^3cd}+\frac{4}{abc^3d}+\frac{4}{abcd^3}.$$

Proof. From $\gamma_1+\gamma_2+\gamma_3+\gamma_4=\pi,\quad a\sin\gamma_i=\lambda,\quad i=1,2,3,4,$ using the equality

$$\cos(\gamma_1+\gamma_2)=-\cos(\gamma_3+\gamma_4),$$

it can be found that

$$\begin{aligned}&4\left(1-\frac{\lambda^2}{a^2}\right)\left(1-\frac{\lambda^2}{b^2}\right)\left(1-\frac{\lambda^2}{c^2}\right)\left(1-\frac{\lambda^2}{d^2}\right)\\ =\ &\left[\left(1-\frac{\lambda^2}{a^2}\right)\left(1-\frac{\lambda^2}{b^2}\right)+\left(1-\frac{\lambda^2}{c^2}\right)\left(1-\frac{\lambda^2}{d^2}\right)+\frac{\lambda^4}{a^2b^2}+\frac{\lambda^4}{c^2d^2}+\frac{2\lambda^4}{abcd}\right]^2\end{aligned}$$

from which it follows that

$$u\lambda^4-v\lambda^6=0.$$

Consequently, $\lambda=\sqrt{\frac{u}{v}}$. Let as remark that by (14), $\lambda\ge\frac{a\sqrt{2}}{2}$. □

In connection with this, let us remark that $\sqrt{u}=4$ area of the chordal quadrangle whose sides have the lengths $\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c},\dfrac{1}{d}$.

Corollary 7. *The value* λ *given by (13) satisfies the following condition*

$$\lambda\le H(a_1,\ldots,a_n)\sin(n-2k)\frac{\pi}{2n}. \tag{21}$$

Proof. Using (13) by the appropriate property of the aritmetic mean we get the inequality

$$n\frac{\frac{\lambda}{a_1}+\ldots+\frac{\lambda}{a_n}}{n}+\frac{1}{6}n\left(\frac{\frac{\lambda}{a_1}+\ldots+\frac{\lambda}{a_n}}{n}\right)^3+\ldots\leq(n-2k)\frac{\pi}{2}$$

or

$$\arcsin\frac{\frac{\lambda}{a_1}+\ldots+\frac{\lambda}{a_n}}{n}\leq(n-2k)\frac{\pi}{2n},$$

from which it follows that (21) is valid. □

Thus, the solution in λ of equation (13) cannot exceed the right-hand side of (21).
If λ is the solution of equation (13), then from (10), that is, from

$$2r\geq\frac{\sum_{i=1}^{n}a_i^2}{\sum_{i=1}^{n}a_i\cos\gamma_i}\quad\text{or}\quad 2r\geq\frac{\sum_{i=1}^{n}a_i^2}{\sum_{i=1}^{n}\sqrt{a_i^2-a_i^2\sin^2\gamma_i}}$$

we have

$$2r\geq\frac{\sum_{i=1}^{n}a_i^2}{\sum_{i=1}^{n}\sqrt{a_i^2-\lambda^2}},\tag{22}$$

$$2r>\frac{\sum_{i=1}^{n}a_i^2}{\sum_{i=1}^{n}a_i\left[\sqrt{1-\frac{1}{2}\left(\frac{\lambda}{a_i}\right)^2}\right]},\tag{23}$$

The equality can appear in (22), but not in (23).
Let us consider the case when

$$\lambda=H(a_1,\ldots,a_n)\sin(n-2k)\frac{\pi}{2n}\tag{24}$$

Of course, we have such case when a k-chordal polygon is equilateral. Namely, then (22) can be written as

$$2r=\frac{a}{\cos(n-2k)\frac{\pi}{2n}},\tag{25}$$

and this is true since by this the diameter of a k-chordal equilateral polygon whose sides have the length a is given.

The following theorem is concerned with the radius of a k-tangential polygon.

Theorem 2. *Let $\underline{A} = A_1 \dots A_n$ be a given k-tangential polygon and let $t_1, \dots, t_n$ be the lengths of its tangents. Then*

$$\left(\frac{1}{t_1} + \cdots + \frac{1}{t_n}\right) \cos\left[(n-2k)\frac{\pi}{2n}\right] > 2k\left(1 - \frac{2k}{n}\right)\frac{1}{r}, \tag{26}$$

where r is the radius of the circle inscribed into $\underline{A}$.

Proof. Let $\beta_1, \dots, \beta_n$ be the angles such that

$$\beta_i = \angle CA_iA_{i+1}, \quad i = 1, \dots, n.$$

Then by Theorem 1 from paper [6]

$$\sum_{i=1}^{n} \cos\beta_i > 2k\cos\beta_j, \quad j = 1, \dots, n.$$

From this (since $r = t_j \operatorname{tg}\beta_j$) it follows that

$$r\sum_{i=1}^{n} \cos\beta_i > 2kt_j\sin\beta_j, \quad j = 1, \dots, n \tag{27}$$

or

$$\frac{r}{2k}\left(\frac{1}{t_1} + \cdots + \frac{1}{t_n}\right)\sum_{i=1}^{n} \cos\beta_i > \sum_{j=1}^{n} \sin\beta_j \tag{28}$$

Since $\sin(\pi x) > 2x$ if $0 < x < \frac{1}{2}$ and $\sin\alpha > \frac{2}{\pi}\alpha$ if $0 < \alpha < \frac{\pi}{2}$ (see proof of Theorem 1. in [6]), we have

$$\sum_{j=1}^{n} \sin\beta_j > \frac{2}{\pi}(\beta_1 + \cdots + \beta_n) = n - 2k. \tag{29}$$

Also we have

$$\sum_{i=1}^{n} \cos\beta_i \le n\cos(n-2k)\frac{\pi}{2n} \tag{30}$$

since the sum $\sum_{i=1}^{n} \cos\beta_i$ is maximal when $\beta_1 = \cdots = \beta_n$. From (28), (29) and (30) we get (26). □

Theorem 3. *Let $\underline{A} = A_1 \dots A_n$ be a k-chordal polygon and let $a_1 \dots a_n$ be the lengths of its sides. If n is even and the lengths $b_1, \dots, b_n$ are such that*

$$a_i^2 + b_i^2 = 4r^2, \quad i = 1, \dots, n$$

where r is the radius of the circle circumscribed to $\underline{A}$, then there is an $(\frac{n}{2}-k)$-chordal polygon with the property that $b_1, \dots, b_n$ are lengths of its sides and that the radius of its circumscribed circle is the same as the radius of the circumcircle of $\underline{A}$.

Proof. If $\underline{A}$ is a k-chordal polygon, then

$$\sum_{i=1}^{n} \beta_i = (n-2k)\frac{\pi}{2}, \quad \beta_i = \angle CA_iA_{i+1}, \quad i = 1, \dots, n$$

where C is the centre of the circle circumscribed to $\underline{A}$.
Let $\underline{B} = B_1 \dots B_n$ be a polygon such that

$$B_i = A_i, \quad i = 1, 3, \dots, n-1$$

$$B_i = A_i', \quad i = 2, 4, \dots, n$$

where C is the midpoint of A_iA_i', $\quad i = 2, 4, \dots, n$. Then the polygon $\underline{B}$ is an $(\frac{n}{2}-k)$-chordal polygon since

$$\sum_{i=1}^{n} \angle CB_iB_{i+1} = \sum_{i=1}^{n} \left(\frac{\pi}{2} - \beta_i\right) = n\frac{\pi}{2} - \sum_{i=1}^{n} \beta_i = n\frac{\pi}{2} - (n-2k)\frac{\pi}{2} = \left[n - 2\left(\frac{n}{2} - k\right)\right]\frac{\pi}{2}. \quad \square$$

Here is an example. See *Figure 3.* If $n = 6$ and $A_1 \dots A_6$ is a l-chordal hexagon, then $B_1 \dots B_6$ is a 2-chordal hexagon.

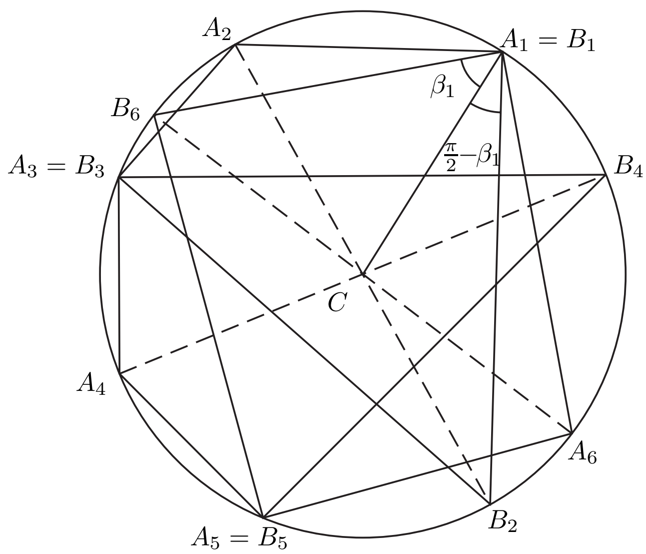

Figure 3.

In the following theorem we shall use the symbol S_j^n introduced in [7] with the following meaning: If $t_1, \ldots, t_n$ are given lengths, then S_j^n is the sum of all $\binom{n}{j}$ products of the form $t_{i_1} \ldots t_{i_j}$ where $i_1, \ldots, i_j$ are different indices of the set $\{1, \ldots, n\}$, that is

$$S_j^n = \sum_{1 \le i_1 < \cdots < i_j \le n} t_{i_1} \ldots t_{i_j}.$$

Also we shall use Theorem 2 proved in [7]:
Let $n \geq 3$ be any given odd number. Then

$$S_1^n r^{n-1} - S_3^n r^{n-3} + S_5^n r^{n-5} - \cdots + (-1)^s S_n^n = 0,$$

$$S_1^{n+1} r^{n-1} - S_3^{n+1} r^{n-3} + S_5^{n+1} r^{n-5} - \cdots + (-1)^s S_n^{n+1} = 0.$$

where $s = (1 + 3 + 5 + \cdots + n) + 1$.

Theorem 4. *Let $n \geq 4$ be an even number. If $\underline{A}$ is a k-tangential polygon whose tangents have the lengths $t_1, \ldots, t_n$, and if $\underline{B}$ is the $\left(\frac{n}{2} - k\right)$-tangential polygon whose tangents have the lengths $\dfrac{1}{t_1}, \ldots, \dfrac{1}{t_n}$, then $r\rho = 1$, where r is the radius of the circle inscribed into $\underline{A}$ and ρ is the radius of the circle inscribed into $\underline{B}$.*

Proof. Let R_i^n be obtained from S_i^n putting $\frac{1}{t_i}$ instead of t_i and let $s = [1 + 3 + 5 + \cdots + (n-1)] + 1$. Then

$$R_1^n \rho^{n-2} - R_3^n \rho^{n-4} + \cdots + (-1)^s R_{n-1}^n = 0, \tag{31}$$

and if the equation

$$S_1^n r^{n-2} - S_3^n r^{n-4} + \cdots + (-1)^s S_{n-1}^n = 0, \tag{32}$$

is divided by $t_1 \ldots t_n$, we obtain

$$R_{n-1}^n r^{n-2} - R_{n-3}^n r^{n-4} + \cdots + (-1)^s R_1^n = 0. \tag{33}$$

For example, if $n = 4$, we have the equation

$$(t_1 + t_2 + t_3 + t_4) r^2 - (t_1 t_2 t_3 + t_2 t_3 t_4 + t_3 t_4 t_1 + t_4 t_1 t_2) = 0,$$

from which, dividing by $t_1 t_2 t_3 t_4$, we get

$$R_3^4 r^2 - R_1^4 = 0 \quad \text{or} \quad R_1^4 \left(\frac{1}{r}\right)^2 - R_3^4 = 0,$$

where

$$R_3^4 = \frac{1}{t_1 t_2 t_3} + \frac{1}{t_2 t_3 t_4} + \frac{1}{t_3 t_4 t_1} + \frac{1}{t_4 t_1 t_2},$$

$$R_1^4 = \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}.$$

From (31) and (33) it is clear that for each r there is ρ such that $r\rho = 1$. Thus we have to prove that

$$r_k \rho_{\frac{n}{k}-k} = 1, \tag{34}$$

where r_k is the radius of the k-tangential n-gon whose tangents have the lengths $t_1, \dots, t_n$ and $\rho_{\frac{n}{k}-k}$ is the radius of the $\left(\frac{n}{2} - k\right)$-tangential n-gon whose tangents have the lengths $\frac{1}{t_1}, \dots, \frac{1}{t_n}$.
The proof is as follows. Let $\beta_1, \dots, \beta_n$ and $\gamma_1, \dots, \gamma_n$ be corresponding angles, that is,

$$\beta_1 + \dots + \beta_n = (n-2k)\frac{\pi}{2},$$

$$\gamma_1 + \dots + \gamma_n = \left[n - \left(\frac{n}{2} - k\right)\right]\frac{\pi}{2},$$

$$t_i = r_k ctg\beta_i, \quad \frac{1}{t_i} = \rho_{\frac{n}{2}-k} ctg\gamma_i, \quad i = 1, \dots, n.$$

From $1 = (r_k ctg\beta_i)(\rho_{\frac{n}{2}-k} ctg\gamma_i)$ we see that $r_k \rho_{\frac{n}{2}-k} = 1$ iff $\gamma_i = \frac{\pi}{2} - \beta_i$. Hence we have

$$\sum_{i=1}^{n}\left(\frac{\pi}{2} - \beta_i\right) = n\frac{\pi}{2} - \sum_{i=1}^{n}\beta_i = n\frac{\pi}{2} - (n-2k)\frac{\pi}{2} = \left[n - 2\left(\frac{n}{2} - k\right)\right]\frac{\pi}{2}.$$

And *Theorem 4* is proved. □

Here are some examples. If $n = 4$, then $r_1\rho_1 = 1$. If $n = 6$, then $r_1\rho_2 = r_2\rho_1 = 1$. If $n = 8$, then $r_1\rho_3 = r_2\rho_2 = r_3\rho_1 = 1$.
Especially, if $t_1 = \dots = t_n = 1$, then

$$r_k = tg\left((n-2k)\frac{\pi}{2n}\right), \quad k = 1, \dots, \frac{n-2}{2},$$

$$\rho_{\frac{n}{2}-k} = tg\left[\left(n - 2\left(\frac{n}{2} - k\right)\right)\frac{\pi}{2n}\right] = tg\frac{k\pi}{n},$$

$$r_k \rho_{\frac{n}{2}-k} = 1,$$

since $tg(n-2k)\frac{\pi}{2n} = tg\left(\frac{\pi}{2} - \frac{k\pi}{n}\right) = ctg\frac{k\pi}{n}$.
So, if $n = 6$ and $k = 1$, the situation is shown in *Figure 4*, where $r_1 = \sqrt{3}, \quad \rho_2 = \frac{1}{\sqrt{3}}$.

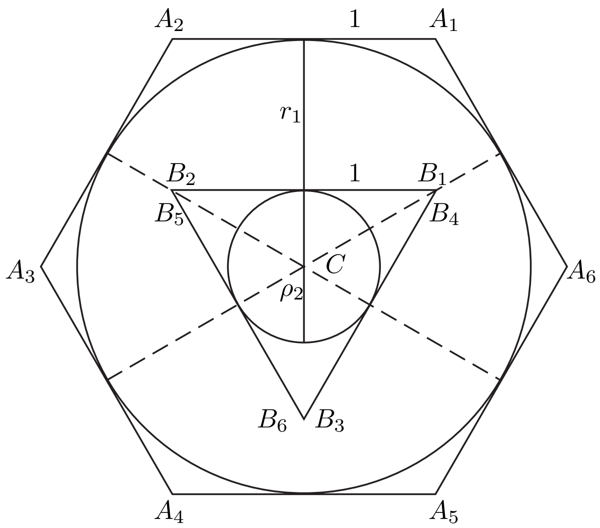

Figure 4.

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