A correction for a result on convergence of Ishikawa iteration for strongly pseudocontractive maps

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Abstract. We give a correction to the main result from [13].

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1. Introduction

Let $X$ be a real Banach space. Let $B$ be a nonempty, convex subset of $X$. Let $T : B \to B$ be a map. Let $x_1 \in B$. We consider the following iteration, see [4]:

\begin{align*}
  x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \\
  y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n = 1, 2, \ldots
\end{align*}

We suppose that $(\alpha_n), (\beta_n) \subset (0, 1)$, and the sequence $(\alpha_n)$ satisfies

\begin{equation}
  0 < w \leq \alpha_n \leq 1.
\end{equation}

For $\beta_n = 0, \forall n \in N$ we get Mann iteration, see [5]. Ishikawa iteration with condition (1) is studied in [13]. In [7] it was proven that two assumptions of the main theorem from [13] are contradictory. In this note we will prove that renouncing to one assumption from [13] and supposing true an assumption à la [3], the above theorem from [13] is true.

The map $J : X \to 2^{X^*}$ given by

\[ Jx := \{ f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\| \}, \forall x \in X, \]

is called the normalized duality mapping. The Hahn-Banach theorem assures that $Jx \neq \emptyset, \forall x \in X$. It is easy to see that we have

\begin{equation}
  \langle j(x), y \rangle \leq \|x\| \|y\|, \forall x, y \in X, \forall j(x) \in J(x).
\end{equation}

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Definition 1. Let $X$ be a real Banach space, let $B$ be a nonempty subset. A map $T : B \rightarrow B$ is called strongly pseudocontractive if for all $x, y \in B$, there exists $j(x - y) \in J(x - y)$ such that

$$\exists \gamma \in (0, 1) : \langle Tx - Ty, j(x - y) \rangle \leq \gamma \|x - y\|^2.$$  \hspace{1cm} (3)

The following Lemma could be found in [6], [12], with different proofs. A particular form of this lemma is in [11].

Lemma 1. [6], [11], [12] If $X$ is a real Banach space, then the following relation is true

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \forall x, y \in X, \forall j(x + y) \in J(x + y).$$  \hspace{1cm} (4)

The following result is from [9]. Three other proofs could be found in [10].

Proposition 1. [9], [10]. Let $(a_n)_n$ be a nonnegative sequence which satisfies

$$a_{n+1} \leq (1 - w)a_n + \sigma_n S,$$  \hspace{1cm} (5)

where $w \in (0, 1), S > 0$ are fixed numbers, $\sigma_n \geq 0, \forall n \in N, \lim_{n \to \infty} \sigma_n = 0$. Then $\lim_{n \to \infty} a_n = 0$.

2. Main result

We are able now to give the following result.

Theorem 1. Let $X$ be a real Banach space, and let $T : X \rightarrow X$ be a continuous, strongly pseudocontractive with bounded range map. If $\lim_{n \to \infty} \|Ty_n - Tx_{n+1}\| = 0$, and $\gamma \in (0, 1/2)$, then the iteration (1) strongly converges to the unique fixed point of $T$.

Proof. The existence follows from [2] and the uniqueness from strongly pseudocontractivity. Let $x^* = Tx^*$. Using (4) for the first inequality, (2) and (3) for the third one, we can see:

$$\|x_{n+1} - x^*\|^2 \leq \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\|^2$$

$$\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Ty_n - x^*, j(x_{n+1} - x^*) \rangle$$

$$\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Ty_n - Tx_{n+1}, j(x_{n+1} - x^*) \rangle$$

$$+ 2\alpha_n \langle Tx_{n+1} - x^*, J(x_{n+1} - x^*) \rangle$$

$$\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \|Ty_n - Tx_{n+1}\| \|x_{n+1} - x^*\|$$

$$+ 2\alpha_n \gamma \|x_{n+1} - x^*\|^2, \forall j(x_{n+1} - x^*) \in J(x_{n+1} - x^*).$$

There results

$$\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \|Ty_n - Tx_{n+1}\| \|x_{n+1} - x^*\|,$$

$$\|x_{n+1} - x^*\|^2 \leq \frac{(1 - \alpha_n)^2}{(1 - 2\alpha_n \gamma)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{(1 - 2\alpha_n \gamma)} \|Ty_n - Tx_{n+1}\| \|x_{n+1} - x^*\|. $$
Because $\gamma \in (0, 1/2)$, $\alpha_n \in (0, 1) \Rightarrow \frac{2(1-\gamma) - \alpha_n}{1 - 2\alpha_n \gamma} \geq 1$ i.e. $- \left( \frac{2(1-\gamma) - \alpha_n}{1 - 2\alpha_n \gamma} \right) \leq -1$, we have

\[
\frac{(1-\alpha_n)^2}{(1-2\alpha_n \gamma)} = \frac{1 - 2\alpha_n + \alpha_n^2}{(1-2\alpha_n \gamma)} = \frac{(1 - 2\alpha_n \gamma) + 2\alpha_n \gamma - 2\alpha_n + \alpha_n^2}{(1-2\alpha_n \gamma)} = 1 - \left( \frac{2(1-\gamma) - \alpha_n}{1 - 2\alpha_n \gamma} \right) \alpha_n \leq 1 - \alpha_n.
\]

(6)

Also, the sequence $(x_n)$ is bounded. We will prove that by induction. Let us denote by $d := \sup\{\|Tx\| : x \in B\} + \|x^*\|$. Because the range of $T$ is bounded we have $d < \infty$. We denote by $M := d + \|x_0 - x^*\| + 1$. Observe that

\[
\|x_1 - x^*\| \leq (1 - \alpha_0) \|x_0 - x^*\| + \alpha_0 \|Ty_0 - x^*\| \\
\leq (1 - \alpha_0)M + \alpha_0(\|Ty_0\| + \|x^*\|) \leq (1 - \alpha_0)M + \alpha_0M = M.
\]

Supposing $\|x_n - x^*\| \leq M$, we will prove that $\|x_{n+1} - x^*\| \leq M$. Indeed we have

\[
\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|Ty_n - x^*\| \\
\leq (1 - \alpha_n)M + \alpha_n(\|Ty_n\| + \|x^*\|) \leq (1 - \alpha_n)M + \alpha_nM = M.
\]

Thus we have

\[
\exists M > 0 : \|x_{n+1} - x^*\| \leq M, \forall n \geq 0.
\]

(7)

Conditions (6), (7) and (8) lead us to

\[
\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n) \|x_n - x^*\|^2 + \|Ty_n - Tx_{n+1}\| \frac{2\alpha_n}{1 - 2\alpha_n \gamma} M.
\]

But $(1 - \alpha_n) \leq (1 - w)$, and $\frac{2\alpha_n}{1 - 2\alpha_n \gamma} \leq \frac{2}{1 - 2\gamma}$. So, we have

\[
\|x_{n+1} - x^*\|^2 \leq (1 - w) \|x_n - x^*\|^2 + \|Ty_n - Tx_{n+1}\| \frac{2}{1 - 2\gamma} M.
\]

Let us denote be $a_n := \|x_n - x^*\|^2$, $\sigma_n := \|Ty_n - Tx_{n+1}\|$, and $S := \frac{2}{1 - 2\gamma} M$. Then we have $\lim_{n \to \infty} a_n = 0$. Thus $\lim_{n \to \infty} x_n = x^*$.

**References**


